Limit multiplicities for principal congruence subgroups of \${\rm GL}({n})\$ and \${\rm SL}({n})\$

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LIMIT MULTIPLICITIES FOR PRINCIPAL CONGRUENCE SUBGROUPS OF $\mathbf{GL}(n)$ AND $\mathbf{SL}(n)$

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Abstract   We study the limiting behavior of the discrete spectra associated to the principal congruence subgroups of a reductive group over a number field. While this problem is well understood in the cocompact case (i.e., when the group is anisotropic modulo the center), we treat groups of unbounded rank. For the groups $\mathbf{GL}(n)$ and $\mathbf{SL}(n)$ we show that the suitably normalized spectra converge to the Plancherel measure (the limit multiplicity property). For general reductive groups we obtain a substantial reduction of the problem. Our main tool is the recent refinement of the spectral side of Arthur's trace formula obtained in [Finis, Lapid, and Müller, *Ann. of Math. (2)* 174(1) (2011), 173–195; Finis and Lapid, *Ann. of Math. (2)* 174(1) (2011), 197–223], which allows us to show that for $\mathbf{GL}(n)$ and $\mathbf{SL}(n)$ the contribution of the continuous spectrum is negligible in the limit.

Keywords: limit multiplicities; lattices in Lie groups; trace formula

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Contents

1. Introduction

Let (for now) G be a connected linear semisimple Lie group with a fixed choice of a Haar measure. Since the group G is of type I, we can write unitary representations of G on separable Hilbert spaces as direct integrals (with multiplicities) over the unitary dual $\Pi(G)$, the set of isomorphism classes of irreducible unitary representations of G with the Fell topology (cf. [30]). An important case is the regular representation of $G \times G$ on $L^2(G)$, which can be decomposed as the direct integral of the tensor products $\pi \otimes \pi^*$ against the *Plancherel measure* μ_{pl} on $\Pi(G)$. The support of the Plancherel measure is called the *tempered dual* $\Pi(G)_{\mathrm{temp}} \subset \Pi(G)$.

Other basic objects of interest are the regular representations R_Γ of G on $L^2(\Gamma \backslash G)$ for lattices Γ in G. We will focus on the discrete part $L^2_{\mathrm{disc}}(\Gamma \backslash G)$ of $L^2(\Gamma \backslash G)$, namely the sum of all irreducible subrepresentations, and we denote by $R_{\Gamma,\mathrm{disc}}$ the corresponding restriction of R_Γ. For any $\pi \in \Pi(G)$, let $m_\Gamma(\pi)$ be the multiplicity of π in $L^2(\Gamma \backslash G)$. Thus,

$$m_\Gamma(\pi) = \dim \mathrm{Hom}_G(\pi, R_\Gamma) = \dim \mathrm{Hom}_G(\pi, R_{\Gamma,\mathrm{disc}}).$$

These multiplicities are known to be finite [60, Theorem 3.3], at least under a weak reduction-theoretic assumption on G and Γ [60, p. 62], which is satisfied if G has no compact factors or if Γ is arithmetic. We define the discrete spectral measure on $\Pi(G)$ with respect to Γ by

$$\mu_\Gamma = \frac{1}{\mathrm{vol}(\Gamma \backslash G)} \sum_{\pi \in \Pi(G)} m_\Gamma(\pi) \delta_\pi,$$

where δ_π is the Dirac measure at π. While one cannot hope to describe the multiplicity functions m_Γ on $\Pi(G)$ explicitly (apart from certain special cases, for example when π belongs to the discrete series), it is feasible and interesting to study asymptotic questions. The limit multiplicity problem concerns the asymptotic behavior of μ_Γ as $\mathrm{vol}(\Gamma \backslash G) \to \infty$.

To make this more explicit, we recall that, up to a closed subset of Plancherel measure zero, the topological space $\Pi(G)_{\mathrm{temp}}$ is homeomorphic to a countable union of Euclidean spaces of bounded dimensions, and that under this homeomorphism the Plancherel density is given by a continuous function. (The same is true in the case of p-adic reductive groups considered below.)[1] We call the relatively quasi-compact subsets of $\Pi(G)$ *bounded* (see § 2 below for a more explicit description). Note that $\mu_\Gamma(A) < \infty$ for bounded sets $A \subset \Pi(G)$ under the reduction-theoretic assumption on G and Γ mentioned above [14]. By definition, a Jordan measurable subset A of $\Pi(G)_{\mathrm{temp}}$ is a bounded set such that $\mu_{\mathrm{pl}}(\partial A) = 0$, where $\partial A = \bar{A} - A^\circ$ is the boundary of A in $\Pi(G)_{\mathrm{temp}}$. A Riemann integrable function on $\Pi(G)_{\mathrm{temp}}$ is a bounded, compactly supported function which is continuous almost everywhere with respect to the Plancherel measure.

Let $\Gamma_1, \Gamma_2, \dots$ be a sequence of lattices in G. We say that the sequence (Γ_n) has the limit multiplicity property if the following two conditions are satisfied.

(1) For any Jordan measurable set $A \subset \Pi(G)_{\mathrm{temp}}$, we have

$$\mu_{\Gamma_n}(A) \to \mu_{\mathrm{pl}}(A) \quad \text{as } n \to \infty.$$

(2) For any bounded set $A \subset \Pi(G) \setminus \Pi(G)_{\mathrm{temp}}$, we have

$$\mu_{\Gamma_n}(A) \to 0 \quad \text{as } n \to \infty.$$

Note that we can rephrase the first condition by requiring that

$$\lim_{n \to \infty} \mu_{\Gamma_n}(f) = \mu_{\mathrm{pl}}(f)$$

for any Riemann integrable function (or, alternatively, for any continuous compactly supported function) f on $\Pi(G)_{\mathrm{temp}}$.

A great deal is known about the limit multiplicity problem for uniform lattices, where R_Γ decomposes discretely. The first results in this direction were proved by DeGeorge and Wallach [27, 32, 72] for normal towers, i.e., descending sequences of finite index normal subgroups of a given uniform lattice with trivial intersection. Subsequently, Delorme [26] completely resolved the limit multiplicity problem for this case in the affirmative. Recently, there has been great progress in proving limit multiplicity for much more general sequences of uniform lattices [1, 2]. In particular, families of non-commensurable lattices were considered for the first time.

In the case of non-compact quotients $\Gamma \backslash G$, where the spectrum also contains a continuous part, much less is known. Here, the limit multiplicity problem has been solved for normal towers of arithmetic lattices and discrete series L-packets $A \subset \Pi(G)$ (with regular parameters) by Rohlfs and Speh [64]. Building on this work, the case of singleton sets A and normal towers of congruence subgroups has been solved by Savin ([67],

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[1] See [26, § 2.3] for the Archimedean case and [75] for the p-adic case.

cf. also [74]). Earlier results on the discrete series had been obtained by DeGeorge [25] and Barbasch and Moscovici [16] for groups of real rank one, and by Clozel [22] for general groups (but with a weaker statement). The limit multiplicity problem for the entire unitary dual has been solved for the principal congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ by Sarnak [65] (cf. [39, p. 173], [29, § 5]). Also, a refined quantitative version of the limit multiplicity property for the non-tempered spectrum of the subgroups $\Gamma_0(N)$ has been proven by Iwaniec [40].[2] A partial result for certain normal towers of congruence arithmetic lattices defined by groups of $\mathbb{Q}$-rank one has been shown by Deitmar and Hoffmann in [29]. Finally, generalizations to the distribution of Hecke eigenvalues have been obtained by Sauvageot [66], Shin [68] and Shin and Templier [69].

In this paper we embark upon a general analysis of the case of non-compact quotients. We consider the entire unitary dual and groups of unbounded rank. The main problem is to show that the contribution of the continuous spectrum is negligible in the limit. This was known up to now only in the case of $\mathrm{GL}(2)$ (or implicitly in the very special situation considered in [64] and [67]). Our approach is based on a careful study of the spectral side of Arthur's trace formula in the recent form given in [33, 34]. As we shall see, this form is crucial for the analysis. Our results are unconditional only for the groups $\mathrm{GL}(n)$ and $\mathrm{SL}(n)$, but we obtain a substantial reduction of the problem in the general case.

Before stating our main result, we shift to an adelic setting which allows one to incorporate Hecke operators into the picture (i.e., to consider the equidistribution of Hecke eigenvalues). Thus, let now G be a reductive group defined over a number field F, and let S be a finite set of places of F containing the set S_∞ of all Archimedean places. Let F_S be the product of the completions F_v for $v \in S$, $\mathbb{A}^S$ the restricted product of the F_v for $v \notin S$, and $\mathbb{A} = F_S \times \mathbb{A}^S$ the ring of adeles of F. In the special case when $S = S_\infty$, we write $F_\infty = F_{S_\infty}$ and $\mathbb{A}_{\mathrm{fin}} = \mathbb{A}^{S_\infty}$. As usual, $G(F_S)^1$ denotes the intersection of the kernels of the homomorphisms $|\chi| : G(F_S) \to \mathbb{R}^{>0}$, where χ ranges over the F-rational characters of G and $|\cdot|$ denotes the normalized absolute value on F_S^*. Similarly, we define the normal subgroup $G(\mathbb{A})^1$ of $G(\mathbb{A})$. Fix a Haar measure on $G(\mathbb{A})$. For any open compact subgroup K of $G(\mathbb{A}^S)$, let $\mu_K = \mu_K^{G,S}$ be the measure on $\Pi(G(F_S)^1)$ given by

$$
\mu_K = \frac{1}{\mathrm{vol}(G(F)\backslash G(\mathbb{A})^1/K)} \sum_{\pi \in \Pi(G(F_S)^1)} \dim \mathrm{Hom}_{G(F_S)^1}(\pi, L^2(G(F)\backslash G(\mathbb{A})^1/K)) \, \delta_\pi
$$

$$
= \frac{\mathrm{vol}(K)}{\mathrm{vol}(G(F)\backslash G(\mathbb{A})^1)} \sum_{\pi \in \Pi(G(\mathbb{A})^1)} \dim \mathrm{Hom}_{G(\mathbb{A})^1}(\pi, L^2(G(F)\backslash G(\mathbb{A})^1)) \, \dim(\pi^S)^K \, \delta_{\pi_S}.
$$

We say that a collection $\mathcal{K}$ of open compact subgroups of $G(\mathbb{A}^S)$ has the limit multiplicity property if $\mu_K \to \mu_{\mathrm{pl}}$ for $K \in \mathcal{K}$ in the sense that,

(1) for any Jordan measurable subset $A \subset \Pi(G(F_S)^1)_{\mathrm{temp}}$, we have $\mu_K(A) \to \mu_{\mathrm{pl}}(A)$, $K \in \mathcal{K}$, and,

(2) for any bounded subset $A \subset \Pi(G(F_S)^1) \setminus \Pi(G(F_S)^1)_{\mathrm{temp}}$, we have $\mu_K(A) \to 0$, $K \in \mathcal{K}$.

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[2]Recall that by Selberg's eigenvalue conjecture the non-tempered spectrum should consist only of the trivial representation in this case.

Here, we write for example $\mu_K(A) \to \mu_{\mathrm{pl}}(A)$ to mean that for every $\epsilon > 0$ there are only finitely many subgroups $K \in \mathcal{K}$ such that $\left|\mu_K(A) - \mu_{\mathrm{pl}}(A)\right| \geqslant \epsilon$. Once again, we can rephrase the first condition by saying that for any Riemann integrable function f on $\Pi(G(F_S)^1)_{\mathrm{temp}}$ we have

$$\mu_K(f) \to \mu_{\mathrm{pl}}(f), \quad K \in \mathcal{K}.$$

Remark 1.1. By a well-known result of Wallach, for any reductive group G, any local factor of an irreducible representation which occurs in the residual spectrum (i.e., the non-cuspidal discrete spectrum) is necessarily non-tempered (see [73, Theorem 4.3] for the Archimedean and [23, Proposition 4.10] for the p-adic case). Therefore, once the limit multiplicity property is established, it automatically holds for the cuspidal spectrum as well.

We note that, when G satisfies the strong approximation property with respect to S_∞ (which is tantamount to saying that G is semisimple, simply connected, and without any F-simple factors H for which $H(F_\infty)$ is compact [61, Theorem 7.12]) and K is an open compact subgroup of $G(\mathbb{A}_{\mathrm{fin}})$, we have

$$G(F)\backslash G(\mathbb{A})/K \simeq \Gamma_K \backslash G(F_\infty)$$

for the lattice $\Gamma_K = G(F) \cap K$ in the connected semisimple Lie group $G(F_\infty)$. Thus, these lattices are incorporated in the adelic setting. A similar connection can be made for general G, where however a single subgroup K will correspond to a finite set of lattices in $G(F_\infty)$.

An important step in the analysis of the limit multiplicity problem is to reduce it to a question about the trace formula. This is non-trivial, not the least because of the complicated nature of the unitary dual. This step was carried out by Delorme in the case where S consists of the Archimedean places [26]. His argument was subsequently extended by Sauvageot to the general case [66], where he also axiomatized the essential property as a "density principle" (see § 2 below). Using the result of Sauvageot, we can recast the limit multiplicity problem as follows. Let $\mathcal{H}(G(F_S)^1)$ be the algebra of smooth, compactly supported bi-K_S-finite functions on $G(F_S)^1$. For any $h \in \mathcal{H}(G(F_S)^1)$, let $\hat{h}$ be the function on $\Pi(G(F_S)^1)$ given by $\hat{h}(\pi) = \mathrm{tr}\,\pi(h)$. Note that we have

$$\mu_K(\hat{h}) = \frac{1}{\mathrm{vol}(G(F)\backslash G(\mathbb{A})^1)}\,\mathrm{tr}\,R_{\mathrm{disc}}(h \otimes \mathbf{1}_K)$$

and

$$\mu_{\mathrm{pl}}(\hat{h}) = h(1).$$

Then we have the following theorem.

Theorem 1.2 (Sauvageot)**.** *Suppose that the collection $\mathcal{K}$ has the property that for any function $h \in \mathcal{H}(G(F_S)^1)$ we have*

$$\mu_K(\hat{h}) \to h(1), \quad K \in \mathcal{K}. \tag{1}$$

Then limit multiplicity holds for $\mathcal{K}$.

We will recall how to obtain this result from Sauvageot's density principle in § 2.

Given this reduction, it is natural to attack assertion (1) via the trace formula. In the cocompact case (i.e., when $G/Z(G)$ is anisotropic over F) one can use the Selberg trace formula. In the general case we use Arthur's (non-invariant) trace formula, which expresses a certain distribution $h \mapsto J(h)$ on $C_c^\infty(G(\mathbb{A})^1)$ geometrically and spectrally [3, 5, 7–10]. The distribution J depends on the choice of a maximal F-split torus $\mathbf{S}_0$ of G and a suitable maximal compact subgroup $\mathbf{K} = \mathbf{K}_S \mathbf{K}^S$ of $G(\mathbb{A})$ (cf. §§ 3 and 4 below). The main terms on the geometric side are the elliptic orbital integrals, most notably the contribution $\mathrm{vol}(G(F)\backslash G(\mathbb{A})^1)h(1)$ of the identity element. The main term on the spectral side is $\mathrm{tr}\, R_{\mathrm{disc}}(h)$.

The relation (1) can now be broken down into the following two statements.

$$\text{For any } h \in \mathcal{H}(G(F_S)^1), \text{ we have } J(h \otimes \mathbf{1}_K) - \mathrm{tr}\, R_{\mathrm{disc}}(h \otimes \mathbf{1}_K) \to 0, \tag{2}$$

and,

$$\text{for any } h \in \mathcal{H}(G(F_S)^1), \text{ we have } J(h \otimes \mathbf{1}_K) \to \mathrm{vol}(G(F)\backslash G(\mathbb{A})^1)h(1). \tag{3}$$

We call these assertions the *spectral* and *geometric limit properties*, respectively.

In the cocompact case, the spectral limit property is trivial, since $J(h) = \mathrm{tr}\, R_{\mathrm{disc}}(h)$. Also, in this case it is easy to see that for any tower $\mathcal{K}$ of normal subgroups K of $\mathbf{K}^S$ and for every $h \in \mathcal{H}(G(F_S)^1)$ we have in fact $J(h \otimes \mathbf{1}_K) = \mathrm{vol}(G(F)\backslash G(\mathbb{A})^1)h(1)$ for almost all $K \in \mathcal{K}$. This is Sauvageot's proof of the limit multiplicity property in this case.

In general, both properties are non-trivial. In this paper, we consider only the simplest collection of normal subgroups of $\mathbf{K}^S$, namely the *principal* congruence subgroups $\mathbf{K}^S(\mathfrak{n})$ of $\mathbf{K}^S$ for non-zero ideals $\mathfrak{n}$ of $\mathfrak{o}_F$ prime to S (see § 3). In this case, the geometric limit property is a consequence of Arthur's analysis of the unipotent contribution to the trace formula in [9] (see § 3, in particular Corollary 3.3). The main task is to prove the spectral limit property for this collection of subgroups. We are able to do this unconditionally for the groups $\mathrm{GL}(n)$ and $\mathrm{SL}(n)$, and consequently obtain the following as our main result.

Theorem 1.3. *Let G be either $\mathrm{GL}(n)$ or $\mathrm{SL}(n)$ over a number field F. Then limit multiplicity holds for the collection of all principal congruence subgroups $\mathbf{K}^S(\mathfrak{n})$ of $\mathbf{K}^S$.*

As explained above, for $G = \mathrm{SL}(n)$ over F and (for simplicity) $S = S_\infty$, the strong approximation theorem [61, Theorem 7.12] allows for the following reformulation of this result in terms of lattices in the semisimple Lie groups $\mathrm{SL}(n, F_\infty)$.

Corollary 1.4. *Limit multiplicity holds for the collection of the principal congruence subgroups*

$$\Gamma(\mathfrak{n}) = \{\gamma \in \mathrm{SL}(n, \mathfrak{o}_F) : \gamma \equiv 1 \pmod{\mathfrak{n}}\}$$

of the lattice $\mathrm{SL}(n, \mathfrak{o}_F)$ in the semisimple Lie group $\mathrm{SL}(n, F_\infty)$.

The key input for our approach to the spectral limit property is the refinement of the spectral expansion of Arthur's trace formula established in [34] (cf. Theorem 4.1 below). This result enables us to set up an inductive argument which relies on two conjectural

properties, one global and one local, which we call (TWN) (tempered winding numbers) and (BD) (bounded degree), respectively. They are stated in § 5, and are expected to hold for any reductive group G over a number field. Theorem 1.3 is proved for any group G satisfying these properties (see Theorem 7.9).

The global property (TWN) is a uniform estimate on the winding number of the normalizing scalars of the intertwining operators in the co-rank one case. For $GL(n)$ and $SL(n)$, this property follows from known, but delicate, properties of the Rankin–Selberg L-functions (Proposition 5.5). In order to describe the local property (BD), recall that in the non-Archimedean case the matrix coefficients of the local intertwining operators are rational functions of q^{-s}, where q is the cardinality of the residue field, and that the degrees of the denominators are bounded in terms of G only. Property (BD) gives an upper bound on the degree of the numerator in terms of the level. This property was studied in [35], where among other things it was proved for the groups $G = GL(n)$ (and implicitly also for $SL(n)$). The import of property (BD) is that it yields a good bound for integrals of logarithmic derivatives of normalized intertwining operators (Proposition 5.16). The Archimedean analog of property (BD) (for a general real reductive group) had been established in [54, Appendix].

The analysis is carried out by induction on the semisimple rank of G. Actually, for the induction step it is necessary to verify that the collection of measures $\{\mu_{\mathbf{K}(\mathfrak{n})}^{G,S_\infty}\}$ is *polynomially bounded* in the sense of Definition 6.2, a property that already shows up in Delorme's work [26]. This property is analyzed in § 6, where we prove Proposition 6.1, a result on real reductive Lie groups which generalizes a part of Delorme's argument, and is (like Delorme's work) based on the Paley–Wiener theorem of Clozel and Delorme [20]. Once we have that the collections $\{\mu_{\mathbf{K}_M(\mathfrak{n})}^{M,S_\infty}\}$ are polynomially bounded for all proper Levi subgroups M of G, we can deduce the spectral limit property for G (Corollary 7.8).

We end this introduction with a few remarks on possible extensions of Theorems 1.3 and 7.9. For general sequences (Γ_n) of distinct irreducible lattices in a semisimple Lie group G, there is an obvious obstruction to the limit multiplicity property, namely the possibility that the lattices Γ_n (or an infinite subsequence thereof) all contain a non-trivial subgroup Δ of the center of G, which forces the corresponding representations R_{Γ_n} to be Δ-invariant. By passing to the quotient G/Δ, we can assume that this is not the case. A less obvious obstruction is that the members of an infinite subsequence of (Γ_n) all contain a non-central normal subgroup of Γ_1 (necessarily of infinite index). In such a case the analog of the geometric limit property (3) fails. Indeed, for $G = SL_2(\mathbb{R})$ we can find a descending sequence of finite index normal subgroups Γ_n of $\Gamma = SL_2(\mathbb{Z})$ such that for all n the multiplicity in $L^2(\Gamma_n \backslash G)$ of either one of the two lowest discrete series representations of G (or equivalently, the genus of the corresponding Riemann surface) is equal to one [59]. Similarly, one can find a descending sequence of finite index normal subgroups Γ_n of $SL_2(\mathbb{Z})$ such that the limiting measure of the sequence (μ_{Γ_n}) has a strictly positive density on the entire complementary spectrum $\Pi(G) \setminus \Pi(G)_{\text{temp}}$ [62].[3]

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[3]It follows from [64] (or alternatively by direct calculation) that the limit multiplicity property holds for the discrete series of $SL_2(\mathbb{R})$ and arbitrary normal towers of subgroups of $SL_2(\mathbb{Z})$, i.e., when the intersection of the normal subgroups is trivial.

By Margulis's normal subgroup theorem, non-central normal subgroups of infinite index do not exist for irreducible lattices Γ in semisimple Lie groups G of real rank at least two and without compact factors ([51, p. 4, Theorem 4]; cf. also [51, IX.6.14]). (The paper [1] is a major outgrowth of the Margulis normal subgroup theorem.) One expects that, for irreducible arithmetic lattices, the limit multiplicity property holds at least for any sequence of distinct *congruence* subgroups not containing non-trivial central elements. In the adelic setting, let G be a reductive group defined over a number field F such that the derived group G^{der} of G is F-simple and simply connected. Then we expect the limit multiplicity property to be true for a collection $\mathcal{K}$ of open compact subgroups of $G(\mathbb{A}^S)$, if $\mathrm{vol}(K \cap G^{\mathrm{der}}(\mathbb{A}^S)) \to 0$ for $K \in \mathcal{K}$ and every non-trivial element of the center of $G(F)$ is contained in only finitely many members of $\mathcal{K}$. For this, a good understanding of the structure of these subgroups seems to be necessary to deal with both the geometric side and the spectral side. We hope to return to this problem in a future paper.

2. Sauvageot's density principle

In this section, we recall the results of Sauvageot [66] and the proof of Theorem 1.2, providing a close link between the limit multiplicity problem and the trace formula. We continue to use the notation introduced before Theorem 1.2. Recall that a bounded subset $A \subset \Pi(G(F_S)^1)$ is a relatively quasi-compact subset. Equivalently, $A \subset \Pi(G(F_S)^1)$ is bounded if the Archimedean infinitesimal characters χ_{π_∞} of the elements $\pi \in A$ are bounded and there exists an open compact subgroup $K \subset G(F_{S-S_\infty})$ such that every $\pi \in A$ contains a non-trivial K-fixed vector.

The main result of [66] (Corollaire 6.2 and Théorème 7.3) is the following.[4]

Theorem 2.1 (Sauvageot)**.** *Let $\epsilon > 0$ be arbitrary. Then, the following hold.*

(1) *For any bounded set $A \subset \Pi(G(F_S)^1) \setminus \Pi_{\mathrm{temp}}(G(F_S)^1)$, there exists $h \in \mathcal{H}(G(F_S)^1)$ such that*

    (a) $\hat{h}(\pi) \geqslant 0$ *for all $\pi \in \Pi(G(F_S)^1)$,*

    (b) $\hat{h}(\pi) \geqslant 1$ *for all $\pi \in A$,*

    (c) $h(1) = \mu_{\mathrm{pl}}(\hat{h}) < \epsilon$.

(2) *For any Riemann integrable function f on $\Pi_{\mathrm{temp}}(G(F_S)^1)$, there exist $h_1, h_2 \in \mathcal{H}(G(F_S)^1)$ such that the following hold.*

    (a) $\left| f(\pi) - \hat{h}_1(\pi) \right| \leqslant \hat{h}_2(\pi)$ *for all $\pi \in \Pi(G(F_S)^1)$, where we extend f by zero to the entire unitary dual $\Pi(G(F_S)^1)$.*

    (b) $h_2(1) = \mu_{\mathrm{pl}}(\hat{h}_2) < \epsilon$.

As in [66], this result easily implies Theorem 1.2. We recall the argument. Let $A \subset \Pi(G(F_S)^1) \setminus \Pi_{\mathrm{temp}}(G(F_S)^1)$ be a bounded set. For any $\epsilon > 0$, let $h \in \mathcal{H}(G(F_S)^1)$ be as in the first part of Theorem 2.1. By assumption, we have $\left| \mu_K(\hat{h}) - h(1) \right| < \epsilon$ for all but finitely many $K \in \mathcal{K}$. For all such K, we have

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[4]See the appendix of [68] for important corrections.

$$\mu_K(A) \leqslant \mu_K(\hat{h}) \leqslant \left| \mu_K(\hat{h}) - h(1) \right| + h(1) < 2\epsilon.$$

Similarly, let f be a Riemann integrable function on $\Pi_{\text{temp}}(G(F_S)^1)$. For any $\epsilon > 0$, let h_1 and h_2 be as in the second part of Theorem 2.1. By assumption, for all but finitely many $K \in \mathcal{K}$ we have $\left| \mu_K(\hat{h}_i) - h_i(1) \right| < \epsilon$, $i = 1, 2$. Using $\mu_{\text{pl}}(\hat{h}_i) = h_i(1)$, we obtain

$$
\begin{aligned}
\left| \mu_K(f) - \mu_{\text{pl}}(f) \right| &\leqslant \left| \mu_K(f) - \mu_K(\hat{h}_1) \right| + \left| \mu_K(\hat{h}_1) - h_1(1) \right| + \left| h_1(1) - \mu_{\text{pl}}(f) \right| \\
&\leqslant \left| \mu_K(\hat{h}_1) - h_1(1) \right| + \mu_K(\hat{h}_2) + h_2(1) \\
&\leqslant \left| \mu_K(\hat{h}_1) - h_1(1) \right| + \left| \mu_K(\hat{h}_2) - h_2(1) \right| + 2h_2(1) < 4\epsilon.
\end{aligned}
$$

Theorem 1.2 follows.

