ON THE SPECTRAL SIDE OF ARTHUR'S TRACE FORMULA II

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Abstract. We derive a refinement of the spectral expansion of Arthur's trace formula. The expression is absolutely convergent with respect to the trace norm.

Contents

1. Introduction

The trace formula is an important tool in studying automorphic forms on arithmetic quotients. It was introduced by Selberg for the case of quotients of the upper half-plane in [Sel56] and subsequently extensively developed by Arthur in his large scale work on the subject. (See [Art05] for a recent survey on the theory.) In essence, the trace formula is an equality between a sum of geometric distributions, such as (possibly weighted) orbital integrals, and a sum of spectral distributions such as traces of representations. In order to apply the trace formula, it is important to have an explicit description of the distributions appearing in it. In [Art82b] Arthur derived an expression for the spectral side of the non-invariant trace formula in terms of certain limits of intertwining operators. In this paper we explicate these terms further and write them as a linear combination of products of first-order derivatives of co-rank one intertwining operators. The exact formula is described in Theorem 1 below. It is used to explicate the spectral expansion of the trace formula in Corollary 2. A key feature of this expansion is its absolute convergence with respect to the trace norm. This relies on earlier work by the third named author and generalizes earlier results in this direction ([Lan90, Mül89, Mül98, Mül00, Mül02, MS04]). Remarkably, Arthur was able

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to finesse this difficulty in his work. This is partly because his emphasis is on comparing trace formulas on different groups. However, for other applications of the trace formula the absolute convergence may be indispensable. An example is the work of the second and third named authors on Weyl's law with remainder for the groups $\mathrm{GL}(n)$ [LM]. (Note that in this case the absolute convergence was already obtained in [MS04] by a different argument, which is special to $\mathrm{GL}(n)$.)

In the scalar case, our formula reduces to a result of Arthur [Art82b, §7]. However, the operator case is more involved and it is not clear how to see directly that Arthur's expression equals ours. Instead we show that the two expressions satisfy identical structural properties. The main difficulty is to show that these properties are sufficiently strong to guarantee uniqueness.

Given a root system of rank n there is an n-dimensional zonotope $\mathcal{Z}$ which is dual to the corresponding hyperplane arrangement. For example, for a root system of type A_n, $\mathcal{Z}$ is the associated permutahedron. The combinatorics of $\mathcal{Z}$ plays a ubiquitous role in the trace formula. The new ingredient here is the hyperplane arrangement of rank $n-1$ which is dual to the monotone path zonotope of $\mathcal{Z}$ in the sense of Billera-Sturmfels ([BS92, BS94]).[1] The latter depends on an auxiliary parameter λ_0 which is also apparent in our formula. In connection with this zonotope we introduce in §4 below a certain graded algebra $\mathcal{T}$ over the polynomial ring $\mathfrak{s}$ with one indeterminate for each positive root. The crucial fact allowing the induction step to go through is that $\mathcal{T}$ is generated as a $\mathfrak{s}$-module by its homogeneous elements of degree $< n$. This property is obtained in the companion paper [FL] as a result of a closer study of the algebra $\mathcal{T}$.

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2. The main result

Let G be a reductive group over a field F. All algebraic subgroups of G considered in the following will be tacitly assumed to be defined over F. Throughout we fix a (not necessarily minimal) parabolic subgroup P_0 of F-co-rank $n = n(P_0)$ in G and a Levi decomposition $P_0 = MU_0$.

We will mostly follow Arthur's notation (cf. [Art82b]) up to some minor differences. In particular, $\mathfrak{a}_M^*$ denotes the n-dimensional vector space over $\mathbb{R}$ spanned by the F-rational characters of M trivial on the center Z_G of G, $\mathfrak{a}_M$ is the dual space spanned by the F-rational co-characters of $Z_M \cap G^{\mathrm{der}}$ and $\mathcal{P}(M)$ denotes the (finite) set of parabolic subgroups of G (defined over F) which contain

[1] Related constructions for the root system of type A_n were considered by Manin-Schechtman [MS89], Lawrence-Naimark [Law97], Bayer-Brandt [BB97] and Felsner-Ziegler [FZ01].

