UNIVERSITÄT LEIPZIG

Joint work with Hector Jardon Sanchez and Samuel Mellick

On Property (T) for Unimodular Random Graphs

Overview

1. Unimodular Random Graphs
2. Why URGs?
3. Property (T) for URGs

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3. Property (T) for URGs

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Example - percolations

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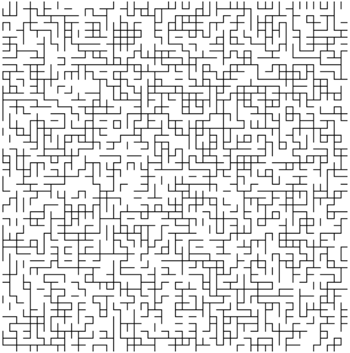

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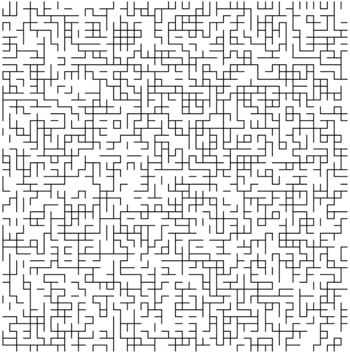

Fix a Cayley graph of a finitely generated group.

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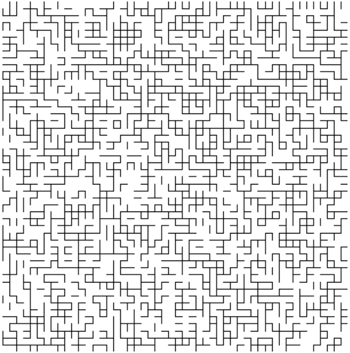

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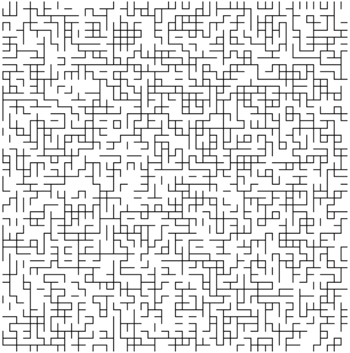

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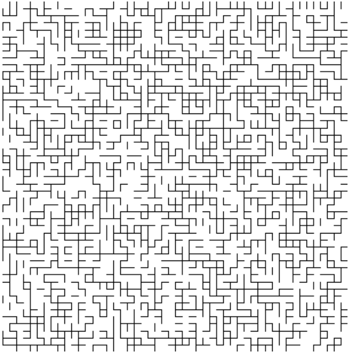

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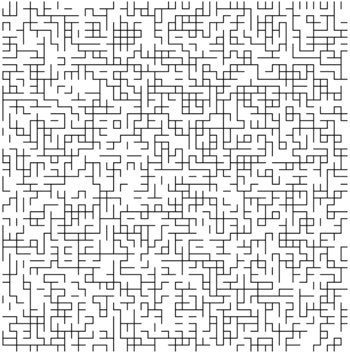

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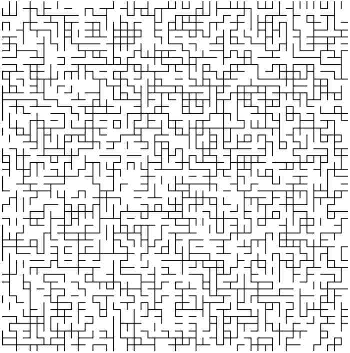

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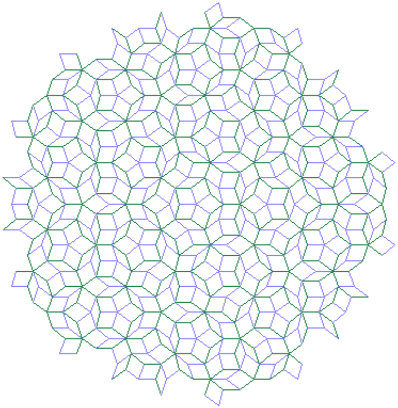

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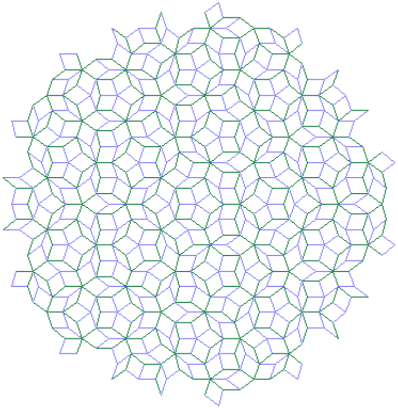