3. The geometric limit property

In this section, we prove the geometric limit property for the principal congruence subgroups $\mathbf{K}^S(\mathfrak{n})$, where S is a finite set of places of F containing S_∞. In fact, we obtain a somewhat more precise estimate (cf. Proposition 3.1 below), which will be useful in the inductive argument of § 7.

3.1. Notation
We will mostly use the notation of [34]. As before, G is a reductive group defined over a number field F and $\mathbb{A}$ is the ring of adeles of F. Denote the adele norm on $\mathbb{A}^\times$ by $|\cdot|_{\mathbb{A}^\times}$. For a finite place v of F, let q_v be the cardinality of the residue field of v. We write $F_\infty = F \otimes \mathbb{R}$ and $\mathbb{A}_{\text{fin}}$ for the ring of finite adeles. As above, we fix a maximal compact subgroup $\mathbf{K} = \prod_v \mathbf{K}_v = \mathbf{K}_\infty \mathbf{K}_{\text{fin}}$ of $G(\mathbb{A}) = G(F_\infty)G(\mathbb{A}_{\text{fin}})$. Let $G(\mathbb{A})^1$ be the intersection of the subgroups $\ker |\chi|_{\mathbb{A}^\times}$ of $G(\mathbb{A})$ as χ ranges over the F-rational characters of G.

Fix once and for all a faithful F-rational representation $\rho : G \to \mathrm{GL}(V)$ and an $\mathfrak{o}_F$-lattice Λ in the representation space V such that the stabilizer of $\hat{\Lambda} = \hat{\mathfrak{o}}_F \otimes \Lambda \subset \mathbb{A}_{\text{fin}} \otimes V$ in $G(\mathbb{A}_{\text{fin}})$ is the group $\mathbf{K}_{\text{fin}}$. (Since the maximal compact subgroups of $\mathrm{GL}(\mathbb{A}_{\text{fin}} \otimes V)$ are precisely the stabilizers of lattices, it is easy to see that such a lattice exists.) For any non-zero ideal $\mathfrak{n}$ of $\mathfrak{o}_F$, let

$$\mathbf{K}(\mathfrak{n}) = \mathbf{K}_G(\mathfrak{n}) = \{ g \in G(\mathbb{A}_{\text{fin}}) : \rho(g)v \equiv v \pmod{\mathfrak{n}\hat{\Lambda}}, \ v \in \hat{\Lambda} \}$$

be the principal congruence subgroup of level $\mathfrak{n}$, a factorizable normal open subgroup of $\mathbf{K}_{\text{fin}}$. The groups $\mathbf{K}(\mathfrak{n})$ form a neighborhood base of the identity element in $G(\mathbb{A}_{\text{fin}})$. We denote by $\mathrm{N}(\mathfrak{n}) = [\mathfrak{o}_F : \mathfrak{n}]$ the ideal norm of $\mathfrak{n}$. Similarly, for a finite set $S \supset S_\infty$ of places of F and an ideal $\mathfrak{n}$ prime to S, let $\mathbf{K}^S(\mathfrak{n}) = \mathbf{K}_G^S(\mathfrak{n})$ be the corresponding open normal subgroup of $\mathbf{K}^S = \prod_{v \notin S} \mathbf{K}_v$.

Throughout, unless otherwise mentioned, all algebraic subgroups of G that we will consider are implicitly assumed to be defined over F.

We fix a maximal F-split torus $\mathbf{S}_0$ of G, and let M_0 be its centralizer, which is a minimal Levi subgroup. We assume that the maximal compact subgroup $\mathbf{K} \subset G(\mathbb{A})$ is admissible

with respect to M_0 [6, § 1]. Denote by A_0 the identity component of $\mathbf{S}_0(\mathbb{R})$, which is viewed as a subgroup of $\mathbf{S}_0(\mathbb{A})$ via the diagonal embedding of $\mathbb{R}$ into F_∞.

Denote by $\mathfrak{a}_0^*$ the $\mathbb{R}$-vector space spanned by the lattice $X^*(M_0)$ of F-rational characters of M_0 (or equivalently by the lattice $X^*(\mathbf{S}_0)$). We write $\mathfrak{a}_0$ for the dual space of $\mathfrak{a}_0^*$, which is spanned by the co-characters of $\mathbf{S}_0$. More generally, for a Levi subgroup $M \supset M_0$ we write $\mathbf{S}_M$ for the split part of the identity component of the center of M, and set $A_M = A_0 \cap Z(M) = A_0 \cap \mathbf{S}_M(\mathbb{R})$.

We will use the notation $A \ll B$ to mean that there exists a constant c (independent of the parameters under consideration) such that $|A| \leqslant cB$. The implied constant may depend on G and ρ, as well as on the field F. If it depends on additional parameters (e.g., ϵ), we write $A \ll_\epsilon B$.

3.2. The geometric side of the trace formula

Arthur's trace formula provides two alternative expressions for a certain distribution J on $G(\mathbb{A})^1$ which depends on the choice of M_0 and $\mathbf{K}$. Let d_0 be the semisimple F-rank of G. For $h \in C_c^\infty(G(\mathbb{A})^1)$, Arthur defines $J(h)$ to be the value at the point $T = T_0$ specified in [6, Lemma 1.1] of a certain polynomial $J^T(h)$ on $\mathfrak{a}_0$ of degree at most d_0. The polynomial $J^T(h)$ depends on the additional choice of a parabolic subgroup P_0 of G with Levi part M_0, which we fix throughout. Consider the equivalence relation on $G(F)$ defined by $\gamma \sim \gamma'$ whenever the semisimple parts of γ and γ' are $G(F)$-conjugate, and denote by $\mathcal{O}$ the set of all resulting equivalence classes. They are indexed by (but are not identical with) the conjugacy classes of semisimple elements of $G(F)$. The coarse geometric expansion [3] is

$$J^T(h) = \sum_{\mathfrak{o} \in \mathcal{O}} J_{\mathfrak{o}}^T(h), \tag{4}$$

where the summands $J_{\mathfrak{o}}^T(h)$ are again polynomials in T of degree at most d_0. Write $J_{\mathfrak{o}}(h) = J_{\mathfrak{o}}^{T_0}(h)$, which depends only on M_0 and $\mathbf{K}$. Then $J_{\mathfrak{o}}(h) = 0$ if the support of h is disjoint from all conjugacy classes of $G(\mathbb{A})$ intersecting $\mathfrak{o}$ (cf. [10, Theorem 8.1]). Let $\Omega \subset G(\mathbb{A})^1$ be a compact set, and denote by $C_\Omega^\infty(G(\mathbb{A})^1)$ the space of smooth functions on $G(\mathbb{A})^1$ supported in Ω. By [10, Lemma 9.1] (together with the descent formula of [6, § 2]), there exists a finite subset $\mathcal{O}(\Omega) \subset \mathcal{O}$ such that for $h \in C_\Omega^\infty(G(\mathbb{A})^1)$ we may restrict summation in (4) to $\mathfrak{o} \in \mathcal{O}(\Omega)$. In particular, the sum is always finite. When $\mathfrak{o}$ consists of the unipotent elements of $G(F)$, we write $J_{\mathrm{unip}}^T(h)$ for $J_{\mathfrak{o}}^T(h)$.

For each $k \geqslant 0$, fix a basis $\mathcal{B}_k$ of $\mathcal{U}(\mathrm{Lie}\, G_\infty \otimes \mathbb{C})_{\leqslant k}$, equipped with the usual filtration, and set

$$\|h\|_k = \sum_{X \in \mathcal{B}_k} \|X \star h\|_{L^1(G(\mathbb{A})^1)}$$

for functions $h \in C_c^\infty(G(\mathbb{A})^1)$, where we view X as a left-invariant differential operator on $G(F_\infty)$. For a compact subset $\Omega \subset G(\mathbb{A})^1$, the norms $\|\cdot\|_k$ give $C_\Omega^\infty(G(\mathbb{A})^1)$ the structure of a Fréchet space. (Note here that it is equivalent to use the seminorms $\sup_{x \in \Omega} |(X \star h)(x)|$ for $X \in \mathcal{U}(\mathrm{Lie}\, G_\infty \otimes \mathbb{C})$ instead of the norms $\|h\|_k$, $k \geqslant 0$.)

Analogously, we set

$$\|h\|_k = \sum_{X \in \mathcal{B}_k} \|X \star h\|_{L^1(G(F_S)^1)}$$

for $k \geqslant 0$ and $h \in C_c^\infty(G(F_S)^1)$. As above, for a compact subset $\Omega_S \subset G(F_S)^1$, these norms give $C_{\Omega_S}^\infty(G(F_S)^1)$ the structure of a Fréchet space.

3.3. An estimate for the unipotent contribution

By [9, Theorem 4.2], the unipotent contribution J_{unip}^T can be split into the contributions of the finitely many $G(\bar{F})$-conjugacy classes of unipotent elements of $G(F)$. By [9, Corollary 4.4], the contribution of the unit element is simply the constant polynomial $\mathrm{vol}(G(F)\backslash G(\mathbb{A})^1)h(1)$. Write

$$J_{\mathrm{unip}-\{1\}}^T(h) = J_{\mathrm{unip}}^T(h) - \mathrm{vol}(G(F)\backslash G(\mathbb{A})^1)h(1), \quad h \in C_c^\infty(G(\mathbb{A})^1).$$

We want to estimate $J_{\mathrm{unip}-\{1\}}(h) = J_{\mathrm{unip}-\{1\}}^{T_0}(h)$ for $h = h_S \otimes \mathbf{1}_{K^S(\mathfrak{n})}$.

Proposition 3.1. *There exists an integer $k \geqslant 0$ such that for any compact subset $\Omega_S \subset G(F_S)^1$ and any integral ideals $\mathfrak{n}_S$ and $\mathfrak{n}$ of $\mathfrak{o}_F$, where $\mathfrak{n}_S$ is a product of prime ideals of places in S and $\mathfrak{n}$ is prime to S, we have*

$$\left| J_{\mathrm{unip}-\{1\}}(h_S \otimes \mathbf{1}_{\mathbf{K}^S(\mathfrak{n})}) \right| \ll_{\Omega_S} \frac{(1 + \log \mathrm{N}(\mathfrak{n}_S\mathfrak{n}))^{d_0}}{\mathrm{N}(\mathfrak{n})} \|h_S\|_k$$

for any bi-$\mathbf{K}_{S-S_\infty}(\mathfrak{n}_S)$-invariant function $h_S \in C_{\Omega_S}^\infty(G(F_S)^1)$.

Remark 3.2. Let $G = \mathrm{GL}(2)$, let $\mathbf{K}(\mathfrak{n})$ be the standard principal congruence subgroups, and assume for simplicity that $S = S_\infty$. Then we have the explicit formula

$$J_{\mathrm{unip}-\{1\}}(h_\infty \otimes \mathbf{1}_{\mathbf{K}(\mathfrak{n})}) = \frac{\mathrm{vol}(M_0(F)\backslash M_0(\mathbb{A})^1)}{\mathrm{N}(\mathfrak{n})} \left(\int_{F_\infty} \int_{\mathbf{K}_\infty} h_\infty(k^{-1}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}k) \log |x|_\infty \ dk \, dx \right.$$
$$\left. + (\gamma_F - \log \mathrm{N}(\mathfrak{n})) \int_{F_\infty} \int_{\mathbf{K}_\infty} h_\infty(k^{-1}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}k) \ dk \, dx \right),$$

where $\gamma_F = c_{0,F}/c_{-1,F}$ is the quotient of the two leading coefficients in the Laurent expansion $\zeta_F(s) = c_{-1,F}(s-1)^{-1} + c_{0,F} + \cdots$ of the Dedekind zeta function of F at $s = 1$ (cf. [36, 43]). This shows that (regarding the dependency on $\mathrm{N}(\mathfrak{n})$) the estimate of Proposition 3.1 is best possible in this case. For general groups, we will give an improved estimate in Proposition 3.8 below.

Proposition 3.1 will be proved below. It has the following consequence.

Corollary 3.3 (Geometric limit property)**.** *For any $h_S \in C_c^\infty(G(F_S)^1)$, we have*

$$\lim_{\mathfrak{n}} J(h_S \otimes \mathbf{1}_{\mathbf{K}^S(\mathfrak{n})}) = \mathrm{vol}(G(F)\backslash G(\mathbb{A})^1)h_S(1).$$

Proof. Fix $h_S \in C_c^\infty(G(F_S)^1)$, and let $\Omega_S \subset G(F_S)^1$ be the support of h_S. Then the support of the test function $h_S \otimes \mathbf{1}_{\mathbf{K}^S(\mathfrak{n})}$ is $\Omega_S \mathbf{K}^S(\mathfrak{n}) \subset \Omega_S \mathbf{K}^S$, and therefore there are only finitely many classes $\mathfrak{o} \in \mathcal{O}$ that contribute to the geometric side of the trace formula (4) for the functions $h_S \otimes \mathbf{1}_{\mathbf{K}^S(\mathfrak{n})}$. Moreover, the only class $\mathfrak{o} \in \mathcal{O}$ for which the union of the $G(\mathbb{A})$-conjugacy classes of elements of $\mathfrak{o}$ meets $G(F_S)\mathbf{K}^S(\mathfrak{n})$ for infinitely many ideals $\mathfrak{n}$ is the unipotent class. For assume that $\mathfrak{o}$ has this property, and let $f \in F[X]$ be the characteristic polynomial of the linear map $\rho(\gamma) - 1 \in \text{End}(V)$ for arbitrary $\gamma \in \mathfrak{o}$. The assumption on $\mathfrak{o}$ implies that every coefficient of f (except the leading coefficient 1) is either arbitrarily close to 0 at some place $v \notin S$ or has absolute value <1 at infinitely many places. Therefore, necessarily, $f = X^{\dim V}$, and γ is unipotent.

As a result, the geometric side reduces to $J_{\text{unip}}(h_S \otimes \mathbf{1}_{\mathbf{K}^S(\mathfrak{n})})$ for all but finitely many ideals $\mathfrak{n}$, and the assertion follows from Proposition 3.1. $\qquad\square$

The proof of Proposition 3.1 consists of a slight extension of Arthur's arguments in [9]. The case where $F = \mathbb{Q}$ and $\mathfrak{n}$ is a power of a fixed prime is in fact already covered by Arthur's arguments. We also remark that, when we restrict the prime divisors of $\mathfrak{n}$ to a fixed finite set, we can appeal directly to Arthur's fine geometric expansion [10] to obtain the geometric limit property (cf. [29, Proposition 1.7]).

We first quote Arthur's asymptotic formula for $J_{\text{unip}-\{1\}}^T$ [9] in a form suitable for our purposes. Let $\mathcal{U} \subset G$ be the unipotent variety of G, so that $\mathcal{U}(F)$ consists of the unipotent elements of $G(F)$. Fix a Euclidean norm $\|\cdot\|$ on $\mathfrak{a}_0$ which is invariant under the Weyl group, and let $d(T) = \min_{\alpha \in \Delta_0}\langle \alpha, T \rangle$ for $T \in \mathfrak{a}_0$. Here, Δ_0 is the set of simple roots of $\mathbf{S}_0$ with respect to P_0. For a parabolic subgroup $P \supset P_0$ with Levi subgroup $M \supset M_0$, write $A_P = A_M$, and set $A_P(T_1) = \{a \in A_M : \log \alpha(a) > \langle \alpha, T_1 \rangle \forall \alpha \in \Delta_P\}$ for $T_1 \in \mathfrak{a}_0$, where Δ_P are the simple roots of $\mathbf{S}_M$ with respect to P (viewed as elements of $\mathfrak{a}_0^*$). As in [3, p. 941], we fix a suitable vector T_1, which depends only on G, P_0, and $\mathbf{K}$, such that $G(\mathbb{A}) = G(F)U_0(\mathbb{A})M_0(\mathbb{A})^1 A_{P_0}(T_1)\mathbf{K}$. Finally, recall the truncation function $F(\cdot, T) = F^G(\cdot, T)$ for $T \in \mathfrak{a}_0$, which is the characteristic function of a certain compact subset of $G(F)\backslash G(\mathbb{A})^1$ depending on T ([3, p. 941], [9, p. 1242]).

Theorem 3.4 [9, Theorem 4.2]. *There exist $k, m \geqslant 0$ and $\epsilon > 0$ with the following property. For any compact set $\Omega \subset G(\mathbb{A})^1$, there exists a constant $d_\Omega \geqslant 0$ such that, for all non-zero ideals $\mathfrak{n}$ of $\mathfrak{o}_F$ with $\mathbf{K}(\mathfrak{n})\Omega\mathbf{K}(\mathfrak{n}) = \Omega$, and any bi-$\mathbf{K}(\mathfrak{n})$-invariant function $h \in C_\Omega^\infty(G(\mathbb{A})^1)$, we have*

$$\left| J_{\text{unip}-\{1\}}^T(h) - \int_{G(F)\backslash G(\mathbb{A})^1} F(x, T) \sum_{\gamma \in \mathcal{U}(F),\, \gamma \neq 1} h(x^{-1}\gamma x)\, dx \right|$$
$$\ll_\Omega \|h\|_k \, \mathrm{N}(\mathfrak{n})^m (1 + \|T\|)^{d_0} e^{-\epsilon d(T)}$$

for all $T \in \mathfrak{a}_0$ with $d(T) \geqslant d_\Omega$.

Note that Theorem 3.4 differs slightly from the formulation in [9]. Namely, we introduced a different sequence of norms on $C_\Omega^\infty(G(\mathbb{A})^1)$ (defining the same topology), we combined all the (finitely many) non-trivial geometric unipotent orbits, made the dependence on $\mathfrak{n}$ explicit, and included the factor $(1 + \|T\|)^{d_0}$ instead of assuming that

$\|T\| \geqslant \epsilon_0 d(T)$ for a suitable $\epsilon_0 > 0$. The last change is allowed, since [9, Theorem 4.2] is based on [9, Theorem 3.1], where one can clearly make the corresponding change (see [9, p. 1249] for the relevant part of Arthur's proof).

Next, we bound the truncated integral

$$\int_{G(F)\backslash G(\mathbb{A})^1} F(x, T) \sum_{\gamma \in \mathcal{U}(F),\, \gamma \neq 1} (h_S \otimes \mathbf{1}_{\mathbf{K}^{S}(\mathfrak{n})})(x^{-1}\gamma x)\, dx$$

in terms of $\mathrm{N}(\mathfrak{n})$. By the dominated convergence theorem, for fixed T the integral approaches zero as $\mathrm{N}(\mathfrak{n}) \to \infty$. We make this quantitative as follows.

Lemma 3.5. *Let $\Omega_S \subset G(F_S)^1$ be a compact set. Then*

$$\int_{G(F)\backslash G(\mathbb{A})^1} F(x, T) \sum_{\gamma \in G(F),\, \gamma \neq 1} \left|(h_S \otimes \mathbf{1}_{\mathbf{K}^{S}(\mathfrak{n})})(x^{-1}\gamma x)\right| dx \ll_{\Omega_S} \frac{\sup |h_S|}{\mathrm{N}(\mathfrak{n})}(1 + \|T\|)^{d_0}, \quad (5)$$

for all bounded measurable functions h_S on $G(F_S)^1$ with support contained in Ω_S and all T with $d(T) \geqslant d_{\Omega_S} = d_{\Omega_S \mathbf{K}^S}$.

The proof of this estimate is based on an elementary estimate for a lattice-point counting problem which we will prove below. We first recall the following standard result from algebraic number theory (cf. [49, p. 102, Theorem 0] for a more precise result).

Lemma 3.6. *Let Λ be a fractional ideal of F, and let $D \subset F_\infty$ be a compact set. Then, for any $a > 0$ and a non-zero (integral) ideal $\mathfrak{n}$ of $\mathfrak{o}_F$, we have*

$$|aD \cap (\mathfrak{n}\Lambda - \{0\})| \ll_{D,\Lambda} \frac{a^{[F:\mathbb{Q}]}}{\mathrm{N}(\mathfrak{n})}.$$

Lemma 3.7. *Let $P = M \ltimes U_P$ be a standard parabolic subgroup of G, let $\mathfrak{u}_P$ be the Lie algebra of U_P, let $\Lambda \subset \mathfrak{u}_P(F)$ be an $\mathfrak{o}_F$-lattice, and let $D \subset \mathfrak{u}_P(F_\infty)$ be a compact set. Then, for all $a \in A_P(T_1)$ and non-zero integral ideals $\mathfrak{n}$, we have*

$$|\mathrm{Ad}(a)D \cap (\mathfrak{n}\Lambda - \{0\})| \ll_{D,\Lambda,T_1} \frac{\delta_P(a)}{\mathrm{N}(\mathfrak{n})}.$$

Proof. Let $e_1, \ldots, e_n$ be an F-basis of $\mathfrak{u}_P(F)$ consisting of eigenvectors with respect to $\mathbf{S}_M$, and let $\alpha_1, \ldots, \alpha_n$ be the associated eigencharacters (i.e., the roots of $\mathbf{S}_M$ on U_P). By passing to a larger Λ and D, if necessary, we can assume that $\Lambda = \sum_i \Lambda_i e_i$ with fractional ideals $\Lambda_1, \ldots, \Lambda_n \subset F$ and $D = \sum_i D_i e_i$ with compact sets $D_1, \ldots, D_n \subset F_\infty$. Since a non-zero vector has at least one non-zero coordinate, we can estimate

$$|\mathrm{Ad}(a)D \cap (\mathfrak{n}\Lambda - \{0\})| \leqslant \sum_{i=1}^n |\alpha_i(a)D_i \cap (\mathfrak{n}\Lambda_i - \{0\})| \prod_{j \neq i} |\alpha_j(a)D_j \cap \mathfrak{n}\Lambda_j|.$$

We now use the estimate of Lemma 3.6 for $|\alpha_i(a)D_i \cap (\mathfrak{n}\Lambda_i - \{0\})|$, while for the other coordinates we use the trivial estimate $|\alpha_j(a)D_j \cap \mathfrak{n}\Lambda_j| \ll_{D_j, \Lambda_j} \alpha_j(a)^{[F:\mathbb{Q}]} + 1$. This gives the desired result, since $\delta_P(a) = \prod_{j=1}^n \alpha_j(a)^{[F:\mathbb{Q}]}$, and the values $\alpha_j(a)$ are bounded away from 0 (in terms of T_1). $\qquad\square$

Proof of Lemma 3.5. By Arthur's discussion in [9, § 5], we can bound the left-hand side of (5) by the product of $(1 + \|T\|)^{d_0}$ and

$$\sup_{a \in A_{P_0}(T_1)} \delta_{P_0}(a)^{-1} \sum_{\gamma \in \mathcal{U}(F),\, \gamma \neq 1} \phi(a^{-1} \gamma a), \tag{6}$$

where

$$\phi(x) = \sup_{y \in \Gamma} \left| (h_S \otimes \mathbf{1}_{\mathbf{K}^S(\mathfrak{n})})(y^{-1} x y) \right|$$

for a compact set $\Gamma \subset G(\mathbb{A})^1$ depending only on G, P_0, and $\mathbf{K}$. Of course, we can assume that $\Gamma = \prod_v \Gamma_v$ with $\Gamma_v = \mathbf{K}_v$ for all $v \notin S'$, where $S' \supset S_\infty$ is a finite set of places of F. In a second step, Arthur bounds (6) by

$$\sum_{P \supset P_0} \sum_{\mu \in M_P(F)} \sup_{a_1 \in A_P(T_1)} \delta_P(a_1)^{-1} \sum_{\nu \in U_P(F):\, \mu\nu \neq 1} \phi_\mu(a_1^{-1} \nu a_1),$$

where M_P is the Levi part of P containing M_0 and

$$\phi_\mu(u) = \sup_{b \in B} \delta_{P_0}(b)^{-1} \phi(b^{-1} \mu u b), \quad u \in U_P(\mathbb{A}),$$

for a fixed compact set $B \subset A_0$.