M as their Levi part. The opposite parabolic of $P \in \mathcal{P}(M)$ (with respect to M) will be denoted by $\overline{P} \in \mathcal{P}(M)$. The simple roots (resp. co-roots) of $P \in \mathcal{P}(M)$ are denoted by $\Delta_P \subset \mathfrak{a}_M^*$ (resp. $\Delta_P^\vee \subset \mathfrak{a}_M$), and the reduced positive roots by Σ_P. We have $|\Delta_P| = |\Delta_P^\vee| = n$ and the lattice generated by Δ_P (resp. $\Delta_P^\vee$) is independent of $P \in \mathcal{P}(M)$ and will be called the root (resp. co-root) lattice of M. Similarly, the cardinality N of Σ_P does not depend on P. For brevity we write $\Delta_0 = \Delta_{P_0}$ and $\Sigma_0 = \Sigma_{P_0}$. We denote by $\mathfrak{a}_{P,+}^*$ the Weyl chamber of $\mathfrak{a}_M^*$ corresponding to $P \in \mathcal{P}(M)$. Also set $\Sigma_{P;Q} = \Sigma_P \cap \Sigma_{\overline{Q}}$ for any $P, Q \in \mathcal{P}(M)$.

We say that the subgroups $P, Q \in \mathcal{P}(M)$ are *adjacent* along $\alpha \in \Delta_P$, denoted $P|^\alpha Q$, if $\Sigma_{P;Q} = \{\alpha\}$. To each $\alpha \in \Delta_P$ there exists a unique $Q \in \mathcal{P}(M)$ such that $P|^\alpha Q$, and we have $\Sigma_Q = \Sigma_P \cup \{-\alpha\} \setminus \{\alpha\}$.

The main object we consider is the following.

Definition 1. A (G, M)-*intertwining family* consists of the data

$$\mathcal{F} = ((\mathcal{F}_P)_{P\in\mathcal{P}(M)}, (\mathcal{F}_{Q|P}(\lambda))_{P,Q\in\mathcal{P}(M)}),$$

where $\mathcal{F}_P$ is a finite dimensional vector space for any $P \in \mathcal{P}(M)$ and for each pair $P, Q \in \mathcal{P}(M)$ of parabolics $\mathcal{F}_{Q|P}(\lambda) : \mathcal{F}_P \to \mathcal{F}_Q$ is an operator valued function depending meromorphically on $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^*$ and satisfying the following properties.

(1) $\mathcal{F}_{P|P} \equiv \mathrm{Id}$ for all $P \in \mathcal{P}(M)$.
(2) For any $P_1, P_2, P_3 \in \mathcal{P}(M)$ we have $\mathcal{F}_{P_3|P_1} \equiv \mathcal{F}_{P_3|P_2} \circ \mathcal{F}_{P_2|P_1}$.
(3) If $P|^\alpha Q$ then $\mathcal{F}_{Q|P}(\lambda)$ depends only on $\langle \lambda, \alpha^\vee \rangle$.

The key example of an intertwining family is given by intertwining operators (in the global case) and normalized intertwining operators (in the local case).

Note that if $\mathcal{F}$ is an intertwining family and $\mu \in \mathfrak{a}_{M,\mathbb{C}}^*$ then the translation $\mathcal{F}_{P|Q}(\cdot + \mu)$ is also an intertwining family. We say that an intertwining family is regular at $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^*$ if $\mathcal{F}_{P|Q}$ is holomorphic near λ for all $P, Q \in \mathcal{P}(M)$.

Suppose that $\mathcal{F}$ is regular at λ and consider the functions

$$c_P(\mathcal{F}; P_0)(\Lambda) = c_P(\Lambda) = \mathcal{F}_{P|P_0}(\lambda)^{-1}\mathcal{F}_{P|P_0}(\lambda + \Lambda) : \mathcal{F}_{P_0} \to \mathcal{F}_{P_0}, \quad P \in \mathcal{P}(M),$$

