UNIVERSITÄT
LEIPZIG

Lecture 2 - The fundamental group functor. Homotopy of maps

Algebraic Topology (SS 2023-24)

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Overview

1. Recap
2. The fundamental group
3. Change of basepoint
4. Extending π_1 to a functor
5. Homotopy of maps

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  - Examples: “all continuous maps”, “all continuous maps which are homeomorphisms”, “Lipschitz-continuous maps”, “group homomorphisms“, “ring homomorphisms” etc.

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- This allows us to define $\pi_0(f)$.

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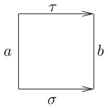

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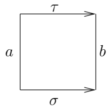

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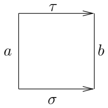

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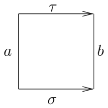

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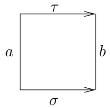

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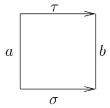

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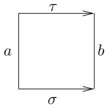

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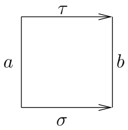

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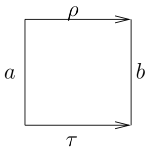

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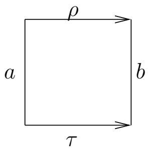

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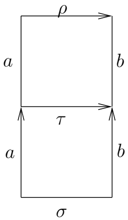

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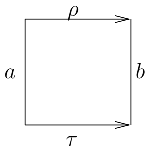

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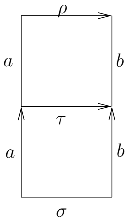

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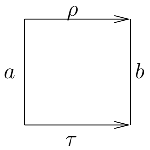

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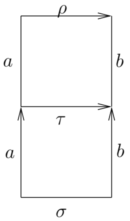

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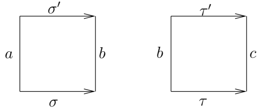

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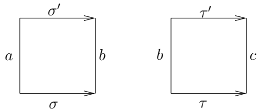

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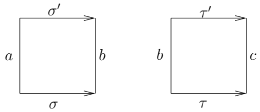

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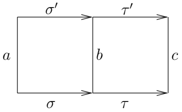

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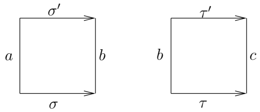

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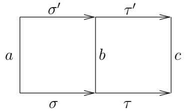

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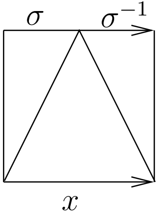

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This finishes the definition of the fundamental group functor.

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- Exercise: Show that $f \simeq g \operatorname{rel} A$ is an equivalence relation.

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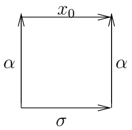

Proposition. If X is contractible then it is simply connected.

Proof.

- Let's see that X is path-connected. If $F\colon \mathrm{id}_X \simeq \mathrm{const}_{x_0}$ is a homotopy then for any $y \in X$ we can consider the path $\sigma\colon y \overset{p}{\rightsquigarrow} x_0$ given by $\sigma(t) := F(y,t)$.
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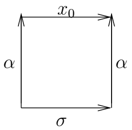

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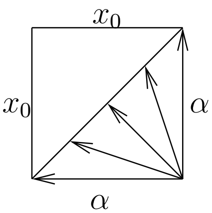

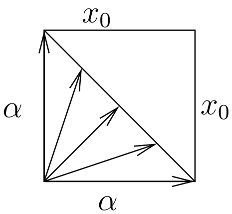

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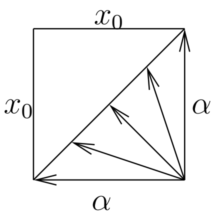

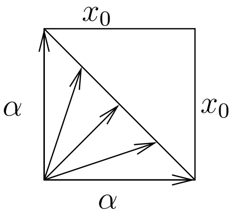

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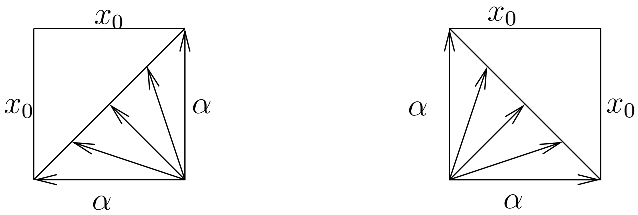

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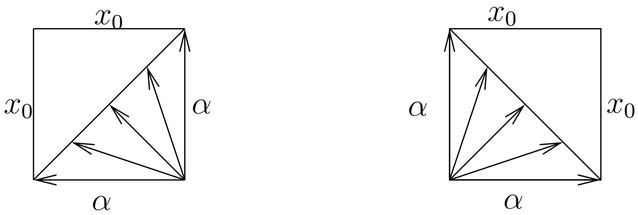

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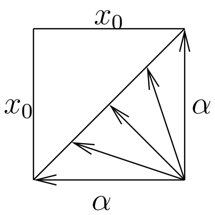

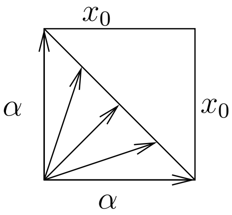

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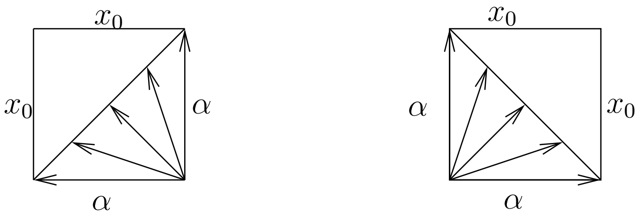

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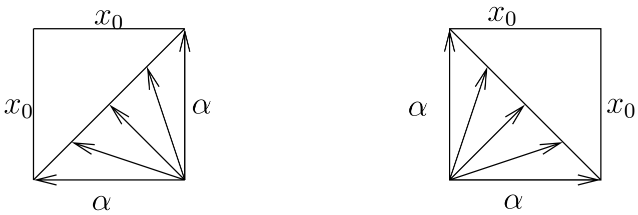

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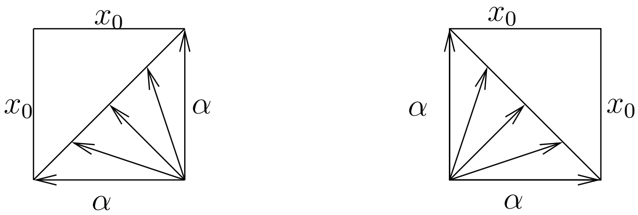

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THANK YOU FOR YOUR ATTENTION!

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