Here, for a given P, we need to sum only over all μ belonging to the intersection of $M_P(F)$ with a compact set that depends only on Ω_S, or equivalently over a finite subset of $M_P(F)$ that depends only on Ω_S. Considering each possibility for μ separately, we see that for all but at most finitely many $\mathfrak{n}$ (depending on Ω_S) only $\mu = 1$ will contribute. Furthermore, from the definition of ϕ_1 we can estimate

$$\phi_1(u) \ll_{\Omega_S} \sup |h_S| \, \mathbf{1}_{\Omega'_S \operatorname{Ad}(\Gamma^S)(\mathbf{K}^S(\mathfrak{n})) \cap U_P(\mathbb{A})}(u), \quad u \in U_P(\mathbb{A}),$$

for a compact set $\Omega'_S \subset G(F_S)^1$ which depends only on Ω_S. Let $\mathfrak{p}_v$ be the prime ideal of the ring of integers of F_v. There exist exponents $e_v \geqslant 0$ for $v \notin S$, with $e_v = 0$ for $v \notin S'$, such that

$$\operatorname{Ad}(\Gamma_v)(\mathbf{K}_v(\mathfrak{p}_v^f)) \subset \mathbf{K}_v(\mathfrak{p}_v^{f-e_v})$$

for $f \geqslant e_v$. Write $\mathfrak{n} = \prod_{v \notin S} \mathfrak{p}_v^{f_v}$. We conclude that $\operatorname{Ad}(\Gamma^S)(\mathbf{K}^S(\mathfrak{n})) \subset \prod_{v \notin S} L_{v, f_v}$, where for $v \notin S$ and $f \geqslant 0$ we set $L_{v,f} = \operatorname{Ad}(\Gamma_v)(\mathbf{K}_v)$ when $f < e_v$ (which implies that $v \in S'$) and $L_{v,f} = \mathbf{K}_v(\mathfrak{p}_v^{f-e_v})$, otherwise. Identify the unipotent radical U_P with its Lie algebra $\mathfrak{u}_P$ via the exponential map. Then the lemma reduces to an application of Lemma 3.7 (with $\mathfrak{n}$ replaced by $\mathfrak{n}' = \prod_{v:\, f_v \geqslant e_v} \mathfrak{p}_v^{f_v - e_v}$). $\qquad\square$

To finish the argument, we follow Arthur's interpolation argument in [9, pp. 1252–1254].

Proof of Proposition 3.1. Let $\Omega_S \subset G(F_S)^1$ be a compact set. Given a bi-$\mathbf{K}_{S-S_\infty}(\mathfrak{n}_S)$-invariant function $h_S \in C^\infty_{\Omega_S}(G(F_S)^1)$, we combine Theorem 3.4 and Lemma 3.5 to obtain

$$\left| J^T_{\operatorname{unip}-\{1\}}(h_S \otimes \mathbf{1}_{\mathbf{K}^S(\mathfrak{n})}) \right| \leqslant c_{\Omega_S} \|h_S\|_k \left(\mathrm{N}(\mathfrak{n}_S \mathfrak{n})^m e^{-\epsilon d(T)} + \mathrm{N}(\mathfrak{n})^{-1} \right) (1 + \|T\|)^{d_0}$$

for all $T \in \mathfrak{a}_0$ with $d(T) \geqslant d_{\Omega_S}$. This implies that

$$\left| J^T_{\mathrm{unip}-\{1\}}(h_S \otimes \mathbf{1}_{\mathbf{K}^S(\mathfrak{n})}) \right| \leqslant 2c_{\Omega_S} \|h_S\|_k \, \mathrm{N}(\mathfrak{n})^{-1}(1 + \|T\|)^{d_0}$$

for all $T \in \mathfrak{a}_0$ with $d(T) \geqslant \max(d_{\Omega_S}, \frac{m+1}{\epsilon} \log \mathrm{N}(\mathfrak{n}_S\mathfrak{n}))$. Applying [7, Lemma 5.2] to the polynomial $J^T_{\mathrm{unip}-\{1\}}(h_S \otimes \mathbf{1}_{\mathbf{K}^S(\mathfrak{n})})$ and the point $T_0 \in \mathfrak{a}_0$, we obtain the assertion. $\qquad\square$

We note that an alternative proof of Corollary 3.3 might be given by replacing $\mathcal{U}(F)$ by $G(F)$ in the arguments above and using [4, p. 267, Theorem 1]. We will give a detailed account of this proof in a future paper, which will treat a somewhat more general situation.

3.4. A refinement

We conclude this section with a refinement of Proposition 3.1 that is close to optimal in its dependence on $\mathrm{N}(\mathfrak{n})$. We will use this refinement to give a certain quantitative refinement of the limit multiplicity property in Theorem 7.10. We may assume that G is isotropic, for otherwise $J_{\mathrm{unip}-\{1\}}(h) = 0$.

Define

$$d_{\min} = \min_{\tilde{\alpha}} \langle \rho, \tilde{\alpha}^\vee \rangle, \tag{7}$$

where ρ is as usual half the sum of the positive roots of G with respect to P_0 (counting multiplicities) and $\tilde{\alpha}$ ranges over the highest roots of the irreducible components of the root system $\Phi(\mathbf{S}_0, G)$. We note that the minimal dimension of a non-trivial geometric unipotent orbit of G containing an element of $G(F)$ is $2d_{\min}$. For split groups, this follows from [24, Lemma 4.3.5] and [76]. In general, note that a non-zero element of the $\tilde{\alpha}$-eigenspace in $\mathfrak{g}$ forms together with the co-root $\tilde{\alpha}^\vee$ and a suitable element of the eigenspace of $-\tilde{\alpha}$ an $\mathfrak{sl}(2)$-triplet. We may then apply the dimension formula [24, Lemma 4.1.3] to compute the dimension of the associated nilpotent orbit as $2\langle \rho, \tilde{\alpha}^\vee \rangle$. The minimality of this orbit follows from the argument of [24, Theorem 4.3.3].

Proposition 3.8. *There exists an integer $k \geqslant 0$ such that for any compact subset $\Omega_S \subset G(F_S)^1$ and any integral ideals $\mathfrak{n}_S$ and $\mathfrak{n}$ of $\mathfrak{o}_F$, where $\mathfrak{n}_S$ is a product of prime ideals of places in S and $\mathfrak{n}$ is prime to S, we have*

$$\left| J_{\mathrm{unip}-\{1\}}(h_S \otimes \mathbf{1}_{\mathbf{K}^S(\mathfrak{n})}) \right| \ll_{\Omega_S} \frac{(1 + \log \mathrm{N}(\mathfrak{n}_S\mathfrak{n}))^{d_0}}{\mathrm{N}(\mathfrak{n})^{d_{\min}}} \|h_S\|_k$$

for any bi-$\mathbf{K}_{S-S_\infty}(\mathfrak{n}_S)$-invariant function $h_S \in C^\infty_{\Omega_S}(G(F_S)^1)$.

We remark that, by Arthur's fine expansion for the unipotent contribution [9], the exponent $d_{\min}$ in the estimate of Proposition 3.8 is optimal. For instance, consider the case where $\mathfrak{n} = \mathfrak{p}^e$ for a prime ideal $\mathfrak{p}$ at a place v of F not contained in S. Then [9] expresses $J_{\mathrm{unip}-\{1\}}(h_S \otimes \mathbf{1}_{\mathbf{K}^S(\mathfrak{n})})$ in terms of certain weighted orbital integrals (including the invariant orbital integral) of $h_S \otimes \mathbf{1}_{\mathbf{K}_v(\mathfrak{p}^e_v)}$ over the non-trivial unipotent orbits in $G(F_{S\cup\{v\}})$ containing elements of $G(F)$. Now it is easy to see that the invariant orbital integral of $\mathbf{1}_{\mathbf{K}_v(\mathfrak{p}^{2e}_v)}$ over a unipotent orbit $\mathfrak{o}$ in $G(F_v)$ is equal to a constant multiple

of q_v^{-de} for e large, where d is the dimension of the geometric orbit associated to $\mathfrak{o}$. Indeed, embed an element of the nilpotent orbit $\log \mathfrak{o} \subset \mathfrak{g} \otimes F_v$ into an $\mathfrak{sl}(2)$-triplet, and let $\mathfrak{g}_i \subset \mathfrak{g} \otimes F_v$, $i \in \mathbb{Z}$, be the associated eigenspaces. Then the explicit formula for the invariant orbital integral $j_{\log \mathfrak{o}}$ of the nilpotent orbit $\log \mathfrak{o}$ [63, Theorem 1] implies the homogeneity relation $j_{\log \mathfrak{o}}(f(\lambda^2 \cdot)) = |\lambda|^{-r} j_{\log \mathfrak{o}}(f)$ for all $f \in C_c^\infty(\mathfrak{g} \otimes F_v)$ and $\lambda \in F_v^\times$, where $r = \dim \mathfrak{g}_1 + 2 \sum_{i \geqslant 2} \dim \mathfrak{g}_i = d$ (using [24, Lemma 4.1.3] for the last equality). This immediately implies the assertion.

A glance at the proof of Proposition 3.1 given above shows that Proposition 3.8 will follow from the following improvement of Lemma 3.7.

Lemma 3.9. *Let $P = M \ltimes U_P$ be a standard parabolic subgroup of G, let $\mathfrak{u}_P$ be the Lie algebra of U_P, let $\Lambda \subset \mathfrak{u}_P(F)$ be an $\mathfrak{o}_F$-lattice, and let $D \subset \mathfrak{u}_P(F_\infty)$ be a compact set. Then, for all $a \in A_P(T_1)$ and non-zero integral ideals $\mathfrak{n}$, we have*

$$|\mathrm{Ad}(a)D \cap (\mathfrak{n}\Lambda - \{0\})| \ll_{D,\Lambda,T_1} \frac{\delta_P(a)}{\mathrm{N}(\mathfrak{n})^{d_{\min}}}.$$

For the proof of this lemma we first need a generalization of Lemma 3.6 to vector spaces.

Lemma 3.10. *Let V be a finite-dimensional F-vector space, let Λ be an $\mathfrak{o}_F$-lattice in V, and let $D \subset V \otimes F_\infty$ be a compact set. Then, for all $a > 0$ and non-zero integral ideals $\mathfrak{n}$, we have*

$$|aD \cap (\mathfrak{n}\Lambda - \{0\})| \ll_{D,\Lambda} \left(\frac{a^{[F:\mathbb{Q}]}}{\mathrm{N}(\mathfrak{n})}\right)^{\dim_F V}.$$

Proof. Choose an F-basis of V, and use Lemma 3.6 for the coordinates, taking into account that $aD \cap (\mathfrak{n}\Lambda - \{0\}) \neq \emptyset$ implies that $a^{[F:\mathbb{Q}]} \gg \mathrm{N}(\mathfrak{n})$. $\square$

The core of the argument is contained in the following lemma on root systems.

Lemma 3.11. *Let Φ be a (possibly non-reduced) root system, and let Φ^+ be a set of positive roots for Φ. Furthermore, let $m_\alpha \geqslant 0$ for $\alpha \in \Phi$ be given, and assume that m_α is invariant under the action of the Weyl group W_Φ on Φ. Set $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} m_\alpha \alpha$. Then, for all $\beta \in \Phi^+$ and all subsets $S \subset \Phi^+$, we have*

$$\sum_{\alpha \notin S} m_\alpha \alpha + \sum_{\alpha \in S} m_\alpha \beta \in d_{\min}\beta + \sum_{\alpha \in \Phi^+} \mathbb{R}^{\geqslant 0}\alpha,$$

where $d_{\min} = \min_{\tilde{\alpha}} \langle \rho, \tilde{\alpha}^\vee \rangle$, $\tilde{\alpha}$ ranging over the highest roots of the irreducible components of Φ.

Proof. It is clearly enough to consider the case where Φ is irreducible. Let $C = \sum_{\alpha \in \Phi^+} \mathbb{R}^{\geqslant 0}\alpha$ be the closed cone spanned by the positive roots. Let $\tilde{\alpha}$ be the highest root of Φ^+. Note that $\langle \alpha, \tilde{\alpha}^\vee \rangle$ for $\alpha \in \Phi^+$ takes the values 0, 1, and 2, and the last only for $\alpha = \tilde{\alpha}$. Let R be the set of all $\alpha \in \Phi^+$ with $\langle \alpha, \tilde{\alpha}^\vee \rangle = 1$.

Inserting the definition of ρ and multiplying the relation by two, we have to show that

$$\sum_{\alpha \notin S} 2m_\alpha \alpha + \sum_{\alpha \in S} 2m_\alpha \beta \in \left(2m_{\tilde{\alpha}} + \sum_{\alpha \in R} m_\alpha \right) \beta + C.$$

It is evidently sufficient to show the modified statement where on the left-hand side we restrict the sums to roots $\alpha \in R \cup \{\tilde{\alpha}\}$. The contribution from $\alpha = \tilde{\alpha}$ is either $2m_{\tilde{\alpha}}\tilde{\alpha}$ or $2m_{\tilde{\alpha}}\beta$, and therefore lies in $2m_{\tilde{\alpha}}\beta + C$ in both cases. It therefore remains to show that

$$\sum_{\alpha \in R-S} 2m_\alpha \alpha + \sum_{\alpha \in R \cap S} 2m_\alpha \beta \in \sum_{\alpha \in R} m_\alpha \beta + C.$$

Note that $\alpha \mapsto -w_{\tilde{\alpha}}(\alpha) = \tilde{\alpha} - \alpha$ defines an involution of R. We may therefore rewrite the last statement as

$$\sum_{\alpha \in R-S} m_\alpha \alpha + \sum_{\alpha \in R \cap S} m_\alpha \beta + \sum_{\alpha \in R : \tilde{\alpha} - \alpha \notin S} m_\alpha (\tilde{\alpha} - \alpha) + \sum_{\alpha \in R : \tilde{\alpha} - \alpha \in S} m_\alpha \beta \in \sum_{\alpha \in R} m_\alpha \beta + C.$$

The only case where $\alpha \in R$ does not contribute a summand $m_\alpha \beta$ to the left-hand side is when $\alpha \notin S$ and $\tilde{\alpha} - \alpha \notin S$, in which case the total contribution is $m_\alpha \alpha + m_\alpha(\tilde{\alpha} - \alpha) = m_\alpha \tilde{\alpha} \in m_\alpha \beta + C$. This finishes the proof of the lemma. $\qquad\square$

Proof of Lemma 3.9. First, note that we may assume that $\langle \alpha, T_1 \rangle \leqslant 0$ for all $\alpha \in \Delta_0$. By extending the lattice Λ to a lattice in $\mathfrak{u}_{P_0}(F)$, while keeping D and a fixed, we may reduce to the case where P is the minimal parabolic subgroup P_0. Let $\Phi = \Phi(\mathbf{S}_0, G)$, and let Φ^+ be the set of positive roots corresponding to P_0. For $\alpha \in \Phi$, let $\mathfrak{u}_\alpha$ be the α-eigenspace in $\mathfrak{g}$, and let $m_\alpha = \dim_F \mathfrak{u}_\alpha$. For $X \in \mathfrak{u}_{P_0}$, let X_α be its projection to $\mathfrak{u}_\alpha$, and write $S(X) = \{\alpha \in \Phi^+ : X_\alpha \neq 0\}$. Evidently, it is enough to show that for each non-empty subset $S \subset \Phi^+$ we can bound

$$\frac{\mathrm{N}(\mathfrak{n})^{d_{\min}}}{\delta_{P_0}(a)} |X \in \mathrm{Ad}(a)D \cap \mathfrak{n}\Lambda : S(X) = S| \tag{8}$$

as a ranges over $A_{P_0}(T_1)$ and $\mathfrak{n}$ over the non-zero ideals of $\mathfrak{o}_F$. For this, we may apply Lemma 3.10 to obtain for (8) the bound

$$\ll \frac{\mathrm{N}(\mathfrak{n})^{d_{\min}}}{\delta_{P_0}(a)} \prod_{\alpha \in S} \left(\frac{\alpha(a)^{[F:\mathbb{Q}]}}{\mathrm{N}(\mathfrak{n})} \right)^{m_\alpha} = \left(\prod_{\alpha \notin S} \alpha(a)^{-[F:\mathbb{Q}]m_\alpha} \right) \mathrm{N}(\mathfrak{n})^{d_{\min} - \sum_{\alpha \in S} m_\alpha}.$$

This is clearly bounded when $\sum_{\alpha \in S} m_\alpha \geqslant d_{\min}$, and we may therefore assume that $\sum_{\alpha \in S} m_\alpha \leqslant d_{\min}$. Note that the existence of a vector $X \in \mathrm{Ad}(a)D \cap \mathfrak{n}\Lambda$ with $S(X) = S$ implies that $\beta(a)^{[F:\mathbb{Q}]} \gg \mathrm{N}(\mathfrak{n})$ for arbitrary $\beta \in S$. Therefore, (8) is

$$\ll \left(\prod_{\alpha \notin S} \alpha(a)^{-[F:\mathbb{Q}]m_\alpha} \right) \beta(a)^{[F:\mathbb{Q}](d_{\min} - \sum_{\alpha \in S} m_\alpha)}.$$

Applying Lemma 3.11, we can write this as a product $\prod_{\alpha \in \Phi^+} \alpha(a)^{-\lambda_\alpha}$ with $\lambda_\alpha \geqslant 0$, and it is therefore bounded for $a \in A_{P_0}(T_1)$. $\qquad\square$

4. Review of the spectral side of the trace formula

We turn to the spectral side of Arthur's trace formula and recall the results of [34], which are based on [7, 8].

4.1. Notation

Let θ be the Cartan involution of $G(F_\infty)$ defining $\mathbf{K}_\infty$. It induces a Cartan decomposition $\mathfrak{g} = \operatorname{Lie} G(F_\infty) = \mathfrak{p} \oplus \mathfrak{k}$ with $\mathfrak{k} = \operatorname{Lie} \mathbf{K}_\infty$. We fix an invariant bilinear form B on $\mathfrak{g}$ which is positive definite on $\mathfrak{p}$ and negative definite on $\mathfrak{k}$. This choice defines a Casimir operator Ω on $G(F_\infty)$, and we denote the Casimir eigenvalue of any $\pi \in \Pi(G(F_\infty))$ by λ_π. Similarly, we obtain a Casimir operator $\Omega_{\mathbf{K}_\infty}$ on $\mathbf{K}_\infty$, and write λ_τ for the Casimir eigenvalue of a representation $\tau \in \Pi(\mathbf{K}_\infty)$ (cf. [14, § 2.3]). The form B induces a Euclidean scalar product $(X, Y) = -B(X, \theta(Y))$ on $\mathfrak{g}$ and all its subspaces.

We write $\mathcal{L}$ for the (finite) set of Levi subgroups containing M_0, i.e., the set of centralizers of subtori of $\mathbf{S}_0$. Let $W_0 = N_{G(F)}(\mathbf{S}_0)/M_0 = N_{G(F)}(M_0)/M_0$ be the Weyl group of $(G, \mathbf{S}_0)$, where $N_{G(F)}(H)$ is the normalizer of H in $G(F)$. For any $s \in W_0$, we choose a representative $w_s \in N_{G(F)}(\mathbf{S}_0) = N_{G(F)}(M_0)$. Note that W_0 acts on $\mathcal{L}$ by $sM = w_s M w_s^{-1}$.

Let now $M \in \mathcal{L}$. Let $W(M) = N_{G(F)}(M)/M$, which can be identified with a subgroup of W_0. Denote by $\mathfrak{a}_M^*$ the $\mathbb{R}$-vector space spanned by the lattice $X^*(M)$ of F-rational characters of M, and let $\mathfrak{a}_{M,\mathbb{C}}^* = \mathfrak{a}_M^* \otimes_\mathbb{R} \mathbb{C}$ be its complexification. We write $\mathfrak{a}_M$ for the dual space of $\mathfrak{a}_M^*$, which is spanned by the co-characters of $\mathbf{S}_M$. It can also be identified with the Lie algebra of the torus $\mathbf{S}_M$. Let $H_M : M(\mathbb{A}) \to \mathfrak{a}_M$ be the homomorphism given by

$$e^{\langle \chi, H_M(m) \rangle} = |\chi(m)|_{\mathbb{A}^\times} = \prod_v |\chi(m_v)|_v$$

for any $\chi \in X^*(M)$, and denote by $M(\mathbb{A})^1 \subset M(\mathbb{A})$ the kernel of H_M. Let $\mathcal{L}(M)$ be the set of Levi subgroups containing M, and let $\mathcal{P}(M)$ be the set of parabolic subgroups of G with Levi part M. We also write $\mathcal{F}(M) = \mathcal{F}^G(M) = \coprod_{L \in \mathcal{L}(M)} \mathcal{P}(L)$ for the (finite) set of parabolic subgroups of G containing M. Note that $W(M)$ acts on $\mathcal{P}(M)$ and $\mathcal{F}(M)$ by $sP = w_s P w_s^{-1}$. Denote by Σ_M the set of reduced roots of $\mathbf{S}_M$ on the Lie algebra of G. For any $\alpha \in \Sigma_M$, we denote by $\alpha^\vee \in \mathfrak{a}_M$ the corresponding co-root. Let $L^2_{\mathrm{disc}}(A_M M(F) \backslash M(\mathbb{A}))$ be the discrete part of $L^2(A_M M(F) \backslash M(\mathbb{A}))$, i.e., the closure of the sum of all irreducible subrepresentations of the regular representation of $M(\mathbb{A})$. We denote by $\Pi_{\mathrm{disc}}(M(\mathbb{A}))$ the countable set of equivalence classes of irreducible unitary representations of $M(\mathbb{A})$ which occur in the decomposition of $L^2_{\mathrm{disc}}(A_M M(F) \backslash M(\mathbb{A}))$ into irreducibles.

For any $L \in \mathcal{L}(M)$, we identify $\mathfrak{a}_L^*$ with a subspace of $\mathfrak{a}_M^*$. We denote by $\mathfrak{a}_M^L$ the annihilator of $\mathfrak{a}_L^*$ in $\mathfrak{a}_M$. We set

$$\mathcal{L}_1(M) = \{L \in \mathcal{L}(M) : \dim \mathfrak{a}_M^L = 1\}$$

(i.e., the set of Levi subgroups containing M as a maximal Levi subgroup) and

$$\mathcal{F}_1(M) = \bigcup_{L \in \mathcal{L}_1(M)} \mathcal{P}(L).$$

Note that the restriction of the scalar product $(\cdot, \cdot)$ on $\mathfrak{g}$ defined above endows $\mathfrak{a}_0 = \mathfrak{a}_{M_0} \subset \mathfrak{g}$ with the structure of a Euclidean space. In particular, this fixes Haar measures on the

spaces $\mathfrak{a}_M^L$ and their duals $(\mathfrak{a}_M^L)^*$. We follow Arthur in the corresponding normalization of Haar measures on the groups $M(\mathbb{A})$ [3, § 1].

4.2. Intertwining operators

Now let $P \in \mathcal{P}(M)$. We write $\mathfrak{a}_P = \mathfrak{a}_M$. Let U_P be the unipotent radical of P. Denote by $\Sigma_P \subset \mathfrak{a}_P^*$ the set of reduced roots of $\mathbf{S}_M$ on the Lie algebra $\mathfrak{u}_P$ of U_P. Let Δ_P be the subset of simple roots of P, which is a basis for $(\mathfrak{a}_P^G)^*$. Write $\mathfrak{a}_{P,+}^*$ for the closure of the Weyl chamber of P, i.e.,

$$\mathfrak{a}_{P,+}^* = \{\lambda \in \mathfrak{a}_M^* : \langle \lambda, \alpha^\vee \rangle \geqslant 0 \text{ for all } \alpha \in \Sigma_P\} = \{\lambda \in \mathfrak{a}_M^* : \langle \lambda, \alpha^\vee \rangle \geqslant 0 \text{ for all } \alpha \in \Delta_P\}.$$

Denote by δ_P the modulus function of $P(\mathbb{A})$. Let $\bar{\mathcal{A}}^2(P)$ be the Hilbert space completion of

$$\{\phi \in C^\infty(M(F)U_P(\mathbb{A})\backslash G(\mathbb{A})) : \delta_P^{-\frac{1}{2}}\phi(\cdot x) \in L_{\mathrm{disc}}^2(A_M M(F)\backslash M(\mathbb{A})) \,\forall x \in G(\mathbb{A})\}$$

with respect to the inner product

$$(\phi_1, \phi_2) = \int_{A_M M(F)U_P(\mathbb{A})\backslash G(\mathbb{A})} \phi_1(g)\overline{\phi_2(g)}\, dg.$$

Let $\alpha \in \Sigma_M$. We say that two parabolic subgroups $P, Q \in \mathcal{P}(M)$ are *adjacent* along α, and write $P|^\alpha Q$, if $\Sigma_P \cap -\Sigma_Q = \{\alpha\}$. Alternatively, P and Q are adjacent if the group $\langle P, Q \rangle$ generated by P and Q belongs to $\mathcal{F}_1(M)$. Any $R \in \mathcal{F}_1(M)$ is of the form $\langle P, Q \rangle$, where P, Q are the elements of $\mathcal{P}(M)$ contained in R; we have $P|^\alpha Q$ with $\alpha^\vee \in \Sigma_P^\vee \cap \mathfrak{a}_M^R$. Interchanging P and Q switches α to $-\alpha$.

For any $P \in \mathcal{P}(M)$, let $H_P : G(\mathbb{A}) \to \mathfrak{a}_P$ be the extension of H_M to a left $U_P(\mathbb{A})$-invariant and right $\mathbf{K}$-invariant map. Denote by $\mathcal{A}^2(P)$ the dense subspace of $\bar{\mathcal{A}}^2(P)$ consisting of its $\mathbf{K}$-finite and $\mathfrak{z}$-finite vectors, where $\mathfrak{z}$ is the center of the universal enveloping algebra of $\mathfrak{g} \otimes \mathbb{C}$. That is, $\mathcal{A}^2(P)$ is the space of automorphic forms ϕ on $U_P(\mathbb{A})M(F)\backslash G(\mathbb{A})$ such that $\delta_P^{-\frac{1}{2}}\phi(\cdot k)$ is a square-integrable automorphic form on $A_M M(F)\backslash M(\mathbb{A})$ for all $k \in \mathbf{K}$. Let $\rho(P, \lambda)$, $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^*$, be the induced representation of $G(\mathbb{A})$ on $\bar{\mathcal{A}}^2(P)$ given by

$$(\rho(P, \lambda, y)\phi)(x) = \phi(xy)e^{\langle \lambda, H_P(xy) - H_P(x) \rangle}.$$

It is isomorphic to $\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}\left(L_{\mathrm{disc}}^2(A_M M(F)\backslash M(\mathbb{A})) \otimes e^{\langle \lambda, H_M(\cdot) \rangle}\right)$.