with values in $\mathrm{End}(\mathcal{F}_{P_0})$. They are holomorphic near 0 and for any adjacent parabolics $P|^\alpha P'$ the restrictions of c_P and $c_{P'}$ to the hyperplane $\langle \Lambda, \alpha^\vee \rangle = 0$ coincide as meromorphic functions. Technically this does not mean that c_P is a (G, M)-family in the sense of [Art81, §6], since $\mathcal{F}$ is not assumed to be regular on $\lambda + \mathrm{i}\mathfrak{a}_M^*$. However, the proof of [Art81, Lemma 6.2] shows that the limit

$$c_M(\mathcal{F}; P_0)(\lambda) = \lim_{\Lambda\to 0} \sum_{P\in\mathcal{P}(M)} \frac{c_P(\Lambda)}{\theta_P(\Lambda)} \in \mathrm{End}(\mathcal{F}_{P_0}) \tag{1}$$

exists, where

$$\theta_P(\Lambda) = \prod_{\alpha\in\Delta_P} \langle \Lambda, \alpha^\vee \rangle .$$

We will establish a formula for $c_M(\mathcal{F})$ in terms of one-dimensional logarithmic derivatives. In order to state the formula in this case, we first consider the most general form of logarithmic derivative relevant here. Since we are considering a non-commutative situation, we have to take the order in which the operators are composed into account. This is dealt with by the following concept.

Definition 2. A *gallery* $\mathcal{G}$ is a sequence $P_0|^{\alpha_1}P_1|\dots|^{\alpha_N}P_N = \overline{P_0}$ of adjacent parabolics. The sequence $\alpha_1,\dots,\alpha_N$ is a (linear) ordering of Σ_0 which completely determines the gallery, and will be simply called the ordering of the gallery.

For any gallery $\mathcal{G} : P_0|^{\alpha_1}P_1|\dots|^{\alpha_N}P_N = \overline{P_0}$ and a multiplicity function $m : \Sigma_0^\vee \to \mathbb{N}$ of degree $\sum_{\alpha\in\Sigma_0} m(\alpha^\vee)$ we write

$$\partial_{\mathcal{G}}^m(\mathcal{F})(\lambda) = \mathcal{F}_{\overline{P_0}|P_0}(\lambda)^{-1}\mathcal{F}_{P_N|P_{N-1}}^{(m(\alpha_N^\vee))}(\lambda)\dots\mathcal{F}_{P_1|P_0}^{(m(\alpha_1^\vee))}(\lambda) : \mathcal{F}_{P_0} \to \mathcal{F}_{P_0},$$

where $\mathcal{F}_{P_i|P_{i-1}}^{(l)}$ denotes the l-th derivative of $\mathcal{F}_{P_i|P_{i-1}}$, the latter viewed as a function in the variable $\langle\cdot,\alpha_i^\vee\rangle$. We note that there are many linear relations (valid for all intertwining families) among the expressions $\partial_{\mathcal{G}}^m(\mathcal{F})$. This matter will be analyzed in more detail in §4 below.

Denote by $\mathfrak{B} = \mathfrak{B}_{P_0}$ the set of ordered bases of $\mathfrak{a}_M$ consisting of elements of $\Sigma_0^\vee$. For $\underline{\beta} \in \mathfrak{B}_{P_0}$ let $\operatorname{vol}(\underline{\beta})$ be the index $[\mathbb{Z}(\Delta_P^\vee) : \mathbb{Z}(\underline{\beta})]$ of the lattice $\mathbb{Z}(\underline{\beta})$ spanned by $\underline{\beta}$ in the co-root lattice $\mathbb{Z}(\Delta_P^\vee)$ of $\mathfrak{a}_M$ and let $\mathbf{1}_{\underline{\beta}} : \Sigma_0^\vee \to \mathbb{N}$ denote the characteristic function of $\underline{\beta}$. Fix $\lambda_0 \in \mathfrak{a}_{P_0,+}^*$ and let $\underline{\beta} \in \mathfrak{B}$. We will see below that there exists a gallery $\mathcal{G}_{\underline{\beta}}^{\mathrm{lex}}$ whose ordering of Σ_0 is induced by the lexicographic order of the coordinate vectors with respect to $\underline{\beta}$ of the normalized co-roots $\frac{\alpha^\vee}{\langle\lambda_0,\alpha^\vee\rangle}$, $\alpha \in \Sigma_0$. In particular, this ordering is compatible with that of $\underline{\beta}$. Our main result is

