UNIVERSITÄT
LEIPZIG

Lecture 1 - Fundamental Group Functor

Algebraic Topology (SS 2023-24)

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Overview

1. Some general point-set topology
2. Categories and functors
3. Definition of the fundamental group functor

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- A convenient way to talk about invariants is with the language of categories and functors.

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  - Examples: “all continuous maps”, “all continuous maps which are homeomorphisms”, “Lipschitz-continuous maps”, “group homomorphisms“, “ring homomorphisms” etc.

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- In this course most of the time $\mathcal{D}$ is AbelianGroups.

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- This means that functors can be used to show that two spaces X and Y are not homeomorphic. A necessary condition for X and Y to be homeomorphic is that the objects $F(X)$ and $F(Y)$ should be isomorphic.

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- This allows us to define $\pi_0(f)$.

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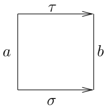

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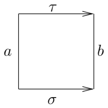

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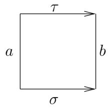

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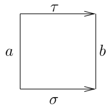

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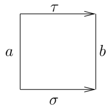

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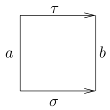

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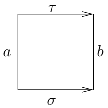

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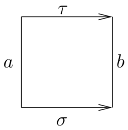

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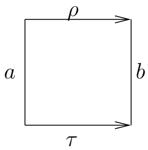

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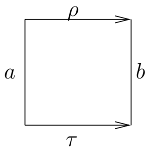

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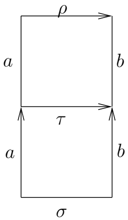

which shows that indeed $\sigma \simeq \rho\,\mathrm{rel}\{0,1\}$, as claimed.

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THANK YOU FOR YOUR ATTENTION!

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