For $P, Q \in \mathcal{P}(M)$, let

$$M_{Q|P}(\lambda) : \mathcal{A}^2(P) \to \mathcal{A}^2(Q), \quad \lambda \in \mathfrak{a}_{M,\mathbb{C}}^*,$$

be the standard *intertwining operator* [8, § 1], which is the meromorphic continuation in λ of the integral

$$[M_{Q|P}(\lambda)\phi](x) = \int_{U_Q(\mathbb{A})\cap U_P(\mathbb{A})\backslash U_Q(\mathbb{A})} \phi(nx)e^{\langle \lambda, H_P(nx) - H_Q(x) \rangle}\, dn, \quad \phi \in \mathcal{A}^2(P),\ x \in G(\mathbb{A}).$$

These operators satisfy the following properties.

(1) $M_{P|P}(\lambda) \equiv \mathrm{Id}$ for all $P \in \mathcal{P}(M)$ and $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^*$.

(2) For any $P, Q, R \in \mathcal{P}(M)$, we have $M_{R|P}(\lambda) = M_{R|Q}(\lambda) \circ M_{Q|P}(\lambda)$ for all $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^*$. In particular, $M_{Q|P}(\lambda)^{-1} = M_{P|Q}(\lambda)$.

(3) $M_{Q|P}(\lambda)^* = M_{P|Q}(-\bar{\lambda})$ for any $P, Q \in \mathcal{P}(M)$ and $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^*$. In particular, $M_{Q|P}(\lambda)$ is unitary for $\lambda \in i\mathfrak{a}_M^*$.

(4) If $P|^\alpha Q$, then $M_{Q|P}(\lambda)$ depends only on $\langle \lambda, \alpha^\vee \rangle$.

Let $\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A}))$, and let $\mathcal{A}_\pi^2(P)$ be the space of all $\phi \in \mathcal{A}^2(P)$ for which the functions $\delta_P^{-\frac{1}{2}} \phi(\cdot g)$, $g \in G(\mathbb{A})$, belong to the π-isotypic subspace of $L^2(A_M M(F) \backslash M(\mathbb{A}))$. For any $P \in \mathcal{P}(M)$, we have a canonical isomorphism of $G(\mathbb{A}_f) \times (\mathfrak{g}_\mathbb{C}, K_\infty)$-modules

$$j_P : \mathrm{Hom}(\pi, L^2(A_M M(F) \backslash M(\mathbb{A}))) \otimes \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\pi) \to \mathcal{A}_\pi^2(P).$$

If we fix a unitary structure on π and endow $\mathrm{Hom}(\pi, L^2(A_M M(F) \backslash M(\mathbb{A})))$ with the inner product $(A, B) = B^*A$ (which is a scalar operator on the space of π), the isomorphism j_P becomes an isometry.

Suppose that $P|^\alpha Q$. The operator $M_{Q|P}(\pi, s) := M_{Q|P}(s\varpi)|_{\mathcal{A}_\pi^2(P)}$, where $\varpi \in \mathfrak{a}_M^*$ is such that $\langle \varpi, \alpha^\vee \rangle = 1$, admits a normalization by a global factor $n_\alpha(\pi, s)$ which is a meromorphic function in s. We may write

$$M_{Q|P}(\pi, s) \circ j_P = n_\alpha(\pi, s) \cdot j_Q \circ (\mathrm{Id} \otimes R_{Q|P}(\pi, s)), \tag{9}$$

where $R_{Q|P}(\pi, s) = \otimes_v R_{Q|P}(\pi_v, s)$ is the product of the locally defined normalized intertwining operators and $\pi = \otimes_v \pi_v$ ([8, § 6]; cf. [57, (2.17)]).

4.3. The spectral side

We now turn to the spectral expansion of Arthur's distribution J on $C_c^\infty(G(\mathbb{A})^1)$. Let $L \supset M$ be Levi subgroups in $\mathcal{L}$, $P \in \mathcal{P}(M)$, and let $m = \dim \mathfrak{a}_L^G$ be the co-rank of L in G. Denote by $\mathfrak{B}_{P,L}$ the set of m-tuples $\underline{\beta} = (\beta_1^\vee, \ldots, \beta_m^\vee)$ of elements of $\Sigma_P^\vee$ whose projections to $\mathfrak{a}_L$ form a basis for $\mathfrak{a}_L^G$. For any $\underline{\beta} = (\beta_1^\vee, \ldots, \beta_m^\vee) \in \mathfrak{B}_{P,L}$, let $\mathrm{vol}(\underline{\beta})$ be the co-volume in $\mathfrak{a}_L^G$ of the lattice spanned by the projection of $\underline{\beta}$ to $\mathfrak{a}_L$, and let

$$\Xi_L(\underline{\beta}) = \{(Q_1, \ldots, Q_m) \in \mathcal{F}_1(M)^m : \beta_i^\vee \in \mathfrak{a}_M^{Q_i}, \ i = 1, \ldots, m\}$$
$$= \{(\langle P_1, P_1' \rangle, \ldots, \langle P_m, P_m' \rangle) : P_i|^{\beta_i} P_i', \ i = 1, \ldots, m\}.$$

For any smooth function f on $\mathfrak{a}_M^*$ and $\mu \in \mathfrak{a}_M^*$, denote by $D_\mu f$ the directional derivative of f along $\mu \in \mathfrak{a}_M^*$. For a pair $P_1|^\alpha P_2$ of adjacent parabolic subgroups in $\mathcal{P}(M)$, write

$$\delta_{P_1|P_2}(\lambda) = M_{P_2|P_1}(\lambda) D_\varpi M_{P_1|P_2}(\lambda) : \mathcal{A}^2(P_2) \to \mathcal{A}^2(P_2),$$

where $\varpi \in \mathfrak{a}_M^*$ is such that $\langle \varpi, \alpha^\vee \rangle = 1$.[5] Equivalently,

$$\delta_{P_1|P_2}(\lambda) = \Phi(\langle \lambda, \alpha^\vee \rangle)^{-1} \Phi'(\langle \lambda, \alpha^\vee \rangle),$$

where Φ is the meromorphic function on $\mathbb{C}$ such that $M_{P_1|P_2}(\lambda) = \Phi(\langle \lambda, \alpha^\vee \rangle)$.

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[5]Note that this definition differs slightly from the definition of $\delta_{P_1|P_2}$ in [34].

For any m-tuple $\mathcal{X} = (\langle P_1, P_1' \rangle, \ldots, \langle P_m, P_m' \rangle) \in \Xi_L(\underline{\beta})$ with $P_i |^{\beta_i} P_i'$, denote by $\Delta_{\mathcal{X}}(P, \lambda)$ the expression

$$\frac{\mathrm{vol}(\underline{\beta})}{m!} M_{P_1'|P}(\lambda)^{-1} \delta_{P_1|P_1'}(\lambda) M_{P_1'|P_2'}(\lambda) \cdots \delta_{P_{m-1}|P_{m-1}'}(\lambda) M_{P_{m-1}'|P_m'}(\lambda) \delta_{P_m|P_m'}(\lambda) M_{P_m'|P}(\lambda).$$

In [34, pp. 179–180], we defined a (purely combinatorial) map $\mathcal{X}_L : \mathfrak{B}_{P,L} \to \mathcal{F}_1(M)^m$ with the property that $\mathcal{X}_L(\underline{\beta}) \in \Xi_L(\underline{\beta})$ for all $\underline{\beta} \in \mathfrak{B}_{P,L}$.[6]

For any $s \in W(M)$, let L_s be the smallest Levi subgroup in $\mathcal{L}(M)$ containing w_s. We recall that $\mathfrak{a}_{L_s} = \{H \in \mathfrak{a}_M \mid sH = H\}$. Set

$$\iota_s = \left| \det(s-1)_{\mathfrak{a}_M^{L_s}} \right|^{-1},$$

a constant which we will not worry much about. For $P \in \mathcal{P}(M)$ and $s \in W(M)$, let $s : \mathcal{A}^2(P) \to \mathcal{A}^2(sP)$ be left translation by w_s^{-1} and $M(P, s) = M_{P|sP}(0)s : \mathcal{A}^2(P) \to \mathcal{A}^2(P)$ as in [8, p. 1309]. $M(P, s)$ is a unitary operator which intertwines $\rho(P, \lambda)$ with itself for $\lambda \in i\mathfrak{a}_{L_s}^*$. Finally, we can state the refined spectral expansion.

Theorem 4.1 ([34])**.** *The spectral side of Arthur's trace formula is given by*

$$J(h) = \sum_{[M]} J_{\mathrm{spec},M}(h), \quad h \in C_c^\infty(G(\mathbb{A})^1),$$

M ranging over the conjugacy classes of Levi subgroups of G (represented by members of $\mathcal{L}$), where

$$J_{\mathrm{spec},M}(h) = \frac{1}{|W(M)|} \sum_{s \in W(M)} \iota_s \sum_{\underline{\beta} \in \mathfrak{B}_{P,L_s}} \int_{i(\mathfrak{a}_{L_s}^G)^*} \mathrm{tr}(\Delta_{\mathcal{X}_{L_s}(\underline{\beta})}(P, \lambda) M(P, s) \rho(P, \lambda, h)) \, d\lambda,$$

with $P \in \mathcal{P}(M)$ arbitrary. The operators are of trace class and the integrals are absolutely convergent.

Note that here the term corresponding to $M = G$ is simply $J_{\mathrm{spec},G}(h) = \mathrm{tr}\, R_{\mathrm{disc}}(h)$.

5. Bounds on intertwining operators

We now introduce the key global and local properties required for the proof of the spectral limit property, and verify them for the groups $\mathrm{GL}(n)$ and $\mathrm{SL}(n)$. These properties will be used in § 7 to provide estimates for the contribution of the continuous spectrum to the spectral side of the trace formula. In view of planned future applications, our estimates are somewhat more precise in their dependence on the relevant parameters than is strictly necessary for the purposes of this paper.

5.1. The level of a compact open subgroup

We begin by introducing the notion of level for open subgroups K of $\mathbf{K}_{\mathrm{fin}}$ (or, more generally, open compact subgroups $K \subset G(\mathbb{A}_{\mathrm{fin}})$), as well as some generalizations. Recall

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[6]The map $\mathcal{X}_L$ depends in fact on the additional choice of a vector $\underline{\mu} \in (\mathfrak{a}_M^*)^m$ which lies outside a prescribed finite set of hyperplanes. For our purposes, the precise definition of $\mathcal{X}_L$ is immaterial.

the definition of the principal congruence subgroups $\mathbf{K}(\mathfrak{n})$ from § 3.1. Let $\mathfrak{n}_K$ be the largest ideal $\mathfrak{n}$ of $\mathfrak{o}_F$ such that $\mathbf{K}(\mathfrak{n}) \subset K$, and define the *level* of K as $\mathrm{level}_G(K) = \mathrm{level}(K) = \mathrm{N}(\mathfrak{n}_K)$. Analogously, we define $\mathrm{level}(K_v)$ for open subgroups $K_v \subset \mathbf{K}_v$. For a smooth representation π of $G(\mathbb{A})$, let $\mathfrak{n}_\pi$ be the largest ideal $\mathfrak{n}$ such that $\pi^{\mathbf{K}(\mathfrak{n})} \neq 0$. Let $\mathrm{level}_G(\pi) = \mathrm{level}(\pi) = \mathrm{N}(\mathfrak{n})$. Thus

$$\mathrm{level}_G(\pi) = \mathrm{level}(\pi) = \min\{\mathrm{level}(K) : \pi^K \neq 0\},$$

where K ranges over the open subgroups of $\mathbf{K}_{\mathrm{fin}}$.

For any Levi subgroup $M \in \mathcal{L}$ and an open subgroup $K \subset \mathbf{K}_{M,\mathrm{fin}} = \mathbf{K}_{\mathrm{fin}} \cap M(\mathbb{A}_{\mathrm{fin}})$, we define $\mathrm{level}_M(K)$ analogously using the principal congruence subgroups $\mathbf{K}(\mathfrak{n}) \cap M(\mathbb{A}_{\mathrm{fin}}) = \mathbf{K}_M(\mathfrak{n})$ of $\mathbf{K}_{M,\mathrm{fin}}$. Equivalently, $\mathrm{level}_M(K)$ is the level of $K \cap M(\mathbb{A}_{\mathrm{fin}})$ with respect to the restriction to M of the faithful G-representation ρ fixed in § 3.1.

More generally, suppose that $M \in \mathcal{L}$ and that $\mathcal{H}$ is a closed factorizable subgroup of $G(\mathbb{A}_{\mathrm{fin}})$. For any open compact subgroup $K \subset M(\mathbb{A}_{\mathrm{fin}})$, let $\mathfrak{n}_{K;\mathcal{H}}$ be the largest ideal $\mathfrak{n}$ of $\mathfrak{o}_F$ such that $\mathbf{K}(\mathfrak{n}) \cap M(\mathbb{A}_{\mathrm{fin}}) \cap \mathcal{H} \subset K$, and define the *relative level* of K with respect to $\mathcal{H}$ as $\mathrm{level}_M(K; \mathcal{H}) = \mathrm{N}(\mathfrak{n}_{K;H})$. Clearly, $\mathrm{level}_M(K; \mathcal{H}_1) \leqslant \mathrm{level}_M(K; \mathcal{H}_2)$ if $\mathcal{H}_1 \subset \mathcal{H}_2$ and $\mathrm{level}_M(K; G(\mathbb{A}_{\mathrm{fin}})) = \mathrm{level}_M(K)$. In particular, $\mathrm{level}_M(K; \mathcal{H}) \leqslant \mathrm{level}_M(K)$ for all $\mathcal{H}$.

As before, for an irreducible admissible representation π of $M(\mathbb{A})$, we write

$$\mathrm{level}_M(\pi; \mathcal{H}) = \min\{\mathrm{level}(K; \mathcal{H}) : \pi^K \neq 0\},$$

where K ranges over the compact subgroups of $\mathbf{K}_{M,\mathrm{fin}}$. Equivalently, $\mathrm{level}_M(\pi; \mathcal{H}) = \mathrm{N}(\mathfrak{n})$, where $\mathfrak{n}$ is the largest ideal such that $\pi^{\mathbf{K}(\mathfrak{n}) \cap M(\mathbb{A}_{\mathrm{fin}}) \cap \mathcal{H}} \neq 0$. Because $\mathcal{H}$ is assumed to be factorizable, it is clear how to define $\mathrm{level}_M(K_v; \mathcal{H})$ and $\mathrm{level}_M(\pi_v; \mathcal{H})$, where v is a finite place of F, K_v an open subgroup of $\mathbf{K}_{M,v}$, and π_v a smooth representation of $M(F_v)$.

When H is an algebraic subgroup of G defined over F, we write simply $\mathrm{level}_M(K; H)$ for $\mathrm{level}_M(K; H(\mathbb{A}_{\mathrm{fin}}))$. However, more general closed subgroups $\mathcal{H} \subset G(\mathbb{A}_{\mathrm{fin}})$ also appear naturally. For any reductive group H, let H^{sc} be the simply connected cover of the derived group H^{der} of H and $p^{\mathrm{sc}} : H^{\mathrm{sc}} \to H$ the natural homomorphism. We write $H(F_v)^+ \subset H^{\mathrm{der}}(F_v)$ for the image of $H^{\mathrm{sc}}(F_v)$ under p^{sc}, and similarly for $H(\mathbb{A})^+ \subset H^{\mathrm{der}}(\mathbb{A})$. Note that $H(\mathbb{A})^+$ contains the derived group of $H(\mathbb{A})$ as an open subgroup, and that it is equal to it if all the F-simple factors of H^{sc} are F-isotropic, in which case it is also the closed subgroup of $H(\mathbb{A})$ generated by its unipotent elements (cf. [61, Theorem 7.1, Proposition 7.6, Theorem 7.6] for the equality and [61, Proposition 3.5, Proposition 3.17] for the openness statement). Also note that, for any normal reductive subgroup H of G, the group $H(\mathbb{A})^+$ is a normal subgroup of $G(\mathbb{A})$, since G acts on H^{sc} by conjugation. Finally, note that the indices $[H^{\mathrm{der}}(F_v) : H(F_v)^+]$ are bounded in terms of H only, as v ranges over the places of F. For simplicity, we will often write by abuse of notation $\mathrm{level}_M(K; H^+)$ instead of $\mathrm{level}_M(K; H(\mathbb{A}_{\mathrm{fin}})^+)$.

Although in this paper our main examples are reductive groups H whose derived group is simply connected, which implies that $H^{\mathrm{sc}} = H^{\mathrm{der}}$ and $H(\mathbb{A})^+ = H^{\mathrm{der}}(\mathbb{A})$, in the general case it is advantageous to work with $H(\mathbb{A})^+$ instead of $H^{\mathrm{der}}(\mathbb{A})$.

Lemma 5.1. *Let G be a reductive group defined over a non-Archimedean local field F, and let M be a Levi subgroup of G defined over F. Then, for any smooth representation*

π of $M(F)$, we have

$$\mathrm{level}_M(\pi; G(F)^+) = \min_\chi \mathrm{level}_M(\pi \otimes \chi),$$

where χ ranges over the characters of $G(F)/G(F)^+$ (viewed as characters of $M(F)$ by restriction).

Proof. It is clear that $\mathrm{level}_M(\pi \otimes \chi; G^+) = \mathrm{level}_M(\pi; G^+)$ for any character χ of $G(F)$ trivial on $G(F)^+$, and that $\mathrm{level}_M(\pi; G^+) \leqslant \mathrm{level}_M(\pi)$. Thus $\mathrm{level}_M(\pi; G^+) \leqslant \min \mathrm{level}_M(\pi \otimes \chi)$.

In the other direction, suppose that $K \subset \mathbf{K}_{M,v}$ is an open subgroup such that $V := \pi^{K \cap G^+} \neq 0$. Then the compact abelian group $A := K/K \cap G(F)^+$ acts on V. Let χ be a character of A such that $V^{A,\chi^{-1}} \neq 0$. We can extend χ to a character of $G(F)/G(F)^+$. Then $(\pi \otimes \chi)^K \neq 0$. $\qquad\square$

We also have the following Archimedean analog of $\mathrm{level}_M(\pi; H^+)$. To define it, recall first Vogan's definition of a norm $\|\cdot\|$ on $\Pi(K_\infty)$ (cf. [19, § 2.2]), where K_∞ is a compact Lie group satisfying the conditions of [71, 0.1.2]. Namely, let K_∞^0 be the connected component of the identity of K_∞. Then $\|\mu\| = \|\chi_\mu + 2\rho\|^2$, where χ_μ denotes the highest weight of an arbitrary irreducible constituent of $\mu|_{K_\infty^0}$ with respect to a maximal torus of K_∞^0 (and the choice of a system of positive roots), and as usual ρ is half of the sum of all positive roots with multiplicities. A lowest K_∞-type of a representation is then a K_∞-type minimizing $\|\cdot\|$.

Let $H \subset G$ be a reductive algebraic subgroup normalized by M. For an irreducible representation π of $M(F_\infty)$, we will write

$$\Lambda_M(\pi; H) = 1 + (\lambda_\pi^H)^2 + \|\tau\|^2,$$

where λ_π^H is the eigenvalue of the Casimir operator of $M(F_\infty) \cap H(F_\infty)$ (which is an element of the center of the universal enveloping algebra of $\mathfrak{g}$) and τ is a lowest $\mathbf{K}_\infty \cap M(F_\infty) \cap H(F_\infty)$-type of π.

For $M \subset H$, we simply write $\Lambda_M(\pi)$ for $\Lambda_M(\pi; H)$. In [57], the parameter $\Lambda_\pi = \min_\tau \sqrt{\lambda_\pi^2 + \lambda_\tau^2}$ was used, where τ ranges over the lowest $\mathbf{K}_\infty$-types of the induced representation $\mathrm{Ind}_{P(F_\infty)}^{G(F_\infty)}(\pi)$ and λ_τ denotes the Casimir eigenvalue of τ. By [56, (5.15)], we have the estimate

$$1 \leqslant \Lambda_M(\pi; H) \leqslant \Lambda_M(\pi) \ll_G 1 + \Lambda_\pi^2 \tag{10}$$

for any H. Using the explicit description of lowest $\mathbf{K}_\infty$-types of irreducible and parabolically induced representations ([70, Theorem 7.16], [71, 6.5.9]), one can also show that

$$\Lambda_\pi^2 \ll_G \Lambda_M(\pi). \tag{11}$$

We will not give any details here, since we will only use this estimate in a side remark (the first part of Remark 5.3 below).

5.2. Bounds for global normalizing factors

For $M \in \mathcal{L}$, $\alpha \in \Sigma_M$, and $\pi \in \Pi_{\text{disc}}(M(\mathbb{A}))$, let $n_\alpha(\pi, s)$ be the global normalizing factor defined by (9).

Let U_α be the unipotent subgroup of G corresponding to α (so that the eigenvalues of $\mathbf{S}_M$ on the Lie algebra of U_α are positive integer multiples of α). Let $M_\alpha \in \mathcal{L}_1(M)$ be the group generated by M and $U_{\pm\alpha}$. Let $\hat{M}_\alpha$ be the group generated by $U_{\pm\alpha}$. It is a connected normal subgroup of M_α defined over F [17, Proposition 4.11]. Moreover, since M has co-rank one in M_α, precisely one simple root β of M_α restricts to α, which implies that the root system of $\hat{M}_\alpha$ is the irreducible component of the root system of M_α containing β. The group $\hat{M}_\alpha$ is therefore F-simple. It is also clearly F-isotropic, and therefore $\hat{M}_\alpha(\mathbb{A})^+$ is the closed subgroup of $G(\mathbb{A})$ generated by $U_{\pm\alpha}(\mathbb{A})$ [18, Proposition 6.2]. It is also the derived group of $\hat{M}_\alpha(\mathbb{A})$.

For any subset $\mathcal{F} \subset \Pi(\mathbf{K}_{M,\infty})$, we denote by $\Pi_{\text{disc}}(M(\mathbb{A}))^{\mathcal{F}}$ the set of all $\pi = \pi_\infty \otimes \pi_{\text{fin}} \in \Pi_{\text{disc}}(M(\mathbb{A}))$ for which π_∞ contains a $\mathbf{K}_{M,\infty}$-type in $\mathcal{F}$. If in addition an open subgroup $K_M \subset \mathbf{K}_{M,\text{fin}}$ is given, we define $\Pi_{\text{disc}}(M(\mathbb{A}))^{\mathcal{F}, K_M}$ as the set of all $\pi \in \Pi_{\text{disc}}(M(\mathbb{A}))^{\mathcal{F}}$ with $\pi_{\text{fin}}^{K_M} \neq 0$.

Definition 5.2. We say that the group G satisfies property (TWN) (tempered winding numbers) if, for any $M \in \mathcal{L}$, $M \neq G$, and any finite subset $\mathcal{F} \subset \Pi(\mathbf{K}_{M,\infty})$, there exists $k > 1$ such that for any $\alpha \in \Sigma_M$ and any $\epsilon > 0$ we have

$$\int_{i\mathbb{R}} \left| \frac{n'_\alpha(\pi, s)}{n_\alpha(\pi, s)} \right| (1 + |s|)^{-k} \, ds \ll_{\mathcal{F}, \epsilon} \Lambda_M(\pi_\infty; \hat{M}_\alpha)^k \, \text{level}_M(\pi; \hat{M}_\alpha^+)^\epsilon$$

for any $\pi \in \Pi_{\text{disc}}(M(\mathbb{A}))^{\mathcal{F}}$. (We recall that $|n_\alpha(\pi, s)| = 1$ for $s \in i\mathbb{R}$.)

Note that $n_\alpha(\pi, s)$ is not changed if we replace G by M_α or any other Levi subgroup containing it. Therefore, property (TWN) is hereditary for Levi subgroups.

Remark 5.3. (1) If we fix an open compact subgroup K_M, then the corresponding bound

$$\int_{i\mathbb{R}} \left| \frac{n'_\alpha(\pi, s)}{n_\alpha(\pi, s)} \right| (1 + |s|)^{-k} \, ds \ll_{K_M} \Lambda_M(\pi_\infty; \hat{M}_\alpha)^k$$

for any $\pi \in \Pi_{\text{disc}}(M(\mathbb{A}))^{\mathcal{F}, K_M}$ and a suitable $k > 1$ depending only on G can be deduced (by invoking (11)) from [57, Theorem 5.3], applied to the groups $M \cap \hat{M}_\alpha \subset \hat{M}_\alpha$. So, the point of (TWN) lies in the dependence of the bound on the level of π.

(2) In fact, we expect that

$$\int_T^{T+1} \left| \frac{n'_\alpha(\pi, it)}{n_\alpha(\pi, it)} \right| \, dt \ll \log(|T| + \Lambda_M(\pi_\infty; \hat{M}_\alpha) + \text{level}_M(\pi; \hat{M}_\alpha^+)) \qquad (12)$$

for all $T \in \mathbb{R}$ and $\pi \in \Pi_{\text{disc}}(M(\mathbb{A}))$.

This would give the following strengthening of (TWN):

$$\int_{i\mathbb{R}} \left| \frac{n'_\alpha(\pi, s)}{n_\alpha(\pi, s)} \right| (1 + |s|)^{-k} \, ds \ll_k \log(\Lambda_M(\pi_\infty; \hat{M}_\alpha) + \text{level}_M(\pi; \hat{M}_\alpha^+))$$

for any $\pi \in \Pi_{\text{disc}}(M(\mathbb{A}))$ and $k > 1$.