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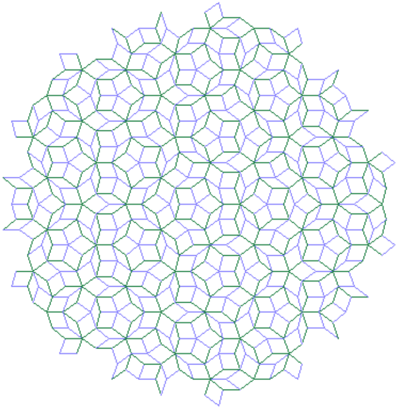

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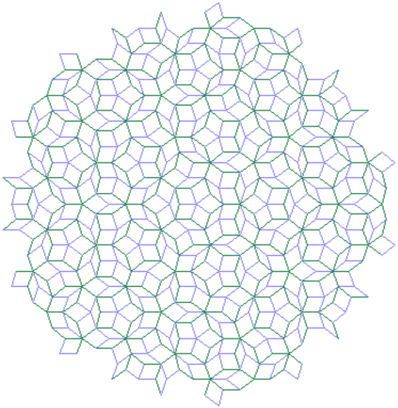

measure: take a very large square, choose root at random.

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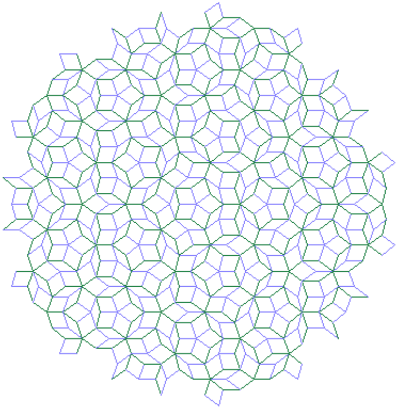

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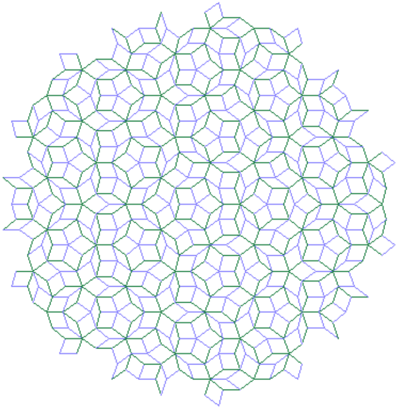

measure: take a very large square, choose root at random. This gives a probability measure on $\mathcal{G}raphs_d$ supported on finite graphs. Take a weak limit of these probability measures.

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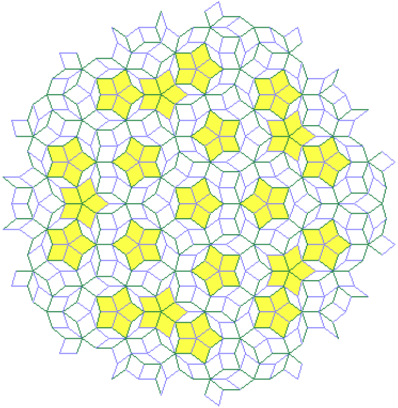

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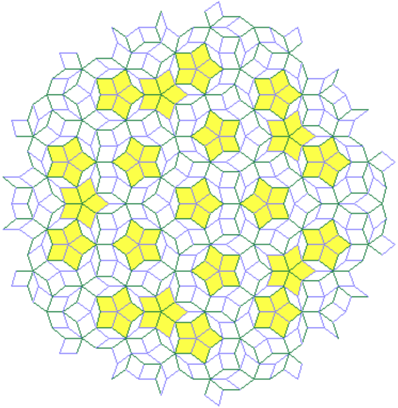

Whenever we see a “5-star”, we glue in five 2-dimensional cells.

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Example - Benjamini-Schramm convergence

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1. Unimodular Random Graphs

2. Why URGs?

3. Property (T) for URGs

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- Then $\lim_{i\to\infty} \frac{b_1(X_i)}{|X_i^0|} = 0$.

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THANK YOU FOR YOUR ATTENTION!

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