Lemma 5.4. *Suppose that $\tilde{G}$ is a connected reductive group over F which satisfies (TWN). Let G be a connected subgroup of $\tilde{G}$ containing $\tilde{G}^{\mathrm{der}}$. Then G also satisfies (TWN). The analogous statement holds for the bound (12).*

Proof. The map $\tilde{M} \mapsto \tilde{M} \cap G$ defines a one-to-one correspondence between the Levi subgroups of $\tilde{G}$ and those of G, and we have $\tilde{M} \cap G \supset \tilde{M} \cap \tilde{G}^{\mathrm{der}} \supset \tilde{M}^{\mathrm{der}}$. Suppose that $M = \tilde{M} \cap G$, and let $\tilde{\pi} \in \Pi_{\mathrm{disc}}(\tilde{M}(\mathbb{A}))$ be realized automorphically on a subspace $V_{\tilde{\pi}} \subset L^2(A_{\tilde{M}} \tilde{M}(F) \backslash \tilde{M}(\mathbb{A}))$. Then the space $V_{\tilde{\pi}}^M = \{\varphi|_{M(\mathbb{A})} : \varphi \in V_{\tilde{\pi}}\}$ is non-trivial, and, if π is an irreducible constituent of the $M(\mathbb{A})$-representation $V_{\tilde{\pi}}^M$, then, for every place v, π_v is a constituent of the restriction of $\tilde{\pi}_v$ to $M(F_v)$. By [11, § 3] for the Archimedean and [21, Theorem 3.3.4] for the non-Archimedean case, we have $n_\alpha(\pi_v, s) = n_\alpha(\tilde{\pi}_v, s)$ for all v. Hence $n_\alpha(\pi, s) = n_\alpha(\tilde{\pi}, s)$. Moreover, $\hat{M}_\alpha = \hat{\tilde{M}}_\alpha$ and $\hat{M}_\alpha \cap M = \hat{\tilde{M}}_\alpha \cap \tilde{M}$, which implies that $\mathrm{level}_{\tilde{M}}(\tilde{\pi}; \hat{\tilde{M}}_\alpha^+) \leqslant \mathrm{level}_M(\pi; \hat{M}_\alpha^+)$ and $\Lambda_{\tilde{M}}(\tilde{\pi}_\infty; \hat{\tilde{M}}_\alpha) \leqslant \Lambda_M(\pi_\infty; \hat{M}_\alpha)$. Therefore the lemma follows from the fact that every $\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A}))$ is equivalent to a constituent of $V_{\tilde{\pi}}^M$ for some $\tilde{\pi} \in \Pi_{\mathrm{disc}}(\tilde{M}(\mathbb{A}))$ with respect to *some* automorphic realization $V_{\tilde{\pi}}$ ([38, Theorem 4.13 and Remark 4.23] applied to $M \subset \tilde{M}$). $\square$

Proposition 5.5. *The estimate (12) holds for $G = \mathrm{GL}(n)$ or $\mathrm{SL}(n)$ with an implied constant depending only on n and F. In particular, the groups $\mathrm{GL}(n)$ and $\mathrm{SL}(n)$ satisfy property (TWN).*

We first need the following two lemmas, which are a direct consequence of the results of Bushnell and Henniart [15] and Jacquet, Piatetski-Shapiro, and Shalika (cf. [42, 52]).[7]

Lemma 5.6. *Let F be a local non-Archimedean field with residue field $\mathbb{F}_q$, let π be an irreducible smooth representation of $\mathrm{GL}(n, F)$, and let $f(\pi)$ be the exponent of the conductor of π as defined in [44]. Then*

$$f(\pi) \leqslant \log_q \mathrm{level}(\pi).$$

Proof. If π is generic, this immediately follows from the characterization of $f(\pi)$ in [44, Théorème 5.1]. In general, write π as the Langlands quotient of the parabolic induction of $\sigma_1 |\det|^{t_1} \otimes \cdots \otimes \sigma_r |\det|^{t_r}$ to $\mathrm{GL}(n, F)$, where $n = n_1 + \cdots + n_r$, $t_1, \ldots, t_r \in \mathbb{R}$, and each σ_i is an irreducible tempered representation of $\mathrm{GL}(n_i, F)$. We then have $f(\pi) = f(\sigma_1) + \cdots + f(\sigma_r)$ [41, Theorem 3.4]. Let π_0 be the parabolic induction of $\sigma_1 \otimes \cdots \otimes \sigma_r$ to $\mathrm{GL}(n, F)$. Then π_0 is irreducible and generic, and we have $f(\pi) = f(\pi_0) \leqslant \log_q \mathrm{level}(\pi_0) \leqslant \log_q \mathrm{level}(\pi)$. $\square$

Lemma 5.7. *Let F be a local non-Archimedean field with residue field $\mathbb{F}_q$. Consider $G = \mathrm{GL}(n)$, $G^{\mathrm{der}} = \mathrm{SL}(n)$, a maximal Levi subgroup $M = \mathrm{GL}(n_1) \times \mathrm{GL}(n_2)$ of G, and an irreducible smooth representation $\pi = \pi_1 \otimes \pi_2$ of $M(F)$. Let $f(\pi_1 \times \tilde{\pi}_2)$ be the exponent of the conductor of $\pi_1 \times \tilde{\pi}_2$ (cf. [15]). Then*

$$f(\pi_1 \times \tilde{\pi}_2) \leqslant n \log_q \mathrm{level}_M(\pi; G^{\mathrm{der}}).$$

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[7]We define levels for $\mathrm{GL}(n)$ and its subgroups in terms of the identity representation ρ of $\mathrm{GL}(n)$ and the standard lattice $\mathfrak{o}_F^n$.

Proof. Since $f(\pi_1 \times \tilde{\pi}_2)$ is not affected by twisting both π_1 and π_2 by the same character, by Lemma 5.1, it suffices to prove that

$$f(\pi_1 \times \tilde{\pi}_2) \leqslant n \log_q \mathrm{level}_M(\pi).$$

The results of [15] give

$$f(\pi_1 \times \tilde{\pi}_2) \leqslant n_1 f(\pi_2) + n_2 f(\pi_1),$$

where $f(\pi_i)$ is the exponent of the conductor of π_i. By Lemma 5.6, we have $f(\pi_i) \leqslant \log_q \mathrm{level}_{\mathrm{GL}(n_i)} \pi_i = \log_q \mathrm{level}_M(\pi; \mathrm{GL}(n_i)) \leqslant \log_q \mathrm{level}_M(\pi)$. The lemma follows. $\qquad\square$

Proof of Proposition 5.5. By Lemma 5.4, it suffices to consider the case of $G = \mathrm{GL}(n)$. The global normalizing factors n_α can be expressed in terms of Rankin–Selberg L-functions whose properties are summarized and analyzed in [58, § 4 and § 5]. More precisely, write $M \simeq \prod_{i=1}^r \mathrm{GL}(n_i)$, where the root α is trivial on $\prod_{i \geqslant 3} \mathrm{GL}(n_i)$, and let $\pi \simeq \otimes \pi_i$ with representations $\pi_i \in \Pi_{\mathrm{disc}}(\mathrm{GL}(n_i, \mathbb{A}))$. Note that then $\hat{M}_\alpha = \mathrm{SL}(n_1 + n_2)$. Let $L(s, \pi_1 \times \tilde{\pi}_2)$ be the completed Rankin–Selberg L-function associated to π_1 and π_2. It satisfies the functional equation

$$L(s, \pi_1 \times \tilde{\pi}_2) = \epsilon\left(\frac{1}{2}, \pi_1 \times \tilde{\pi}_2\right) N(\pi_1 \times \tilde{\pi}_2)^{\frac{1}{2}-s} L(1-s, \tilde{\pi}_1 \times \pi_2),$$

where $\left|\epsilon(\frac{1}{2}, \pi_1 \times \tilde{\pi}_2)\right| = 1$ and

$$N(\pi_1 \times \tilde{\pi}_2) = d_F^{n_1 n_2} \prod_v q_v^{f_v(\pi_{1,v} \times \tilde{\pi}_{2,v})}$$

is the conductor. Here, the local exponents $f_v(\pi_{1,v} \times \tilde{\pi}_{2,v})$ are as in Lemma 5.7 above, and d_F is the absolute value of the discriminant of F. We can then write

$$n_\alpha(\pi, s) = \frac{L(s, \pi_1 \times \tilde{\pi}_2)}{\epsilon(\frac{1}{2}, \pi_1 \times \tilde{\pi}_2) N(\pi_1 \times \tilde{\pi}_2)^{\frac{1}{2}-s} L(s+1, \pi_1 \times \tilde{\pi}_2)}.$$

The proof of Propositions 4.5 and 5.1 in [58] gives

$$\int_T^{T+1} \left|\frac{n_\alpha'(\pi, \mathrm{i}t)}{n_\alpha(\pi, \mathrm{i}t)}\right| \, dt \ll \log(|T| + \nu(\pi_1 \times \tilde{\pi}_2))$$

with

$$\nu(\pi_1 \times \tilde{\pi}_2) = N(\pi_1 \times \tilde{\pi}_2)(2 + c(\pi_1 \times \tilde{\pi}_2))$$

and the quantity $c(\pi_1 \times \tilde{\pi}_2) \geqslant 0$ of [58, (4.21)], which depends only on the Archimedean factors of π_1 and π_2. Moreover, $c(\pi_1 \times \tilde{\pi}_2)$ is by definition invariant under twisting by characters of $\mathrm{GL}(n_1 + n_2, F_\infty)$ (viewed as characters of $\mathrm{GL}(n_1, F_\infty) \times \mathrm{GL}(n_2, F_\infty)$ by restriction). Combining [58, Lemma 4.2] and [58, Lemma 5.4][8] shows that $\log(2 + c(\pi_1 \times \tilde{\pi}_2)) \ll \min_\chi \log \Lambda_M(\pi_\infty \otimes \chi; \mathrm{GL}(n_1 + n_2))$, where χ ranges over the characters of $\mathrm{GL}(n_1 + n_2, F_\infty)$.

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[8]Note that in [58, Lemma 4.2] it is not necessary to assume that the π_i are generic; it suffices that they are unitary. Also, [58, Lemma 5.4] deals explicitly only with the real places v of F, but complex places can be dealt with in the same way.

Since the central character $\omega(\pi_\infty \otimes \chi)$ of the representation $\pi_\infty \otimes \chi$ is simply $\omega(\pi_\infty)\chi^{n_1+n_2}$, by an appropriate choice of χ we can achieve that $\omega(\pi_\infty \otimes \chi)$ belongs to a finite set of characters that depends only on $n_1 + n_2$ and F_∞. We can then clearly bound the contribution of $\mathrm{Lie}(Z(\mathrm{GL}(n_1+n_2, F_\infty)))$ to the Casimir eigenvalue of $\pi_\infty \otimes \chi$ in terms of $n_1 + n_2$ and $[F : \mathbb{Q}]$, and similarly for the central contribution to $\|\tau\|$, where τ is a lowest $\mathbf{K}_\infty$-type of $\pi_\infty \otimes \chi$. Altogether, we obtain the estimate $\log(2 + c(\pi_1 \times \tilde{\pi}_2)) \ll \log \Lambda_M(\pi_\infty; \hat{M}_\alpha)$. On the other hand, by Lemma 5.7, we have $\log N(\pi_1 \times \tilde{\pi}_2) \ll 1 + \log \mathrm{level}_M(\pi; \hat{M}_\alpha)$. The proposition follows. $\qquad \square$

Remark 5.8. For a general group G, the normalizing factors are given, at least up to local factors, by quotients of automorphic L-functions associated to the irreducible constituents of the adjoint representation of the L-group $^L M$ of M on the Lie algebra of the unipotent radical of the corresponding parabolic subgroup of $^L G$ [48]. To argue as above, we would need to know that these L-functions have finitely many poles and satisfy a functional equation with the associated conductor bounded polynomially in $\mathrm{level}(\pi)$ for any $\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A}))$. Unfortunately, the finiteness of the number of poles and the expected functional equation are not known in general (although they are known in some important cases, e.g. [37]). It is possible that for classical groups these properties are within reach using the work of Arthur [12] and Mœglin [53], and the same may apply to the exceptional group G_2 [45]. However, this may require further work, and we will not pursue this matter any further.

5.3. Degrees of normalized local intertwining operators

Let $M \in \mathcal{L}$ and $P, Q \in \mathcal{P}(M)$ adjacent along $\alpha \in \Sigma_M$. Recall that, in the p-adic case, for any irreducible representation π_v of $M(F_v)$ and any open subgroup K_v of $G(F_v)$ the restriction $R_{Q|P}(\pi_v, s)^{K_v}$ of the operator $R_{Q|P}(\pi_v, s)$ to the finite-dimensional space of K_v-fixed vectors in $I_P^G(\pi_v)$ is a rational function in q^{-s}. More precisely, there exists a polynomial f, depending only on π_v, and whose degree is bounded in terms of G only, such that $f(q^{-s})R_{Q|P}(\pi_v, s)^{K_v}$ is polynomial in q^{-s}. We have $U_Q \cap U_{\bar{P}} = U_\alpha$, where $\{\alpha\} = \Sigma_Q \cap \Sigma_{\bar{P}}$. Note that the operator $R_{Q|P}$ is obtained by induction from the intertwining operator $R_{Q \cap M_\alpha | P \cap M_\alpha}^{M_\alpha}$ between representations of the group $M_\alpha(F_v)$. Let P_α be the maximal parabolic subgroup of M_α with unipotent radical U_α (and Levi subgroup M), and let $\bar{P}_\alpha$ be the corresponding opposite parabolic subgroup.

Definition 5.9. We say that G satisfies property (BD) (bounded degree) if there exists a constant c (depending only on G and ρ), such that for any

- Levi subgroup $M \in \mathcal{L}$, $M \neq G$,
- $\alpha \in \Sigma_M$,
- finite place v of F,
- open subgroup $K_v \subset \mathbf{K}_{M_\alpha, v}$,
- smooth irreducible representation π_v of $M(F_v)$, the degrees of the numerators of the

linear operators $R^{M_\alpha}_{\bar{P}_\alpha | P_\alpha}(\pi_v, s)^{K_v}$ are bounded by

$$\begin{cases} c \log_{q_v} \mathrm{level}_{M_\alpha}(K_v; \hat{M}^+_\alpha), & \text{if } \mathbf{K}_v \text{ is hyperspecial,} \\ c(1 + \log_{q_v} \mathrm{level}_{M_\alpha}(K_v; \hat{M}^+_\alpha)), & \text{otherwise.} \end{cases}$$

Remark 5.10. (1) Property (BD) is by definition hereditary for Levi subgroups.

(2) Property (BD) is discussed in more detail in [35]. Conjectures 1 and 2 of [35] for all Levi subgroups of G amount to the slightly weaker statement obtained by replacing $\mathrm{level}_{M_\alpha}(K_v; \hat{M}^+_\alpha)$ by $\mathrm{level}_{M_\alpha}(K_v; M^{\mathrm{der}}_\alpha)$, but the above formulation is more natural (and we intend to use it in a future paper). Note that Conjectures 1 and 2 of [35] for the F-simple factors of the groups L^{sc}, $L \in \mathcal{L}$, imply property (BD) in its current formulation.

(3) Note also that by [50, Lemma 1.6] (cf. also [35, Proposition 3] and the remark following it) we may replace $\mathrm{level}_{M_\alpha}(K_v; \hat{M}^+_\alpha)$ by the inverse of the volume of $(p^{\mathrm{sc}}_{\hat{M}_\alpha})^{-1}(K_v)$ in $\hat{M}^{\mathrm{sc}}_\alpha(F_v)$.

Let now G_M be the subgroup of G generated by the unipotent subgroups U_α, $\alpha \in \Sigma_M$. In other words, if P and $\bar{P}$ are (arbitrary) opposite parabolic subgroups in $\mathcal{P}(M)$, then G_M is the subgroup of G generated by U and $\bar{U}$. By [17, Proposition 4.11], G_M is a connected semisimple normal subgroup of G defined over F. Clearly, all non-central normal subgroups of G_M are F-isotropic, and therefore $G_M(\mathbb{A})^+$ is the closed subgroup of $G(\mathbb{A})$ generated by the groups $U_\alpha(\mathbb{A})$, $\alpha \in \Sigma_M$. It is also the derived group of $G_M(\mathbb{A})$.

Property (BD) has the following consequence for the operators $R_{Q|P}(\pi_v, s)$, which is a slight strengthening of [35, Proposition 6].[9]

Lemma 5.11. *Let G satisfy property (BD). Then, for any*

- *Levi subgroup $M \in \mathcal{L}$, $M \neq G$,*
- *any adjacent parabolic subgroups $P, Q \in \mathcal{P}(M)$,*
- *finite place v of F,*
- *open subgroup $K_v \subset \mathbf{K}_v$,*
- *smooth irreducible representation π_v of $M(F_v)$,*

the degrees of the numerators of the linear operators $R_{Q|P}(\pi_v, s)^{K_v}$ are bounded by

$$\begin{cases} c \log_{q_v} \mathrm{level}(K_v; G^+_M), & \text{if } \mathbf{K}_v \text{ is hyperspecial,} \\ c(1 + \log_{q_v} \mathrm{level}(K_v; G^+_M)), & \text{otherwise.} \end{cases}$$

---

[9]We take the opportunity to correct an inaccuracy in the proof (not the statement) of [35, Proposition 5]. Using the notation of [35] freely, in the last sentence of the proof it is stated that the matrix coefficients of $M_{Q_{i+1}|Q_i}(\sigma, \langle \lambda, \alpha^\vee_i \rangle)^K$ are given by those of $M_{\overline{Q'}|Q'}(\sigma, \langle \lambda, \alpha^\vee_i \rangle)^{K \cap M_R}$. Correctly, they are given by the matrix coefficients of $M_{\overline{Q'}|Q'}(\sigma, \langle \lambda, \alpha^\vee_i \rangle)^{\gamma K \gamma^{-1} \cap M_R}$, where γ ranges over K_0. However, we are free to consider instead of K the largest principal congruence subgroup K'_n of K'_0 contained in K, which is a normal subgroup of K_0. The same correction applies to the proof of [35, Lemma 20].

Proof. The matrix coefficients of $R_{Q|P}(\pi_v, s)^{K_v}$ are linear combinations of the matrix coefficients of $R_{Q\cap M_\alpha|P\cap M_\alpha}^{M_\alpha}(\pi_v, s)^{\gamma K_v \gamma^{-1}\cap M_\alpha(F_v)}$, $\gamma \in \mathbf{K}_v$. Property (BD) implies that the degrees of the latter coefficients are bounded by $c \log_{q_v} \text{level}(\gamma K_v \gamma^{-1}\cap M_\alpha(F_v); \hat{M}_\alpha^+)$ in the hyperspecial case, and by $c(1 + \log_{q_v} \text{level}(\gamma K_v \gamma^{-1}\cap M_\alpha(F_v); \hat{M}_\alpha^+))$ otherwise. Since $G_M(\mathbb{A}_{\text{fin}})^+$ is normal in $G(\mathbb{A}_{\text{fin}})$ and $G_M(\mathbb{A}_{\text{fin}})^+ \supset \hat{M}_\alpha(\mathbb{A}_{\text{fin}})^+$, we obtain the required estimate. $\qquad\square$

The proof of Lemma 5.11 yields in addition the following estimate.

Lemma 5.12. *Let* $\alpha \in \Sigma_M$. *Then* $\text{level}_M(\pi; \hat{M}_\alpha^+) \leqslant \text{level}_G(I_P^G(\pi); G_M^+)$. *In other words, if* K *is an open subgroup of* $\mathbf{K}_{\text{fin}}$ *such that* $I_P^G(\pi)^K \neq 0$, *then* $\text{level}_M(\pi; \hat{M}_\alpha^+) \leqslant \text{level}(K; G_M^+)$.

Proof. Suppose that $I_P^G(\pi)^K \neq 0$. Since the finite part of $I_P^G(\pi)$ is $I_{P(\mathbb{A}_{\text{fin}})\cap \mathbf{K}_{\text{fin}}}^{\mathbf{K}_{\text{fin}}} \pi|_{\mathbf{K}_{M,\text{fin}}}$ as a $\mathbf{K}_{\text{fin}}$-module, π must have a non-zero vector fixed under $\text{proj}_M(\gamma K \gamma^{-1}\cap P(\mathbb{A}_{\text{fin}}))$ for some $\gamma \in \mathbf{K}_{\text{fin}}$, where $\text{proj}_M : P \to M$ denotes the canonical projection. Suppose that $K \supset \mathbf{K}(\mathfrak{n}) \cap G_M(\mathbb{A}_{\text{fin}})^+$. Since $G_M(\mathbb{A}_{\text{fin}})^+$ is normal in $G(\mathbb{A})$ and $G_M(\mathbb{A}_{\text{fin}})^+ \supset \hat{M}_\alpha(\mathbb{A}_{\text{fin}})^+$, we get

$$\text{proj}_M(\gamma K \gamma^{-1}\cap P(\mathbb{A}_{\text{fin}})) \supset \text{proj}_M(\mathbf{K}(\mathfrak{n})\cap G_M(\mathbb{A}_{\text{fin}})^+\cap P(\mathbb{A}_{\text{fin}})) \supset \mathbf{K}_M(\mathfrak{n})\cap \hat{M}_\alpha(\mathbb{A}_{\text{fin}})^+.$$

The lemma follows. $\qquad\square$

Remark 5.13. Note that for $\mathbf{K}_v$ hyperspecial and $I_{P(F_v)}^{G(F_v)}(\pi_v)^{\mathbf{K}_v\cap G_M(F_v)^+} \neq 0$ the operator $R_{Q|P}(\pi_v, s)^{\mathbf{K}_v\cap G_M(F_v)^+}$ is independent of s. This implies that the assertion of Lemma 5.11 is satisfied unconditionally if $\mathbf{K}_v$ is hyperspecial and $\text{level}(K_v; G_M^+) = 1$. (In particular, the special case of property (BD) where $\mathbf{K}_v$ is hyperspecial and $\text{level}_{M_\alpha}(K_v; \hat{M}_\alpha^+) = 1$ is always true.) To see this, note that via $p^{\text{sc}}|_{G_M^{\text{sc}}}$ we can pull back $\mathbf{K}_v$ to a hyperspecial maximal compact subgroup of $G_M^{\text{sc}}(F_v)$, M to a proper Levi subgroup $\tilde{M}$ of G_M^{sc}, P and Q to adjacent parabolic subgroups $\tilde{P}$ and $\tilde{Q}$ with Levi subgroup $\tilde{M}$, and π to a smooth representation $\tilde{\pi}$ of $\tilde{M}(F_v)$, which is a finite direct sum of smooth irreducible representations. Then we can reduce to the property that $R_{\tilde{Q}|\tilde{P}}$ acts as a constant on the one-dimensional space of unramified vectors in the induction to $G_M^{\text{sc}}(F_v)$ of any unramified irreducible summand of $\tilde{\pi}$.

We have the following analog of Lemma 5.4.

Lemma 5.14. *Suppose that* $\tilde{G}$ *satisfies property (BD) and that* G *is a subgroup of* $\tilde{G}$ *containing* $\tilde{G}^{\text{der}}$. *Then* G *also satisfies (BD).*

Proof. Let $\tilde{M} \in \mathcal{L}^{\tilde{G}}$, and let $\tilde{P}, \tilde{Q} \in \mathcal{P}^{\tilde{G}}(\tilde{M})$ be adjacent along α. Then $M = \tilde{M}\cap G \in \mathcal{L}^G$ and the groups $P = \tilde{P}\cap G \in \mathcal{P}^G(M)$ and $Q = \tilde{Q}\cap G \in \mathcal{P}^G(M)$ are adjacent along α. If $\tilde{\pi}_v$ is an irreducible representation of $\tilde{M}(F_v)$, then we can identify the restriction of $I_{\tilde{P}}^{\tilde{G}}(\tilde{\pi}_v)$ to $G(F_v)$ with $I_P^G(\tilde{\pi}_v|_{M(F_v)})$. Moreover, if π_v is a constituent of $\tilde{\pi}_v|_{M(F_v)}$, then $R_{Q|P}(\pi_v, s)$ is the restriction of $R_{\tilde{Q}|\tilde{P}}(\tilde{\pi}_v, s)$. The lemma follows immediately. $\qquad\square$

Finally, the main result of [35] can be phrased as follows.

Theorem 5.15. *The groups* $\mathrm{GL}(n)$ *and* $\mathrm{SL}(n)$ *satisfy (BD).*

Proof. The $\mathrm{GL}(n)$ case follows directly from [35, Theorem 1], taking into account that Levi subgroups of $\mathrm{GL}(n)$ are products of groups $\mathrm{GL}(n_i)$. For $\mathrm{SL}(n)$, we may use Lemma 5.14. $\qquad\square$

5.4. Logarithmic derivatives of normalized local intertwining operators

The relevance of property (BD) to the trace formula lies in the following consequence, which we will prove in the remainder of this section.

Proposition 5.16. *Suppose that* G *satisfies (BD). Let* $M \in \mathcal{L}$, *and let* P, $Q \in \mathcal{P}(M)$ *be adjacent parabolic subgroups. Then, for all open compact subgroups* K *of* $G(\mathbb{A}_{\mathrm{fin}})$ *and all* $\tau \in \Pi(\mathbf{K}_\infty \cap G_M)$, *we have*

$$\int_{i\mathbb{R}} \left\| R_{Q|P}(\pi, s)^{-1} R'_{Q|P}(\pi, s) \Big|_{I_P^G(\pi)^{\tau, K}} \right\| (1 + |s|^2)^{-1} \, ds \ll 1 + \log(\|\tau\| + \mathrm{level}(K; G_M^+)).$$

We remark that the dependence of the bound on τ is not essential for the limit multiplicity problem, but it is relevant for other asymptotic questions.

The key ingredient for the proof of Proposition 5.16 is the following generalization of the classical Bernstein inequality due to Borwein and Erdélyi.

Proposition 5.17 [13]. *Let* $\mathbf{S}^1$ *be the unit circle in* $\mathbb{C}$, *and let* $z_1, \ldots, z_m \notin \mathbf{S}^1$. *Suppose that* $f(z)$ *is a rational function on* $\mathbb{C}$ *such that* $\sup_{z \in \mathbf{S}^1} |f(z)| \leqslant 1$ *and* $(z - z_1) \ldots (z - z_m) f(z)$ *is a polynomial of degree* n. *Then*

$$\left| f'(z) \right| \leqslant \max \left(\max(n - m, 0) + \sum_{j:|z_j|>1} \frac{|z_j|^2 - 1}{|z_j - z|^2}, \sum_{j:|z_j|<1} \frac{1 - |z_j|^2}{|z_j - z|^2} \right), \quad z \in \mathbf{S}^1.$$

Note that in [13, Theorem 1] this inequality is stated explicitly only for $n \leqslant m$. However, as explained on [13, p. 418], the case $n > m$ follows from the previous case by passing to

$$\frac{f(z)(|w_1| - 1) \ldots (|w_{n-m}| - 1)}{(z - w_1) \ldots (z - w_{n-m})},$$

where $w_1, \ldots, w_{n-m} \in \mathbb{C}$ are auxiliary parameters with $|w_i| > 1$, and then letting $w_i \to \infty$.

We will need a vector-valued version of Proposition 5.17 which is a direct consequence. For the next two lemmas, let V be a normed space over $\mathbb{C}$.

Corollary 5.18. *Let* $z_1, \ldots, z_m \in \mathbb{C} \setminus \mathbf{S}^1$. *Suppose that* $A : \mathbb{C} \setminus \{z_1, \ldots, z_m\} \to V$ *is such that* $(z - z_1) \ldots (z - z_m) A(z)$ *is a polynomial in* $z \in \mathbb{C}$ *of degree* n *with coefficients in* V. *Assume that* $\|A(z)\| \leqslant 1$ *for all* $z \in \mathbf{S}^1$. *Then*

$$\left\| A'(z) \right\| \leqslant \max \left(\max(n - m, 0) + \sum_{j:|z_j|>1} \frac{|z_j|^2 - 1}{|z_j - z|^2}, \sum_{j:|z_j|<1} \frac{1 - |z_j|^2}{|z_j - z|^2} \right), \quad z \in \mathbf{S}^1.$$

Consequently,

$$\int_{\mathbf{S}^1} \left\| A'(z) \right\| \ |dz| \leqslant 2\pi \max(m, n). \tag{13}$$

This follows from Proposition 5.17 by applying it to $f(z) = (A'(z), w)$ for any linear form w on V with $\|w\| \leqslant 1$.

We remark that, when $V = \mathbb{C}$, i.e., when $A(z)$ is scalar valued, the bound (13), at least with 2π replaced by 8, can be easily proved directly without appealing to Proposition 5.17 (see [34, Lemma 1]). However, we do not know a direct proof of (13) (even with 2π replaced by an arbitrary constant) in the general case.

Analogously, we have the following.

Lemma 5.19. *Let* $z_j = u_j + \mathrm{i}v_j \in \mathbb{C}$, $j = 1, \ldots, m$, *and let* $b(z) = (z - z_1)\ldots(z - z_m)$. *Suppose that* $A : \mathbb{C} \setminus \{z_1, \ldots, z_m\} \to V$ *is such that* $\|A(z)\| \leqslant 1$ *for all* $z \in \mathrm{i}\mathbb{R}$ *and* $b(z)A(z)$ *is a polynomial in* $z \in \mathbb{C}$ *(necessarily of degree* $\leqslant m$*) with coefficients in* V. *Then*

$$\int_{\mathrm{i}\mathbb{R}} \left\| A'(z) \right\| \frac{|dz|}{1 + |z|^2} \leqslant 2\pi \sum_{j=1}^{m} \frac{|u_j| + 1}{(|u_j| + 1)^2 + v_j^2} \leqslant 2\pi m.$$

Proof. For any $w \in \mathbb{C}$, let $\phi_w(z) = \frac{z + \bar{w}}{z - w}$, and set

$$\phi_{\gtrless}(z) = \prod_{j : \mathrm{Re}\, z_j \gtrless 0} \phi_{z_j}(z).$$

Applying [13, Theorem 4],[10] we conclude as before that

$$\left\| A'(z) \right\| \leqslant \max(\left| \phi'_>(z) \right|, \left| \phi'_<(z) \right|) \leqslant \left| \phi'_>(z) \right| + \left| \phi'_<(z) \right|, \quad z \in \mathrm{i}\mathbb{R}.$$

It remains to observe that for any $w = u + \mathrm{i}v \in \mathbb{C} \setminus \mathrm{i}\mathbb{R}$ we have

$$\int_{\mathrm{i}\mathbb{R}} \left| \phi'_w(z) \right| \frac{|dz|}{1 + |z|^2} = 2\pi \frac{|u| + 1}{(|u| + 1)^2 + v^2}.$$

Indeed, we have $\left| \phi'_w(z) \right| = \frac{2|u|}{|z - w|^2} = \frac{2|u|}{u^2 + (t - v)^2}$ for $z = \mathrm{i}t$, $t \in \mathbb{R}$, so

$$\int_{\mathrm{i}\mathbb{R}} \left| \phi'_w(z) \right| \frac{|dz|}{1 + |z|^2} = \int_{\mathbb{R}} \frac{2\,|u|}{(u^2 + (t - v)^2)(1 + t^2)} \ dt.$$

By the residue theorem, this is equal to

$$2\pi \left(\frac{|u|}{u^2 + (\mathrm{i} - v)^2} + \frac{1}{1 + (v + \mathrm{i}\,|u|)^2} \right) = \frac{2\pi}{v + \mathrm{i}(|u| - 1)} \left(\frac{|u|}{v - \mathrm{i}(|u| + 1)} + \frac{1}{v + \mathrm{i}(|u| + 1)} \right)$$
$$= \frac{2\pi(|u| + 1)}{v^2 + (|u| + 1)^2},$$

as claimed. □

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[10] This result is misstated on [34, p. 190].

Proof of Proposition 5.16. Replacing K by its largest factorizable subgroup does not change $\mathrm{level}(K; G_M^+)$. We may therefore assume that K is factorizable. Write $K = \prod K_v$, and set $K^v = \prod_{w \neq v} K_w$. Also set $N_v = \mathrm{level}_v(K_v; G_M^+)$, so that $\prod N_v = \mathrm{level}(K; G_M^+)$. Let S' (respectively, S'') be the finite set of finite places such that $\mathbf{K}_v$ is not hyperspecial (respectively, $N_v \neq 1$). Of course S' depends only on $\mathbf{K}_{\mathrm{fin}}$. Note that, by Remark 5.13, $R_v(\pi_v, s)^{K_v}$ is independent of s if v is finite and $v \notin S' \cup S''$. We have

$$R(\pi, s)^{-1} R'(\pi, s)|_{I(\pi)^{\tau, K}} = R_\infty(\pi_\infty, s)^{-1} R_\infty'(\pi_\infty, s)|_{I(\pi_\infty)^\tau} \otimes \mathrm{Id}_{I(\pi^\infty)^K}$$
$$+ \sum_{v \in S' \cup S''} R_v(\pi_v, s)^{-1} R_v'(\pi_v, s)\Big|_{I(\pi_v)^{K_v}} \otimes \mathrm{Id}_{I(\pi^v)^{\tau, K^v}}.$$

Recall that the operators $R_v(\pi_v, s)$ are unitary for $\mathrm{Re}\, s = 0$.

Consider first the case where $v \in S' \cup S''$. Write $R_v(\pi_v, s)\big|_{I(\pi_v)^{K_v}} = A_v(q_v^{-s})$. Then

$$\int_{i\mathbb{R}} \left\| R_v(\pi_v, s)^{-1} R_v'(\pi_v, s)\Big|_{I(\pi_v)^{K_v}} \right\| \frac{ds}{1 + |s|^2} = \int_{i\mathbb{R}} \left\| R_v'(\pi_v, s)\Big|_{I(\pi_v)^{K_v}} \right\| \frac{ds}{1 + |s|^2}$$

$$\leqslant 2 \sum_{n=0}^{\infty} \left(1 + \frac{4\pi^2 n^2}{(\log q_v)^2} \right)^{-1} \int_0^{\frac{2\pi i}{\log q_v}} \left\| R_v'(\pi_v, s)\Big|_{I(\pi_v)^{K_v}} \right\| ds$$

$$\leqslant 2 \left(1 + \frac{1}{4} \log q_v \right) \int_0^{\frac{2\pi i}{\log q_v}} \left\| R_v'(\pi_v, s)|_{I(\pi_v)^{K_v}} \right\| ds = \left(2 + \frac{1}{2} \log q_v \right) \int_{\mathbf{S^1}} \left\| A_v'(z) \right\| |dz|.$$
$$(14)$$

By Lemma 5.11, property (BD) for G implies that A_v satisfies the conditions of Corollary 5.18 (with respect to the operator norm) with m bounded in terms of G only and

$$n \ll \begin{cases} 1 + \log_{q_v} N_v, & \text{if } v \in S', \\ \log_{q_v} N_v, & \text{otherwise.} \end{cases}$$

By Corollary 5.18, (14) is therefore $\ll (\log q_v)(\log_{q_v} N_v) = \log N_v$, if $v \notin S'$ and $\ll 1 + \log N_v$, otherwise.

Regarding the Archimedean contribution, it follows from [54, Proposition A.2] that the operator $R_\infty(\pi_\infty, s)\big|_{I(\pi_\infty)^\tau}$ satisfies the conditions of Lemma 5.19 with $b(s) = \prod_{j=1}^r \prod_{k=1}^m (s - \rho_j + ck)$, where

- $c > 0$ depends only on M,
- r is bounded in terms of G only,
- $m \ll 1 + \|\tau\|$.

(In addition, the real parts $\mathrm{Re}\, \rho_j$ are bounded from above in terms of G only, but we will not need to use this fact.) Note that although [54, Proposition A.2] gives the bound $m \ll 1 + \|\tilde{\tau}\|$ on the $\tilde{\tau}$-isotypic subspace, where $\tilde{\tau} \in \Pi(\mathbf{K}_\infty)$, on [54, p. 88] it is explicitly stated that we may in fact consider the isotypic subspace for a representation of $\mathbf{K}_\infty \cap G^{\mathrm{der}}(F_\infty)$, and it is clear from the definition of $R_\infty(\pi_\infty, s)$ that we may even replace G^{der} by G_M.

Write $\rho_j = u_j + \mathrm{i}v_j$. By Lemma 5.19, we infer that

$$\int_{\mathrm{i}\mathbb{R}} \left\| R_\infty(\pi_\infty, s)^{-1} R'_\infty(\pi_\infty, s) \Big|_{I(\pi_\infty)^\tau} \right\| (1 + |s|^2)^{-1} \, ds$$

$$= \int_{\mathrm{i}\mathbb{R}} \left\| R'_\infty(\pi_\infty, s) \Big|_{I(\pi_\infty)^\tau} \right\| (1 + |s|^2)^{-1} \, ds$$

$$\ll \sum_{j=1}^r \sum_{k=1}^m \frac{|u_j - ck| + 1}{(|u_j - ck| + 1)^2 + v_j^2} \ll 1 + \log(1 + \|\tau\|).$$

Altogether,

$$\int_{\mathrm{i}\mathbb{R}} \left\| R(\pi, s)^{-1} R'(\pi, s) \Big|_{I(\pi)^{\tau, K}} \right\| (1 + |s|)^{-2} \, ds \ll 1 + \log(1 + \|\tau\|) + \sum_{v \text{ finite}} \log N_v$$

$$= 1 + \log(1 + \|\tau\|) + \log \mathrm{level}(K; G_M^+),$$

as required. $\qquad\square$

6. Polynomially bounded collections of measures

As a preparation for our proof of the spectral limit property in § 7, we prove in this section a result on real reductive Lie groups (Proposition 6.1 below) which extends an argument of Delorme in [26]. Let temporarily G_∞ be the group of real points of a connected reductive group defined over $\mathbb{R}$, or, slightly more generally, the quotient of such a group by a connected subgroup of its center (like the group $G(F_\infty)^1$ to which we will apply our results in § 7). Let K_∞ be a maximal compact subgroup of G_∞, and let θ be the associated Cartan involution. We will consider Levi subgroups M and parabolic subgroups P defined over $\mathbb{R}$. All Levi subgroups are implicitly assumed to be θ-stable. We factorize any Levi subgroup M as a direct product $M = A_M \times M^1$, where A_M is the largest central subgroup of M isomorphic to a power of $\mathbb{R}^{>0}$, and let $\mathfrak{a}_M = \mathrm{Lie}\, A_M$. We identify representations of M^1 with representations of M on which A_M acts trivially. Fix a minimal θ-stable Levi subgroup M_0. As in § 4.1, we fix an invariant bilinear form B on $\mathrm{Lie}\, G_\infty$, which induces Euclidean norms on all its subspaces and therefore Hermitian norms on the spaces $\mathfrak{a}_{M,\mathbb{C}}^*$. For each $r > 0$ and each finite set $\mathcal{F} \subset \Pi(K_\infty)$, we define $\mathcal{H}(G_\infty)_{r,\mathcal{F}}$ as the space of all smooth functions f on G_∞ with support contained in the compact set $K_\infty \exp(\{x \in \mathfrak{a}_0 : \|x\| \leqslant r\}) K_\infty$ whose translates $f(k_1 \cdot k_2)$, $k_1, k_2 \in K_\infty$, span a finite-dimensional space that decomposes under the action of $K_\infty \times K_\infty$ as a sum of representations $\tau_1 \otimes \tau_2$ with $\tau_1, \tau_2 \in \mathcal{F}$. We let $\mathcal{H}(G_\infty)_r$ be the union of the spaces $\mathcal{H}(G_\infty)_{r,\mathcal{F}}$ over all finite sets $\mathcal{F} \subset \Pi(K_\infty)$. The union of the spaces $\mathcal{H}(G_\infty)_r$ for all $r > 0$ is then the space $\mathcal{H}(G_\infty)$ introduced in § 1.

As before, for $f \in C^\infty(G_\infty)$ and $k \geqslant 0$, let

$$\|f\|_k = \sum_{X \in \mathcal{B}_k} \|X \star f\|_{L^1(G_\infty)}.$$

These norms endow $\mathcal{H}(G_\infty)_{r,\mathcal{F}}$ with the structure of a Fréchet space.

Let $\mathrm{Irr}(G_\infty)$ be the set of all irreducible admissible representations of G_∞ up to infinitesimal equivalence. The unitary dual $\Pi(G_\infty)$ can be viewed as a subset of $\mathrm{Irr}(G_\infty)$ in a natural way. For $\pi \in \mathrm{Irr}(G_\infty)$, denote its infinitesimal character by χ_π and its Casimir eigenvalue (which depends only on χ_π) by λ_π. For any $\mu \in \Pi(K_\infty)$, let $\mathrm{Irr}(G_\infty)_\mu$ be the set of irreducible representations containing μ as a K_∞-type. More generally, for any subset $\mathcal{F}$ of $\Pi(K_\infty)$, we write $\mathrm{Irr}(G_\infty)_{\mathcal{F}} = \cup_{\tau \in \mathcal{F}} \mathrm{Irr}(G_\infty)_\tau$.

We write $\mathcal{D}$ for the set of all conjugacy classes of pairs (M, δ) consisting of a Levi subgroup M of G_∞ and a discrete series representation δ of M^1. For any $\underline{\delta} \in \mathcal{D}$, let $\mathrm{Irr}(G_\infty)_{\underline{\delta}}$ be the set of all irreducible representations which arise by the Langlands quotient construction from the (tempered) irreducible constituents of $I_M^L(\delta)$ for Levi subgroups $L \supset M$. Here, I_M^L denotes (unitary) induction from an arbitrary parabolic subgroup of L with Levi subgroup M to L. We then have a disjoint decomposition

$$\mathrm{Irr}(G_\infty) = \coprod_{\underline{\delta} \in \mathcal{D}} \mathrm{Irr}(G_\infty)_{\underline{\delta}},$$

and consequently

$$\Pi(G_\infty) = \coprod_{\underline{\delta} \in \mathcal{D}} \Pi(G_\infty)_{\underline{\delta}}.$$

Recall the definition of the norm $\|\cdot\|$ on $\Pi(K_\infty)$ given in § 5.1. We call a finite subset $\mathcal{F} \subset \Pi(K_\infty)$ saturated if, for each $\mu \in \mathcal{F}$, all $\mu' \in \Pi(K_\infty)$ with $\|\mu'\| \leqslant \|\mu\|$ are also contained in $\mathcal{F}$.

For $\pi \in \mathrm{Irr}(G_\infty)$, we write $\underline{\delta}(\pi)$ for the unique element $\underline{\delta} \in \mathcal{D}$ with $\pi \in \mathrm{Irr}(G_\infty)_{\underline{\delta}}$. We introduce a partial order on $\mathcal{D}$ as in [19, § 2.3], using the lowest K_∞-types of $I_M^G(\delta)$: $\underline{\delta} \prec \underline{\delta}'$ if and only if $\|\mu\| < \|\mu'\|$ for lowest K_∞-types μ and μ' of $I_M^G(\delta)$ and $I_{M'}^G(\delta')$, respectively.

When $\mathcal{F} \subset \Pi(K_\infty)$ is finite and saturated, Vogan's theory of lowest K_∞-types implies that

$$\mathrm{Irr}(G_\infty)_{\mathcal{F}} = \cup_{\underline{\delta} \in \mathcal{D}_{\mathcal{F}}} \mathrm{Irr}(G_\infty)_{\underline{\delta}} \tag{15}$$

for a finite subset $\mathcal{D}_{\mathcal{F}} \subset \mathcal{D}$ (cf. [20, Proposition D.1]).

For $\underline{\delta} \in \mathcal{D}$, we let $\mathcal{F}(\underline{\delta})$ be the finite saturated set of all $\mu' \in \Pi(K_\infty)$ with $\|\mu'\| \leqslant \|\mu\|$ for a lowest K_∞-type μ of $I_M^G(\delta)$.

Proposition 6.1. *Let $\mathfrak{M}$ be a set of Borel measures on $\Pi(G_\infty)$. Then the following conditions on $\mathfrak{M}$ are equivalent.*

(1) *For all $\underline{\delta} \in \mathcal{D}$, there exists $N_{\underline{\delta}} > 0$ such that*

$$\nu(\{\pi \in \Pi(G_\infty)_{\underline{\delta}} : |\lambda_\pi| \leqslant R\}) \ll_{\underline{\delta}} (1 + R)^{N_{\underline{\delta}}}$$

for all $\nu \in \mathfrak{M}$ and $R > 0$.

(2) *There exists $r > 0$ such that $\sup_{\nu \in \mathfrak{M}} |\nu(\hat{f})|$ is a continuous seminorm on $\mathcal{H}(G_\infty)_{r, \mathcal{F}}$ for any finite set $\mathcal{F} \subset \Pi(K_\infty)$.*

(3) *$\sup_{\nu \in \mathfrak{M}} |\nu(\hat{f})|$ is a continuous seminorm on $\mathcal{H}(G_\infty)_{r, \mathcal{F}}$ for any $r > 0$ and any finite set $\mathcal{F} \subset \Pi(K_\infty)$.*

(4) *For each finite set $\mathcal{F} \subset \Pi(K_\infty)$, there exists an integer $k = k(\mathcal{F})$ such that $\sup_{\nu \in \mathfrak{M}} \nu(g_{k,\mathcal{F}}) < \infty$, where $g_{k,\mathcal{F}}$ is the non-negative function on $\Pi(G_\infty)$ defined by*

$$g_{k,\mathcal{F}}(\pi) = \begin{cases} (1+|\lambda_\pi|)^{-k}, & \text{if } \pi \in \Pi(G_\infty)_{\mathcal{F}}, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 6.2. We call a collection $\mathfrak{M}$ of measures satisfying the equivalent conditions of Proposition 6.1 *polynomially bounded.*

We begin the proof of Proposition 6.1. Let $\mathfrak{M}$ be a collection of Borel measures on $\Pi(G_\infty)$. Evidently the third condition of the proposition implies the second one. Note that, if z_{G_∞} is the Casimir element in the center of $\mathcal{U}(\text{Lie}\, G_\infty \otimes \mathbb{C})$, then $\widehat{z_{G_\infty} f}(\pi) = \lambda_\pi \hat{f}(\pi)$. Since also $\left|\hat{f}(\pi)\right| \leqslant \|f\|_{L^1(G_\infty)}$, it follows that for any $k \geqslant 0$ we have

$$\hat{f}(\pi) \ll_k \|f\|_{2k}\, g_{k,\mathcal{F}}(\pi)$$

for all $f \in \mathcal{H}(G_\infty)_{\mathcal{F}}$ and $\pi \in \Pi(G_\infty)$. We infer that the fourth condition of the proposition implies the third one.

For $k \geqslant 0$ and $\underline{\delta} \in \mathcal{D}$, let $g_{k,\underline{\delta}} = (1+|\lambda_\pi|)^{-k}$ for $\pi \in \Pi(G_\infty)_{\underline{\delta}}$, and extend this function by zero to all of $\Pi(G_\infty)$.

For a given $\underline{\delta} \in \mathcal{D}$, consider the following two statements.

There exists $N_{\underline{\delta}} > 0$ such that

$$\nu(\{\pi \in \Pi(G_\infty)_{\underline{\delta}} : |\lambda_\pi| \leqslant R\}) \ll_{\underline{\delta}} (1+R)^{N_{\underline{\delta}}} \quad \text{for all } \nu \in \mathfrak{M} \text{ and } R \geqslant 0. \quad (16a)$$

There exists an integer $k = k_{\underline{\delta}} > 0$ such that $\sup_{\nu \in \mathfrak{M}} \nu(g_{k,\underline{\delta}}) < \infty$. $\qquad (16b)$

It is easy to see that these statements are equivalent: if $(16a)$ is satisfied, then we can bound

$$\nu(g_{k,\underline{\delta}}) \leqslant \sum_{m \geqslant 0} \nu(\{\pi \in \Pi(G_\infty)_{\underline{\delta}} : m \leqslant |\lambda_\pi| \leqslant m+1\})(m+1)^{-k} \ll_{\underline{\delta}} \sum_{m \geqslant 0} \frac{(m+2)^{N_{\underline{\delta}}}}{(m+1)^k},$$

which is bounded independently of $\nu \in \mathfrak{M}$ for $k \geqslant N_{\underline{\delta}} + 2$. On the other hand, we clearly have

$$(1+R)^{-k}\nu(\{\pi \in \Pi(G_\infty)_{\underline{\delta}} : |\lambda_\pi| \leqslant R\}) \leqslant \nu(g_{k,\underline{\delta}}),$$

which gives the other implication.

Observe now that the first condition of the proposition is just $(16a)$ for all $\underline{\delta}$. Moreover, by (15), the fourth condition is equivalent to $(16b)$ for all $\underline{\delta}$. Therefore, the first and fourth conditions of the proposition are equivalent.

It remains to show that the second condition implies the first (or the fourth) one. This step is somewhat more difficult, and it requires some preliminary results, namely the classification of tempered and admissible representations of G_∞ and the Paley–Wiener theorem. We first recall Vogan's classification of irreducible admissible representations.

For (M, δ) as above, and $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^*$, consider the induced representation $\pi_{\delta,\lambda}$ (with respect to any parabolic subgroup containing M as a Levi subgroup). Its semisimplification depends only on the K_∞-conjugacy class of the triple (M, δ, λ). Vogan defines the R-group R_δ of δ, a finite group of exponent two, as well as its subgroup $R_{\delta,\lambda}$. The dual group $\hat{R}_\delta$ acts simply transitively on the set $A(\delta)$ of lowest K_∞-types of $\pi_{\delta,\lambda}$. We then have a decomposition of the representation $\pi_{\delta,\lambda}$ as a direct sum of $\left| R_{\delta,\lambda} \right|$ many representations $\pi_{\delta,\lambda}(\mu)$, where μ is an orbit of $R_{\delta,\lambda}^\perp$ in $A(\delta)$ [71, 6.5.10, 6.5.11]:

$$\pi_{\delta,\lambda} = \bigoplus_{\mu \in A(\delta)/R_{\delta,\lambda}^\perp} \pi_{\delta,\lambda}(\mu).$$

We call the $\pi_{\delta,\lambda}(\mu)$ *basic representations*. Each basic representation $\pi_{\delta,\lambda}(\mu)$ has a unique irreducible subquotient $\bar{\pi}_{\delta,\lambda}(\mu)$ containing a K_∞-type in the orbit μ. Alternatively, this subquotient can also be constructed as a Langlands quotient [71, 6.6.14, 6.6.15]. This construction sets up a bijection $\bar{\pi}_{\delta,\lambda}(\mu) = \pi \mapsto \sigma_\pi = \pi_{\delta,\lambda}(\mu)$ between infinitesimal equivalence classes of irreducible admissible representations π and basic representations σ_π, where the latter are interpreted as elements of the Grothendieck group of admissible representations [71, 6.5.13]. By definition, the parameterization is compatible with the disjoint decomposition of $\mathrm{Irr}(G_\infty)$ according to the elements of $\mathcal{D}$.

The distributions $\mathrm{tr}\, \sigma_\pi$ for $\pi \in \mathrm{Irr}(G_\infty)$ form a basis of the Grothendieck group of admissible representations. More precisely, we have the following relations expressing the characters of irreducible representations $\pi \in \mathrm{Irr}(G_\infty)$ in terms of the characters of basic representations:

$$\mathrm{tr}\,\pi(\phi) = \mathrm{tr}\,\sigma_\pi(\phi) + \sum_{\pi': \underline{\delta}(\pi) \prec \underline{\delta}(\pi'),\, \chi_\pi = \chi_{\pi'}} n(\pi, \pi')\,\mathrm{tr}\,\sigma_{\pi'}(\phi), \tag{17}$$

with certain integers $n(\pi, \pi')$ [71, 6.6.7]. Note that here the sum on the right-hand side is finite. For our purposes, all we need to know about the integers $n(\pi, \pi')$ is the following uniform boundedness property [26, Proposition 2.2].

Lemma 6.3 (Vogan). *We have*

$$\sum_{\pi': \underline{\delta}(\pi) \prec \underline{\delta}(\pi'),\, \chi_\pi = \chi_{\pi'}} \left| n(\pi, \pi') \right| \ll 1$$

for all $\pi \in \mathrm{Irr}(G_\infty)$.

For the Paley–Wiener theorem, we need to group the basic representations into series of induced representations, which gives a slightly different parameterization. We use the concept of a non-degenerate limit of discrete series introduced in [46, 47]. Let $\underline{\delta} \in \mathcal{D}$ with representative (M, δ). Whenever L is a Levi subgroup containing M and the irreducible constituents δ' of $I_M^L(\delta)$ are non-degenerate limits of discrete series of the group L^1, we call the resulting pairs (L, δ') *affiliated* with the class $\underline{\delta}$ [20, Définition 2]. These representations are precisely those irreducible constituents of the representations $I_M^L(\delta)$ for $L \supset M$, which are not themselves irreducibly induced from any smaller Levi subgroup. For fixed (M, δ), the Levi subgroups $L \supset M$ appearing in this construction are precisely

those for which $\mathfrak{a}_L^*$ is the fixed space of $\mathfrak{a}_M^*$ under one of the subgroups $R_{\delta,\lambda} \subset R_\delta$ (where we regard R_δ as a subgroup of $W(A_M)_\delta \subset W(A_M) = N_K(A_M)/C_K(A_M)$ as in [20, § 2.1]).

We can then rewrite any representation $\pi_{\delta,\lambda}(\mu)$ in the form $\pi_{\delta',\lambda}$, where $L \supset M$ is a Levi subgroup with $\lambda \in \mathfrak{a}_{L,\mathbb{C}}^* \subset \mathfrak{a}_{M,\mathbb{C}}^*$ and (L, δ') is affiliated with $\mathcal{D}$, such that the intermediate induction to the largest Levi subgroup $L_{\mathrm{Re}\,\lambda}$ with $\mathrm{Re}\,\lambda \in \mathfrak{a}_{L_{\mathrm{Re}\,\lambda}}^*$ is irreducible (and tempered). (To see this, combine [71, 6.6.14, 6.6.15] with [20, (2.1), (2.2)].) Note that the tempered dual of G_∞ can be parameterized as either the set of all basic representations $\pi_{\delta,\lambda}(\mu)$ with $\mathrm{Re}\,\lambda = 0$ (which are always irreducible), or as the set of all *irreducible* induced representations $\pi_{\delta',\lambda}$, $\mathrm{Re}\,\lambda = 0$, where δ' is a non-degenerate limit of discrete series.

Recall the definition of the Paley–Wiener space $\mathcal{PW}(\mathfrak{a})_r$, $r > 0$, of a Euclidean vector space $\mathfrak{a}$. It is the space of all entire functions F on the complexified dual $\mathfrak{a}_{\mathbb{C}}^*$ such that the Paley–Wiener norms

$$\|F\|_{r,n} = \sup_{\lambda \in \mathfrak{a}_{\mathbb{C}}^*} (1 + \|\lambda\|)^n e^{-r\|\mathrm{Re}\,\lambda\|} |F(\lambda)|, \quad n \geqslant 0,$$

are finite. The Paley–Wiener norms endow $\mathcal{PW}(\mathfrak{a})_r$ with the structure of a Fréchet space, and the Fourier transform is a topological isomorphism between $\mathcal{PW}(\mathfrak{a})_r$ and the Fréchet space of smooth functions on $\mathfrak{a}$ supported on the ball of radius r around 0.

Now let $\underline{\delta} \in \mathcal{D}$. Consider the finite set $\mathcal{D}'(\underline{\delta})$ of all pairs (M, δ), where M is a standard Levi subgroup of G_∞, $\delta \in \Pi(M^1)$ a non-degenerate limit of discrete series, and (M, δ) is affiliated with $\underline{\delta}$. The Paley–Wiener space $\mathcal{PW}_{r,\underline{\delta}}$ is then defined as the space of all elements $F = (F_{(M,\delta)}) \in \prod_{(M,\delta) \in \mathcal{D}'(\underline{\delta})} \mathcal{PW}(\mathfrak{a}_M)_r$ fulfilling the following conditions.

(1) Whenever the triples (M, δ, λ) and (M', δ', λ') are conjugate by an element of K_∞, we have $F_{(M',\delta')}(\lambda') = F_{(M,\delta)}(\lambda)$.

(2) Whenever for $M \subset M'$ we have a decomposition

$$I_M^{M'}(\delta_M) = \bigoplus_{i=1}^m \delta_{M'}^{(i)}$$

with (M, δ_M), $(M', \delta_{M'}^{(i)}) \in \mathcal{D}'(\underline{\delta})$, the corresponding identity

$$F_{(M,\delta_M)}(\lambda) = \sum_{i=1}^m F_{(M',\delta_{M'}^{(i)})}(\lambda), \quad \lambda \in \mathfrak{a}_{M',\mathbb{C}}^* \subset \mathfrak{a}_{M,\mathbb{C}}^*,$$

holds.

For any finite saturated set $\mathcal{F} \subset \Pi(K_\infty)$, the space $\mathcal{PW}_{r,\mathcal{F}}$ is defined as $\prod_{\underline{\delta} \in \mathcal{D}_\mathcal{F}} \mathcal{PW}_{r,\underline{\delta}}$. These Paley–Wiener spaces have in a natural way the structure of Fréchet spaces, and we define for each $n \geqslant 0$ the Paley–Wiener norm $\|F\|_{r,n}$ of $F \in \mathcal{PW}_{r,\mathcal{F}}$ to be the maximum of the norms $\|F_{(M,\delta)}\|_{r,n}$, where $(M, \delta) \in \mathcal{D}'(\underline{\delta})$, $\underline{\delta} \in \mathcal{D}_\mathcal{F}$. (Cf. [20, Appendice C] for a concrete combinatorial description of these spaces.)

We can now state the Paley–Wiener theorem of Clozel and Delorme [20, Théorème 1, Théorème 1].

Theorem 6.4 (Clozel–Delorme). *For any finite saturated set $\mathcal{F} \subset \Pi(K_\infty)$ and any $r > 0$, the natural continuous map of Fréchet spaces $T_{r,\mathcal{F}} : \mathcal{H}(G_\infty)_{r,\mathcal{F}} \to \mathcal{PW}_{r,\mathcal{F}}$ given by $f \mapsto (\operatorname{tr}\pi_{\delta,\lambda}(f))$ is surjective.*

Remark 6.5. By the open mapping theorem, a continuous surjection of Fréchet spaces is automatically open. For the surjections $T_{r,\mathcal{F}}$ of Theorem 6.4 this means concretely that for every integer $k \geqslant 0$ there exists an integer n (depending on k, r, and $\mathcal{F}$) with the following property: for any $F \in \mathcal{PW}_{r,\mathcal{F}}$ there exists $\phi \in \mathcal{H}(G_\infty)_{r,\mathcal{F}}$ such that $T_{r,\mathcal{F}}(\phi) = F$ and $\|\phi\|_k \ll_{r,\mathcal{F},k} \|F\|_{r,n}$.

We now turn to the proof of Proposition 6.1, which is an extension of an argument of Delorme (cf. the proof of [26, Proposition 3.3]). As in [26], the proof is based on the existence of certain test functions on G_∞; however, in comparison to Delorme's argument, we also need to bound the seminorms of these functions. We therefore recall the construction in some detail. The first elementary lemma [31, Lemma 6.3] asserts the existence of functions with certain properties of the Fourier transform.

Lemma 6.6 (Duistermaat–Kolk–Varadarajan). *Let $\mathfrak{a}$ be a Euclidean vector space, let W be a finite group acting on $\mathfrak{a}$, and let $r > 0$. Then, for any $t \geqslant 1$, there exists a function $\hat{h}(t, \cdot) \in \mathcal{PW}(\mathfrak{a})_r^W$ with the following properties.*

(1) $\hat{h}(t, \lambda) \in \mathbb{R}^{\geqslant 0}$ *for all $\lambda \in \mathfrak{a}_\mathbb{C}^*$ for which there exists an element $w \in W$ with $w(\lambda) = -\bar{\lambda}$.*

(2) $\left|\hat{h}(t, \lambda)\right| \geqslant 1$ *for all $\lambda \in \mathfrak{a}_\mathbb{C}^*$ with $\|\lambda\| \leqslant t$.*

(3) *For all $m \geqslant 0$, we have $\left\|\hat{h}(t, \cdot)\right\|_{r,m} \ll_{r,m} t^m$. In particular, for all $a > 0$ and $m \geqslant 0$, we have*

$$\hat{h}(t, \lambda) \ll_{r,m,a} \frac{t^m}{(1 + \|\operatorname{Im}\lambda\|)^m}$$

for all $\lambda \in \mathfrak{a}_\mathbb{C}^*$ with $\|\operatorname{Re}\lambda\| \leqslant a$.

The following lemma is a strengthening of [26, Proposition 3.2] (we have added the third assertion).

Lemma 6.7. *Let $r > 0$, let $\underline{\delta} \in \mathcal{D}$ with representative (M, δ), and let $k \geqslant 0$ be an integer. Then there exist an integer $r_{k,\underline{\delta}} \geqslant 0$ (depending on k and $\underline{\delta}$) and for each $t \geqslant 1$ a function $\phi_{\underline{\delta}}^{t,k} \in \mathcal{H}(G_\infty)_{r,\mathcal{F}(\underline{\delta})}$ with the following properties.*

(1) $\operatorname{tr}\sigma_\pi(\phi_{\underline{\delta}}^{t,k}) = 0$ *for all $\pi \in \operatorname{Irr}(G_\infty)$ with $\underline{\delta}(\pi) \neq \underline{\delta}$.*

(2) $\operatorname{tr}\sigma(\phi_{\underline{\delta}}^{t,k}) = [R_\delta : R_{\delta,\lambda}]\hat{h}(t, \lambda)$ *for all basic representations $\sigma = \pi_{\delta,\lambda}(\mu)$, $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^*$, $\mu \in A(\delta)$.*

(3) $\left\|\phi_{\underline{\delta}}^{t,k}\right\|_k \ll_{k,\underline{\delta}} t^{r_{k,\underline{\delta}}}$.

Proof. We apply the Paley–Wiener theorem to the following element $F^t = F_{(M',\delta')}^t$ of $\mathcal{PW}_{r,\mathcal{F}}$, where $\mathcal{F} = \mathcal{F}(\underline{\delta})$. Apply Lemma 6.6 to the vector space $\mathfrak{a}_M$ and the

group $W(A_M) = N_K(A_M)/C_K(A_M)$ to obtain functions $\hat{h}(t,\cdot) \in \mathcal{PW}(\mathfrak{a}_M)_r^{W(A_M)}$. Set $F^t_{(M',\delta')}(\lambda) = 0$ whenever (M',δ') is not affiliated with $\underline{\delta}$. On the other hand, if (M',δ') is affiliated with $\underline{\delta}$, then we have a decomposition $I_M^{M'}(\delta) = \oplus_{i=1}^{S}\delta_i'$, where $\delta_1' = \delta'$ and S is a divisor of $|R_\delta|$. We then set $F^t_{(M',\delta')}(\lambda) = \frac{|R_\delta|}{S}\hat{h}(t,\lambda)$. By the $W(A_M)$-invariance of $\hat{h}(t,\cdot)$ and the transitivity of induction, this defines an element F^t of $\mathcal{PW}_{r,\mathcal{F}}$. (Note that the commutativity of the group R_δ implies that, if we have a decomposition $I_M^{M''}(\delta) = \oplus_{i=1}^{S''}\delta_i''$ with (M'',δ_i'') affiliated with $\underline{\delta}$ and $M'' \supset M'$, then S'' is a multiple of S, and each $I_{M'}^{M''}(\delta_i')$ splits into S''/S many irreducible constituents δ_j'', cf. [20, pp. 204–205].) Moreover, by the third assertion of Lemma 6.6, the Paley–Wiener norms of F^t satisfy $\left\|F^t\right\|_{r,n} \ll_{r,\underline{\delta},n} t^n$ for every $n \geqslant 0$. The Paley–Wiener theorem (Theorem 6.4) and Remark 6.5 provide for every $k \geqslant 0$ an integer $r_{k,\underline{\delta}} \geqslant 0$ and a preimage $\phi_{\underline{\delta}}^{t,k} \in \mathcal{H}(G_\infty)_{r,\mathcal{F}}$ of F^t under $T_{r,\mathcal{F}}$ with $\left\|\phi_{\underline{\delta}}^{t,k}\right\|_k \ll_{r,\underline{\delta},k} \left\|F^t\right\|_{r,r_{k,\underline{\delta}}} \ll_{r,\underline{\delta},k} t^{r_{k,\underline{\delta}}}$. Therefore, $\phi_{\underline{\delta}}^{t,k}$ satisfies the third property of the lemma. The first property is clear by construction. Finally, write $\sigma = \pi_{\delta,\lambda}(\mu) \simeq \pi_{\delta',\lambda}$, where δ' is a non-degenerate limit of discrete series of $(M')^1$, $M' \supset M$ is a Levi subgroup with $\lambda \in \mathfrak{a}_{M',\mathbb{C}}^* \subset \mathfrak{a}_{M,\mathbb{C}}^*$, and (M',δ') is affiliated with $\mathcal{D}$. The number S of irreducible constituents of $I_M^{M'}(\delta)$ is then equal to $|R_{\delta,\lambda}|$. Therefore $\operatorname{tr}\sigma(\phi_{\underline{\delta}}^{t,k}) = F^t_{(M',\delta')}(\lambda) = [R_\delta : R_{\delta,\lambda}]\hat{h}(t,\lambda)$, which establishes the second property and finishes the proof. $\qquad\square$

Corollary 6.8. *The test functions $\phi_{\underline{\delta}}^{t,k} \in \mathcal{H}(G_\infty)_{r,\mathcal{F}(\underline{\delta})}$ above satisfy the following additional properties.*

(1) $\operatorname{tr}\sigma_\pi(\phi_{\underline{\delta}}^{t,k}) \geqslant 1$ *for all $\pi \in \Pi(G_\infty)_{\underline{\delta}}$ with $|\lambda_\pi| \leqslant t^2 - c_{\underline{\delta}}$, where $c_{\underline{\delta}} \geqslant 0$ is a constant depending only on $\underline{\delta}$.*

(2) *For all $m \geqslant 0$, we have*

$$0 \leqslant \operatorname{tr}\sigma_\pi(\phi_{\underline{\delta}}^{t,k}) \ll_{\underline{\delta},m} \frac{t^{2m}}{(1+|\lambda_\pi|)^m}$$

for all $\pi \in \Pi(G_\infty)_{\underline{\delta}}$.

Proof. Let $\bar{\pi}_{\delta,\lambda}(\mu) \simeq \pi \in \Pi(G_\infty)$, and let $\sigma_\pi = \pi_{\delta,\lambda}(\mu)$. Since π is unitary, we need to have $w(\lambda) = -\bar{\lambda}$ for an element $w \in W(A_M)_\delta$ [26, (2.1)]. By Lemma 6.7, the trace $\operatorname{tr}\sigma_\pi(\phi_{\underline{\delta}}^{t,k})$ is an integer multiple of $\hat{h}(t,\lambda)$. By the first property of Lemma 6.6, it is therefore non-negative real.

Furthermore, the Casimir eigenvalue of π can be computed as $\lambda_\pi = -\|\operatorname{Im}\lambda\|^2 + \|\operatorname{Re}\lambda\|^2 + \|\chi_\delta\|^2 - c_M$ for a constant c_M (cf. [14, § 3.2, (2)]). Again by unitarity, we have $\|\operatorname{Re}\lambda\| \leqslant \|\rho_P\|$, where ρ_P is half the sum of the positive roots of a parabolic P with Levi subgroup M [26, (2.2)]. Therefore, we obtain $|\lambda_\pi| \geqslant \|\lambda\|^2 - c_{\underline{\delta}}$ for a constant $c_{\underline{\delta}}$, which we may take to be non-negative.

To show the first assertion, $|\lambda_\pi| \leqslant t^2 - c_{\underline{\delta}}$ implies that $\|\lambda\| \leqslant t$, and by the second property of Lemma 6.6 we obtain $\hat{h}(t,\lambda) \geqslant 1$. For the second assertion, we use the last property of Lemma 6.6 (with $2m$ instead of m) and the boundedness of $\|\operatorname{Re}\lambda\|$ together with the obvious fact that $[R_\delta : R_{\delta,\lambda}]$ is bounded by $|R_\delta|$. $\qquad\square$

End of proof of Proposition 6.1. It remains to prove the equivalent statements (16a) or (16b) for any $\underline{\delta}$ assuming the second condition of the proposition. We will prove them by induction on $\underline{\delta}$, i.e., for a given $\underline{\delta}$, we assume that (16b) is satisfied for all $\underline{\delta}' \prec \underline{\delta}$ and are going to prove (16a) for $\underline{\delta}$. For this, consider the test functions $\phi_{\underline{\delta}}^{t,k}$ constructed in Lemma 6.7 for $t = (1 + c_{\underline{\delta}} + R)^{1/2} \geqslant 1$, where $R > 0$ is a parameter. By assumption, for a suitable $r > 0$, for each finite set $\mathcal{F}$ the supremum $\sup_{\nu \in \mathfrak{M}} \left| \nu(\hat{f}) \right|$ is a continuous seminorm on $\mathcal{H}(G_\infty)_{r,\mathcal{F}}$. This means that for a suitable value of k (depending on $\mathcal{F}$) we have $\nu(\hat{f}) \ll_{\mathcal{F}} \|f\|_k$ for all $\nu \in \mathfrak{M}$ and $f \in \mathcal{H}(G_\infty)_{r,\mathcal{F}}$. Taking $f = \hat{\phi}_{\underline{\delta}}^{t,k}$ and using the third assertion of Lemma 6.7, we obtain that there exists an integer $m_{\underline{\delta}} \geqslant 0$ such that

$$\nu(\hat{\phi}_{\underline{\delta}}^{t,k}) \ll_{\underline{\delta}} (1 + R)^{m_{\underline{\delta}}}, \quad \nu \in \mathfrak{M}. \tag{18}$$

Write

$$\nu(\hat{\phi}_{\underline{\delta}}^{t,k}) = \int \operatorname{tr} \pi(\phi_{\underline{\delta}}^{t,k}) d\nu(\pi).$$

Inserting (17) into this equation, we obtain

$$\nu(\hat{\phi}_{\underline{\delta}}^{t,k}) = \int (\operatorname{tr} \sigma_\pi)(\phi_{\underline{\delta}}^{t,k}) d\nu(\pi) + \int \left[\sum_{\pi': \underline{\delta}(\pi) \prec \underline{\delta}(\pi'), \, \chi_\pi = \chi_{\pi'}} n(\pi, \pi') \operatorname{tr} \sigma_{\pi'}(\phi_{\underline{\delta}}^{t,k}) \right] d\nu(\pi).$$

By the first assertion of Corollary 6.8, the first integral provides an upper bound for the measure of the set $\{\pi \in \Pi(G_\infty)_{\underline{\delta}} : |\lambda_\pi| \leqslant R\}$:

$$\nu(\{\pi \in \Pi(G_\infty)_{\underline{\delta}} : |\lambda_\pi| \leqslant R\}) \leqslant \int (\operatorname{tr} \sigma_\pi)(\phi_{\underline{\delta}}^{t,k}) d\nu(\pi).$$

Regarding the second integral, only the π' with $\underline{\delta}(\pi') = \underline{\delta}$ can contribute, and we can estimate their contribution using the second assertion of Corollary 6.8:

$$0 \leqslant \operatorname{tr} \sigma_{\pi'}(\phi_{\underline{\delta}}^{t,k}) \ll_{\underline{\delta},m} \frac{t^{2m}}{(1 + |\lambda_{\pi'}|)^m} = \frac{(1 + c_{\underline{\delta}} + R)^m}{(1 + |\lambda_\pi|)^m},$$

since π and π' have the same infinitesimal character. Combining this inequality with (18) and using Lemma 6.3, we obtain

$$\nu(\{\pi \in \Pi(G_\infty)_{\underline{\delta}} : |\lambda_\pi| \leqslant R\}) \leqslant \nu(\hat{\phi}_{\underline{\delta}}^{t,k}) - \int \left[\sum_{\pi': \underline{\delta}(\pi) \prec \underline{\delta}(\pi'), \, \chi_\pi = \chi_{\pi'}} n(\pi, \pi') \operatorname{tr} \sigma_{\pi'}(\phi_{\underline{\delta}}^{t,k}) \right] d\nu(\pi)$$

$$\ll_{\underline{\delta},m} (1 + R)^{m_{\underline{\delta}}} + (1 + R)^m \sum_{\underline{\delta}' \prec \underline{\delta}} \nu(g_{m,\underline{\delta}'})$$

for all $\nu \in \mathfrak{M}$. For suitable m, the sum $\sum_{\underline{\delta}' \prec \underline{\delta}} \nu(g_{m,\underline{\delta}'})$ is bounded independently of ν by the induction hypothesis. We conclude that (16a) holds for $\underline{\delta}$, which finishes the induction and thereby the proof of the proposition. $\qquad \square$

We remark that the proof simplifies for the groups $\mathrm{GL}(n)$, since in this case the tempered basic representations $\pi_{\delta,\lambda}$, $\operatorname{Re} \lambda = 0$, are always irreducible, the R-groups are trivial, and the sets $\mathcal{D}'(\underline{\delta})$ are therefore singletons. The Paley–Wiener space $\mathcal{PW}_{r,\delta}$ is then just the space of W_δ-invariant functions in $\mathcal{PW}(\mathfrak{a}_M)_r$, where (M, δ) is a representative of $\underline{\delta}$ and W_δ denotes the stabilizer of δ inside the Weyl group $W(A_M)$.

7. The spectral limit property

We now come back to the situation of §§ 1–5. Before treating the spectral limit property (2), we consider first the question whether the collection of measures $\{\mu_K^{G,S_\infty}\}_{K \in \mathcal{K}}$ on $G(F_\infty)^1$ associated to a set $\mathcal{K}$ of open subgroups K of $\mathbf{K}_{\text{fin}}$ is polynomially bounded. We conjecture that this is true for the set of all open subgroups of $\mathbf{K}_{\text{fin}}$ (or even for the set of all open compact subgroups of $G(\mathbb{A}_{\text{fin}})$). Note that each finite set $\mathcal{K}$ is known to have this property [55]. So, as in the case of property (TWN) above, the issue is to control the dependence on K.

Remark 7.1. Deitmar and Hoffmann [28] have shown unconditionally that for any G the collection of measures $\{\mu_{\mathbf{K}(\mathfrak{n}),\text{cusp}}^{G,S_\infty}\}$, where $\mu_{\mathbf{K}(\mathfrak{n}),\text{cusp}}^{G,S_\infty}$ is the analog of $\mu_{\mathbf{K}(\mathfrak{n})}^{G,S_\infty}$ for the *cuspidal* spectrum, is polynomially bounded. (In fact, they obtain a more precise statement.) However, for our argument we need to know the corresponding statement for the full discrete spectrum.

Our results in this direction are Lemmas 7.6 and 7.7 below, which we will use for a conditional proof of the spectral limit property for principal congruence subgroups in Corollary 7.8, thereby finishing our argument. Recall the spectral expansion of Theorem 4.1, which expresses Arthur's distribution $J(h)$ as a sum of contributions $J_{\text{spec},M}(h)$ associated to the conjugacy classes of Levi subgroups M of G. Also recall properties (TWN) and (BD) from § 5. They are hereditary for Levi subgroups.

Fix $M \in \mathcal{L}$, $M \neq G$. The technical heart of our argument is contained in the following lemma and its corollary. We freely use the notation introduced in § 4. We denote by $\|\cdot\|_{1,\mathfrak{H}}$ the trace norm of an operator on a Hilbert space $\mathfrak{H}$. Extending the notation of § 6, for a finite set $\mathcal{F} \subset \Pi(\mathbf{K}_\infty)$ and an open subgroup K_S of $\mathbf{K}_{S-S_\infty}$ let $\mathcal{H}(G(F_S)^1)_{\mathcal{F},K_S}$ be the space of all bi-K_S-invariant functions $f \in \mathcal{H}(G(F_S)^1)$ whose translates $f(k_1 \cdot k_2)$, $k_1, k_2 \in \mathbf{K}_\infty$, span a space that decomposes under the action of $\mathbf{K}_\infty \times \mathbf{K}_\infty$ as a sum of representations $\tau_1 \otimes \tau_2$ with $\tau_1, \tau_2 \in \mathcal{F}$. Recall the norms $\|\cdot\|_k$ on $C_c^\infty(G(F_S)^1)$ introduced in § 3.2.

Lemma 7.2. *Suppose that G satisfies properties (TWN) and (BD). Let $M \in \mathcal{L}$, and let $P \in \mathcal{P}(M)$. Furthermore, let $S \supset S_\infty$ be a finite set of places of F. Then, for any finite set $\mathcal{F} \subset \Pi(\mathbf{K}_\infty)$ and any sufficiently large $N > 0$, there exists an integer $k \geqslant 0$ such that, for any*

- *open subgroup K_S of $\mathbf{K}_{S-S_\infty}$,*
- *open compact subgroup K of $G(\mathbb{A}^S)$,*
- *$\epsilon > 0$,*
- *$s \in N_G(M)/M$,*
- *$\underline{\beta} \in \mathfrak{B}_{P,L_s}$,*
- *$\mathcal{X} \in \Xi_{L_s}(\underline{\beta})$,*

we have

$$\int_{i(\mathfrak{a}_{L_s}^G)^*} \|\Delta_\mathcal{X}(P, \lambda) M(P, s) \rho(P, \lambda, h \otimes \mathbf{1}_K)\|_{1, \bar{\mathcal{A}}^2(P)} \ d\lambda \ll_{\mathcal{F}, N, \epsilon}$$

$$\text{vol}(K) \, \text{level}(K_S K; G_M^+)^\epsilon \, \|h\|_k \sum_{\substack{\tau \in \mathcal{F}, \\ \pi \in \Pi_{\text{disc}}(M(\mathbb{A}))}} \Lambda_M(\pi_\infty)^{-N} \dim \mathcal{A}_\pi^2(P)^{K_S K, \tau} \quad (19)$$

for all $h \in \mathcal{H}(G(F_S)^1)_{\mathcal{F}, K_S}$. *Consequently,*

$$J_{\text{spec}, M}(h \otimes \mathbf{1}_K)$$

$$\ll_{\mathcal{F}, N, \epsilon} \text{vol}(K) \, \text{level}(K_S K; G_M^+)^\epsilon \, \|h\|_k \sum_{\substack{\tau \in \mathcal{F}, \\ \pi \in \Pi_{\text{disc}}(M(\mathbb{A}))}} \Lambda_M(\pi_\infty)^{-N} \dim \mathcal{A}_\pi^2(P)^{K_S K, \tau}. \quad (20)$$

Note here that for all $N \geqslant N_0$, where N_0 depends only on G, the right-hand sides of (19) and (20) are finite [55].

Proof. We argue as in [34, § 5] (cf. also [57]). First, note that we may omit $M(P, s)$ on the left-hand side of (19), since it is a unitary operator which commutes with $\rho(P, \lambda, h \otimes \mathbf{1}_K)$, and hence does not affect the trace norm. Let Δ be the operator $\text{Id} - \Omega + 2\Omega_{\mathbf{K}_\infty}$, where Ω (respectively, $\Omega_{\mathbf{K}_\infty}$) is the Casimir operator of $G(F_\infty)$ (respectively, $\mathbf{K}_\infty$). For any $k > 0$, we bound the left-hand side of (19) by

$$\int_{i(\mathfrak{a}_{L_s}^G)^*} \left\| \Delta_\mathcal{X}(P, \lambda) \rho(P, \lambda, \Delta)^{-2k} \right\|_{1, \bar{\mathcal{A}}^2(P)^{K_S K, \mathcal{F}}} \left\| \rho(P, \lambda, \Delta^{2k} \star h \otimes \mathbf{1}_K) \right\| \ d\lambda$$

$$\leqslant \text{vol}(K) \left\| \Delta^{2k} \star h \right\|_{L^1(G(F_S)^1)} \int_{i(\mathfrak{a}_{L_s}^G)^*} \left\| \Delta_\mathcal{X}(P, \lambda) \rho(P, \lambda, \Delta)^{-2k} \right\|_{1, \bar{\mathcal{A}}^2(P)^{K_S K, \mathcal{F}}} \ d\lambda.$$

Consider the integral on the right-hand side.[11] For any $\pi \in \Pi_{\text{disc}}(M(\mathbb{A}))$ and $\tau \in \Pi(\mathbf{K}_\infty)$, the operator $\rho(P, \lambda, \Delta)$ acts by the scalar $\mu(\pi, \lambda, \tau) = 1 + \|\lambda\|^2 - \lambda_\pi + 2\lambda_\tau - e_P$ on $\mathcal{A}_\pi^2(P)^\tau$, where λ_π and λ_τ are the Casimir eigenvalues of π and τ, respectively, and e_P is a constant depending only on P (cf. [14, § 3.2, (2)]). Since it is easy to see that $e_P \leqslant 0$, using (10), we get

$$\mu(\pi, \lambda, \tau)^2 \geqslant \frac{1}{4}(1 + \|\lambda\|^2 + \lambda_\pi^2 + \lambda_\tau^2) \gg_G \|\lambda\|^2 + \Lambda_M(\pi_\infty).$$

Therefore,

$$\int_{i(\mathfrak{a}_{L_s}^G)^*} \left\| \Delta_\mathcal{X}(P, \lambda) \rho(P, \lambda, \Delta)^{-2k} \right\|_{1, \bar{\mathcal{A}}^2(P)^{K_S K, \mathcal{F}}} \ d\lambda \leqslant$$

$$\sum_{\tau \in \mathcal{F}} \sum_{\pi \in \Pi_{\text{disc}}(M(\mathbb{A}))} \int_{i(\mathfrak{a}_{L_s}^G)^*} \|\Delta_\mathcal{X}(P, \lambda)\|_{1, \mathcal{A}_\pi^2(P)^{K_S K, \tau}} \, \mu(\pi, \lambda, \tau)^{-2k} \ d\lambda.$$

Estimating $\|A\|_1 \leqslant \dim V \|A\|$ for any linear operator A on a finite-dimensional Hilbert space V, we bound the previous expression by

$$\sum_{\tau \in \mathcal{F}} \sum_{\pi \in \Pi_{\text{disc}}(M(\mathbb{A}))} \dim \mathcal{A}_\pi^2(P)^{K_S K, \tau} \int_{i(\mathfrak{a}_{L_s}^G)^*} \|\Delta_\mathcal{X}(P, \lambda)\|_{\mathcal{A}_\pi^2(P)^{K_S K, \tau}} \, \mu(\pi, \lambda, \tau)^{-2k} \ d\lambda.$$

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[11]In the corresponding formula [34, (5.1)] the restriction to the K_0-fixed part was mistakenly omitted.

Using the definition of $\Delta_\chi(P, \lambda)$, we can bound the above by a constant multiple of

$$\sum_{\substack{\tau \in \mathcal{F}, \\ \pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A}))}} \Lambda_M(\pi_\infty)^{-k/2} \dim \mathcal{A}^2_\pi(P)^{K_S K, \tau} \int_{\mathrm{i}(\mathfrak{a}^G_{L_s})^*} (1 + \|\lambda\|)^{-k} \prod_{i=1}^m \left\| \delta_{P_i | P'_i}(\lambda) \big|_{\mathcal{A}^2_\pi(P'_i)^{K_S K, \tau}} \right\| d\lambda. \tag{21}$$

We estimate the integral over $\mathrm{i}(\mathfrak{a}^G_{L_s})^*$. Let $\mathcal{F}_M \subset \Pi(\mathbf{K}_{M,\infty})$ be the finite set of all irreducible components of restrictions of elements of $\mathcal{F}$ to $\mathbf{K}_{M,\infty}$. Then, by Frobenius reciprocity, only those $\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A}))$ with $\pi_\infty \in \Pi(M(F_\infty))^{\mathcal{F}_M}$ can contribute to (21).

Let $\underline{\beta} = (\beta^\vee_1, \dots, \beta^\vee_m)$, and introduce the new coordinates $s_i = \langle \lambda, \beta^\vee_i \rangle$, $i = 1, \dots, m$, on $(\mathfrak{a}^G_{L_s, \mathbb{C}})^*$. By (9), we can write

$$\delta_{P_i | P'_i}(\lambda) = \frac{n'_{\beta_i}(\pi, s_i)}{n_{\beta_i}(\pi, s_i)} \mathrm{Id} + j_{P'_i} \circ (\mathrm{Id} \otimes R(\pi, s_i)^{-1} R'(\pi, s_i)) \circ j^{-1}_{P'_i}.$$

Property (TWN) and Proposition 5.16 (which is based on property (BD)), together with Lemma 5.12, yield the estimate

$$\int_{\mathrm{i}(\mathfrak{a}^G_{L_s})^*} (1 + \|\lambda\|)^{-k} \prod_{i=1}^m \left\| \delta_{P_i | P'_i}(\lambda) \big|_{\mathcal{A}^2_\pi(P'_i)^{K_S K, \tau}} \right\| d\lambda \ll_{\epsilon, \mathcal{F}} \Lambda_M(\pi_\infty; G_M)^N \mathrm{level}(K_S K; G^+_M)^\epsilon \tag{22}$$

for any $\epsilon > 0$ and sufficiently large N and k (depending possibly on τ). Altogether, we obtain (19), and, using Theorem 4.1, also (20). □

Remark 7.3. Note that the improved estimate (12) yields the following improvement of (22):

$$\int_{\mathrm{i}(\mathfrak{a}^G_{L_s})^*} (1 + \|\lambda\|)^{-m-\epsilon} \prod_{i=1}^m \left\| \delta_{P_i | P'_i}(\lambda) \big|_{\mathcal{A}^2_\pi(P'_i)^{K_S K, \tau}} \right\| d\lambda$$

$$\ll_\epsilon \log(\Lambda_M(\pi_\infty; G_M) + \|\tau\| + \mathrm{level}(K_S K; G^+_M))^m, \tag{23}$$

where the implied constant does not depend on τ.

Applying Lemma 7.2 to the principal congruence subgroups $\mathbf{K}^S(\mathfrak{n})$, and assuming polynomial boundedness of the collection $\{\mu^{M, S_\infty}_{\mathbf{K}_M(\mathfrak{n})}\}$, we obtain the following result.

Corollary 7.4. *Suppose that G satisfies properties (TWN) and (BD). Furthermore, let $M \in \mathcal{L}$, $M \neq G$, and assume that the set of measures $\{\mu^{M, S_\infty}_{\mathbf{K}_M(\mathfrak{n})}\}$ is polynomially bounded. Let $S \supset S_\infty$ be a finite set of places of F. Then, for any finite set $\mathcal{F} \subset \Pi(\mathbf{K}_\infty)$, there exists an integer $k \geqslant 1$ such that, for any open subgroup $K_S \subset \mathbf{K}_{S-S_\infty}$ and any $\epsilon > 0$, we have*

$$J_{\mathrm{spec}, M}(h \otimes \mathbf{1}_{\mathbf{K}^S(\mathfrak{n})}) \ll_{K_S, \mathcal{F}, \epsilon} \|h\|_k \, \mathrm{N}(\mathfrak{n})^{(\dim M - \dim G)/2 + \epsilon} \tag{24}$$

for all $h \in \mathcal{H}(G(F_S)^1)_{\mathcal{F}, K_S}$ and all integral ideals $\mathfrak{n}$ of $\mathfrak{o}_F$ prime to S.

Proof. Fix $P = M \ltimes U \in \mathcal{P}(M)$ and an ideal $\mathfrak{n}_0$ such that $K_S \supset \mathbf{K}_{S-S_\infty}(\mathfrak{n}_0)$. We have

$$\dim \mathcal{A}^2_\pi(P)^{\mathbf{K}(\mathfrak{n}_0\mathfrak{n}),\tau} = m_\pi \dim \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\pi)^{\mathbf{K}(\mathfrak{n}_0\mathfrak{n}),\tau}$$
$$= m_\pi \dim \mathrm{Ind}_{P(F_\infty)}^{G(F_\infty)}(\pi_\infty)^\tau \dim \mathrm{Ind}_{P(\mathbb{A}_{\mathrm{fin}})}^{G(\mathbb{A}_{\mathrm{fin}})}(\pi_{\mathrm{fin}})^{\mathbf{K}(\mathfrak{n}_0\mathfrak{n})},$$

where

$$m_\pi = \dim \mathrm{Hom}(\pi, L^2_{\mathrm{disc}}(A_M M(F)\backslash M(\mathbb{A}))).$$

Note that here the factor $\dim \mathrm{Ind}_{P(F_\infty)}^{G(F_\infty)}(\pi_\infty)^\tau$ is bounded by $(\dim \tau)^2$. On the other hand, since $\mathbf{K}(\mathfrak{n}_0\mathfrak{n})$ is a normal subgroup of $\mathbf{K}_{\mathrm{fin}}$, we have

$$\dim \mathrm{Ind}_{P(\mathbb{A}_{\mathrm{fin}})}^{G(\mathbb{A}_{\mathrm{fin}})}(\pi_{\mathrm{fin}})^{\mathbf{K}(\mathfrak{n}_0\mathfrak{n})} \leqslant [\mathbf{K}_{\mathrm{fin}} : (\mathbf{K}_{\mathrm{fin}} \cap P(\mathbb{A}_{\mathrm{fin}}))\mathbf{K}(\mathfrak{n}_0\mathfrak{n})] \dim \pi_{\mathrm{fin}}^{\mathbf{K}_M(\mathfrak{n}_0\mathfrak{n})}.$$

Using the factorization $\mathbf{K}_{\mathrm{fin}} \cap P(\mathbb{A}_{\mathrm{fin}}) = (\mathbf{K}_{\mathrm{fin}} \cap M(\mathbb{A}_{\mathrm{fin}}))(\mathbf{K}_{\mathrm{fin}} \cap U(\mathbb{A}_{\mathrm{fin}}))$, we can write

$$[\mathbf{K}_{\mathrm{fin}} : (\mathbf{K}_{\mathrm{fin}} \cap P(\mathbb{A}_{\mathrm{fin}}))\mathbf{K}(\mathfrak{n}_0\mathfrak{n})] = \mathrm{vol}(\mathbf{K}_M(\mathfrak{n}_0\mathfrak{n}))\mathrm{vol}(\mathbf{K}(\mathfrak{n}_0\mathfrak{n}))^{-1}$$
$$[\mathbf{K}(\mathfrak{n}_0\mathfrak{n}) \cap P(\mathbb{A}_{\mathrm{fin}}) : (\mathbf{K}(\mathfrak{n}_0\mathfrak{n}) \cap M(\mathbb{A}_{\mathrm{fin}}))(\mathbf{K}(\mathfrak{n}_0\mathfrak{n}) \cap U(\mathbb{A}_{\mathrm{fin}}))]$$
$$[\mathbf{K}_{\mathrm{fin}} \cap U(\mathbb{A}_{\mathrm{fin}}) : \mathbf{K}(\mathfrak{n}_0\mathfrak{n}) \cap U(\mathbb{A}_{\mathrm{fin}})]^{-1}.$$

The index $[\mathbf{K}(\mathfrak{n}_0\mathfrak{n}) \cap P(\mathbb{A}_{\mathrm{fin}}) : (\mathbf{K}(\mathfrak{n}_0\mathfrak{n}) \cap M(\mathbb{A}_{\mathrm{fin}}))(\mathbf{K}(\mathfrak{n}_0\mathfrak{n}) \cap U(\mathbb{A}_{\mathrm{fin}}))]$ is bounded independently of $\mathfrak{n}$. Furthermore, identifying U with its Lie algebra $\mathfrak{u}$ via the exponential map, which is an isomorphism of affine varieties, one sees that

$$\mathrm{N}(\mathfrak{n}_0\mathfrak{n})^{-\dim U} \ll [\mathbf{K}_{\mathrm{fin}} \cap U(\mathbb{A}_{\mathrm{fin}}) : \mathbf{K}(\mathfrak{n}_0\mathfrak{n}) \cap U(\mathbb{A}_{\mathrm{fin}})]^{-1} \ll \mathrm{N}(\mathfrak{n}_0\mathfrak{n})^{-\dim U}.$$

(We will only need the upper bound.) Therefore

$$\dim \mathrm{Ind}_{P(\mathbb{A}_{\mathrm{fin}})}^{G(\mathbb{A}_{\mathrm{fin}})}(\pi_{\mathrm{fin}})^{\mathbf{K}(\mathfrak{n}_0\mathfrak{n})} \ll \mathrm{N}(\mathfrak{n}_0\mathfrak{n})^{-\dim U} \mathrm{vol}(\mathbf{K}(\mathfrak{n}_0\mathfrak{n}))^{-1}\mathrm{vol}(\mathbf{K}_M(\mathfrak{n}_0\mathfrak{n})) \dim \pi_{\mathrm{fin}}^{\mathbf{K}_M(\mathfrak{n}_0\mathfrak{n})}.$$

Incorporating the above into Lemma 7.2, we obtain that for sufficiently large $N > 0$ there exists $k \geqslant 0$ such that

$$J_{\mathrm{spec},M}(h \otimes \mathbf{1}_{\mathbf{K}^S(\mathfrak{n})}) \ll_{\mathfrak{n}_0,\mathcal{F},\epsilon,N} \|h\|_k$$
$$\mathrm{N}(\mathfrak{n})^{-\dim U+\epsilon}\mathrm{vol}(\mathbf{K}_M(\mathfrak{n}_0\mathfrak{n})) \sum_{\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A}))^{\mathcal{F}_M}} \left(1 + |\lambda_{\pi_\infty}|\right)^{-N} m_\pi \dim \pi_{\mathrm{fin}}^{\mathbf{K}_M(\mathfrak{n}_0\mathfrak{n})}.$$

By assumption, the set of measures $\{\mu_{\mathbf{K}_M(\mathfrak{n}_0\mathfrak{n})}^{M,S_\infty}\}$ is polynomially bounded. Therefore the fourth condition of Proposition 6.1 yields the existence of an integer N, depending only on $\mathcal{F}_M$, such that

$$\mu_{\mathbf{K}_M(\mathfrak{n}_0\mathfrak{n})}^{M,S_\infty}(g_{N,\mathcal{F}_M}) = \frac{\mathrm{vol}(\mathbf{K}_M(\mathfrak{n}_0\mathfrak{n}))}{\mathrm{vol}(M(F)\backslash M(\mathbb{A})^1)} \sum_{\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A}))^{\mathcal{F}_M}} \left(1 + |\lambda_{\pi_\infty}|\right)^{-N} m_\pi \dim \pi_{\mathrm{fin}}^{\mathbf{K}_M(\mathfrak{n}_0\mathfrak{n})}$$

is bounded independently of $\mathfrak{n}$. This proves the assertion, since $\dim U = (\dim G - \dim M)/2$. $\qquad\square$

Remark 7.5. As before, (12) implies a slightly improved version of (24) in which the expression $\mathrm{N}(\mathfrak{n})^{(\dim M-\dim G)/2+\epsilon}$ is replaced by $(1 + \log \mathrm{N}(\mathfrak{n}))^m \mathrm{N}(\mathfrak{n})^{(\dim M-\dim G)/2}$.

We are now in a position to prove that under the assumption that G satisfies (TWN) and (BD) the collection of measures $\{\mu_{\mathbf{K}_M(\mathfrak{n})}^{M,S_\infty}\}$ on $\Pi(M(F_\infty)^1)$ is polynomially bounded for any $M \in \mathcal{L}$.[12]

Lemma 7.6. *Let G be anisotropic modulo the center. Then the collection of measures $\{\mu_K^{G,S_\infty}\}$, where K ranges over the open subgroups of $\mathbf{K}_{\mathrm{fin}}$, is polynomially bounded.*

Proof. In this case, the trace formula for a test function $h \in C_c^\infty(G(F_\infty)^1)$ can be written as

$$\mathrm{vol}(G(F)\backslash G(\mathbb{A})^1)\mu_K^{G,S_\infty}(\hat{h}) = \int_{G(F)\backslash G(\mathbb{A})^1} \sum_{\gamma \in G(F)} (h \otimes \mathbf{1}_K)(g^{-1}\gamma g)dg.$$

Since g can be integrated over a compact set C which is independent of h and K, for all h supported in a fixed set $\Omega_\infty \subset G(F_\infty)^1$ we can bound the absolute value of the right-hand side by $A_{\Omega_\infty} \sup |h|$, where A_{Ω_∞} is the constant

$$A_{\Omega_\infty} = \mathrm{vol}(G(F)\backslash G(\mathbb{A})^1) \sup_{g \in C} \sum_{\gamma \in G(F)} (\mathbf{1}_{\Omega_\infty \mathbf{K}_{\mathrm{fin}}})(g^{-1}\gamma g).$$

Therefore, $\sup \left|\mu_K^{G,S_\infty}(\hat{h})\right|$ is a continuous seminorm on every space $\mathcal{H}(G(F_\infty)^1)_{r,\mathcal{F}}$, and by Proposition 6.1 we obtain the assertion. $\qquad\square$

Lemma 7.7. *Suppose that G satisfies (TWN) and (BD). Then, for each $M \in \mathcal{L}$, the collection of measures $\{\mu_{\mathbf{K}_M(\mathfrak{n})}^{M,S_\infty}\}$, $\mathfrak{n}$ ranging over the integral ideals of $\mathfrak{o}_F$, is polynomially bounded.*

Proof. We use induction on the semisimple rank of M. The base of the induction is Lemma 7.6. For the induction step, we can assume the assertion for all groups in $\mathcal{L}\backslash\{G\}$, and have the task to establish it for G itself. Fix $r > 0$, and apply the trace formula to $h \otimes \mathbf{1}_{\mathbf{K}(\mathfrak{n})}$, where $h \in \mathcal{H}(G(F_\infty)^1)_{r,\mathcal{F}}$. By Theorem 4.1, we have

$$\mathrm{vol}(G(F)\backslash G(\mathbb{A})^1)\mu_{\mathbf{K}(\mathfrak{n})}^{G,S_\infty}(\hat{h}) = J(h \otimes \mathbf{1}_{\mathbf{K}(\mathfrak{n})}) - \sum_{[M],\, M \neq G} J_{\mathrm{spec},M}(h \otimes \mathbf{1}_{\mathbf{K}(\mathfrak{n})}).$$

Now, for each single choice of $\mathfrak{n}$ the absolute value $\left|J(h \otimes \mathbf{1}_{\mathbf{K}(\mathfrak{n})})\right|$ is a continuous seminorm by the work of Arthur [3]. Moreover, as in the proof of Corollary 3.3, for all $\mathfrak{n}$ outside of a finite set depending only on r we have $J(h \otimes \mathbf{1}_{\mathbf{K}(\mathfrak{n})}) = J_{\mathrm{unip}}(h \otimes \mathbf{1}_{\mathbf{K}(\mathfrak{n})})$. Therefore it follows from our analysis of $J_{\mathrm{unip}}(h \otimes \mathbf{1}_{\mathbf{K}(\mathfrak{n})})$ in Proposition 3.1 that $\sup_{\mathfrak{n}} \left|J(h \otimes \mathbf{1}_{\mathbf{K}(\mathfrak{n})})\right|$ is a continuous seminorm on $\mathcal{H}(G(F_\infty)^1)_{r,\mathcal{F}}$. By Corollary 7.4 (with $S = S_\infty$), we obtain that the spectral terms $\sup_{\mathfrak{n}} \left|J_{\mathrm{spec},M}(h \otimes \mathbf{1}_{\mathbf{K}(\mathfrak{n})})\right|$ for $M \neq G$ are also continuous seminorms on $\mathcal{H}(G(F_\infty)^1)_{r,\mathcal{F}}$. By Proposition 6.1, we conclude that the collection $\{\mu_{\mathbf{K}(\mathfrak{n})}^{G,S_\infty}\}$ is polynomially bounded. $\qquad\square$

As before, let S be a finite set of places of the field F containing S_∞.

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[12]Variants of Lemma 7.6 have been previously established in [26, Proposition 3.3] and [28].

Corollary 7.8 (Spectral limit property)**.** *Suppose that G satisfies (TWN) and (BD). Then we have the spectral limit property for the set of subgroups $\mathbf{K}^S(\mathfrak{n})$, where $\mathfrak{n}$ ranges over the integral ideals of $\mathfrak{o}_F$ prime to S.*

Proof. From Lemma 7.7, we get that, for each $M \in \mathcal{L}$, the collection of measures $\{\mu^{M,S_\infty}_{\mathbf{K}_M(\mathfrak{n})}\}$ is polynomially bounded. Therefore we can apply Corollary 7.4 to conclude that for each $h \in \mathcal{H}(G(F_S)^1)$ we have

$$J_{\mathrm{spec},M}(h \otimes \mathbf{1}_{\mathbf{K}^S(\mathfrak{n})}) \to 0$$

for all $M \neq G$. Hence (by Theorem 4.1),

$$J(h \otimes \mathbf{1}_{\mathbf{K}^S(\mathfrak{n})}) - \mathrm{tr}\, R_{\mathrm{disc}}(h \otimes \mathbf{1}_{\mathbf{K}^S(\mathfrak{n})}) \to 0,$$

which is the spectral limit property (2). □

Theorem 7.9. *Suppose that G satisfies (TWN) and (BD). Then limit multiplicity holds for the set of subgroups $\mathbf{K}^S(\mathfrak{n})$, where $\mathfrak{n}$ ranges over the integral ideals of $\mathfrak{o}_F$ prime to S.*

Proof. The geometric limit property (3) has been established in Corollary 3.3, and the spectral limit property (2) in Corollary 7.8. By Theorem 1.2, we obtain the result. □

This also finishes the proof of Theorem 1.3, since (TWN) (respectively, (BD)) has been verified for the groups $\mathrm{GL}(n)$ and $\mathrm{SL}(n)$ in Proposition 5.5 (respectively, Theorem 5.15).

Repeating the argument above, and combining Corollary 7.4 with the improved geometric estimate of Proposition 3.8, we obtain the following quantitative statement. Recall the definition of $d_{\min}$ in (7).

Theorem 7.10. *Suppose that G satisfies (TWN) and (BD). Then, for any $r > 0$ and any finite set $\mathcal{F} \subset \Pi(\mathbf{K}_\infty)$, there exists an integer $k \geqslant 0$ such that for any open subgroup $K_S \subset \mathbf{K}_{S-S_\infty}$ and $\epsilon > 0$ we have*

$$\left| \mu^{G,S}_{\mathbf{K}^S(\mathfrak{n})}(\hat{h}) - h(1) \right| \ll_{r,K_S,\mathcal{F},\epsilon} \mathrm{N}(\mathfrak{n})^{-d_{\min}+\epsilon} \|h\|_k \qquad (25)$$

for all $h \in \mathcal{H}(G(F_S)^1)_{r,\mathcal{F},K_S}$ and all integral ideals $\mathfrak{n}$ of $\mathfrak{o}_F$ prime to S.

Note here that $d_{\min} \leqslant (\dim G - \dim M)/2$ for any proper $M \in \mathcal{L}$, since $\dim G - \dim M$ is the dimension of the Richardson orbit associated to a parabolic subgroup $P \in \mathcal{P}(M)$. If we also assume (12), then we can further improve the right-hand side of (25) to $\frac{(1+\log \mathrm{N}(\mathfrak{n}))^{d_0}}{\mathrm{N}(\mathfrak{n})^{d_{\min}}} \|h\|_k$ (see Remark 7.5).

Remark 7.11. A natural problem is to deduce from Theorem 7.10 an estimate for the difference $\left| \mu^{G,S}_{\mathbf{K}^S(\mathfrak{n})}(A) - \mu_{\mathrm{pl}}(A) \right|$ for suitable subsets $A \subset \Pi(G)$. This would require a quantitative version of the density principle (Theorem 2.1). We will not discuss this aspect here.

Acknowledgements

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