Theorem 1. *Suppose that $\lambda_0 \in \mathfrak{a}_{P_0,+}^*$ is strongly regular (see Definition 4 below). For any intertwining family $\mathcal{F}$ we have an equality of meromorphic functions*

$$c_M(\mathcal{F})(\lambda) = \frac{(-1)^n}{n!}\sum_{\underline{\beta}\in\mathfrak{B}_{P_0}} \operatorname{vol}(\underline{\beta})\partial_{\mathcal{G}_{\underline{\beta}}^{\mathrm{lex}}}^{\mathbf{1}_{\underline{\beta}}}(\mathcal{F})(\lambda). \tag{2}$$

The proof will occupy §§3-5 below. It is indirect and uses induction on n. By translation, it is enough to prove (2) at $\lambda = 0$ under the assumption that $\mathcal{F}$ is regular at 0. Henceforth, we will always assume that $\mathcal{F}$ is regular at 0 and simplify the notation by writing $c_M(\mathcal{F}) = c_M(\mathcal{F})(0)$ and similarly for $\partial_{\mathcal{G}}^m(\mathcal{F})$.

Remark 1. The case $n = 1$ is straightforward. In this case $\mathcal{P}(M) = \{P_0, \overline{P_0}\}$ and (2) reduces to the identity

$$\lim_{\lambda\to 0}\left(\frac{1}{\lambda} - \frac{\mathcal{F}_{\overline{P_0}|P_0}(0)^{-1}\mathcal{F}_{\overline{P_0}|P_0}(\lambda)}{\lambda}\right) = -\mathcal{F}_{\overline{P_0}|P_0}(0)^{-1}\mathcal{F}'_{\overline{P_0}|P_0}(0).$$

The case $n = 2$ is already non-evident. In this case there are exactly two galleries (cf. Lemma 2 below), and for both of them the coefficient of $\partial_{\mathcal{G}}^{\mathbf{1}_{\underline{\beta}}}(\mathcal{F})$ on the right-hand side of (2) is $\frac{1}{2}\operatorname{vol}(\underline{\beta})$, regardless of λ_0.

Remark 2. Consider the case where

(1) $\mathcal{F}_P$ does not depend on P.

(2) The operators $\mathcal{F}_{Q|P}(\lambda)$ act as scalars.
(3) There exist meromorphic functions $\phi_\alpha : \mathbb{C} \to \mathbb{C}$, one for each $\alpha \in \Sigma_0$, such that $\mathcal{F}_{P'|P}(\lambda) = \phi_\alpha(\langle\lambda, \alpha^\vee\rangle)$ for all $P|^\alpha P'$.

Then $\partial^m_{\mathcal{G}}(\mathcal{F})$ does not depend on $\mathcal{G}$ and (2) reduces to [Art82b, Lemma 7.1] applied to

$$c_\alpha(t) = \begin{cases} 1, & \alpha \in \Sigma_0, \\ \phi_{-\alpha}(0)^{-1}\phi_{-\alpha}(-t), & \text{otherwise.} \end{cases}$$

Unfortunately, the method in [ibid., §7] does not apply in the general case.

Remark 3. In general, it is not a priori clear that the right-hand side of (2) is independent of the choice of λ_0. The strong regularity condition on λ_0 can be lifted once this independence on λ_0 is established. However, we will not consider this matter here since it does not really limit the applicability of the Theorem.

Remark 4. The concept of an intertwining family and both sides of (2) make sense in the more general context of simplicial hyperplane arrangements. Our proof establishes the identity (2) also for non-crystallographic root systems. In addition, it can easily be adapted to the case of simplicial arrangements of rank at most four.

Finally, as in [Art82b, §7] it is useful to have a slightly more general formulation of Theorem 1. Let $\mathcal{L}(M)$ denote the set of all Levi subgroups of G containing M. For $L \in \mathcal{L}(M)$ let $P_0^L = P_0 \cap L$ (a parabolic subgroup of L). We write Δ_0^L for $\Delta_{P_0^L}$ and view it as a subset of Σ_0. Suppose that $L \in \mathcal{L}(M)$ and $Q \in \mathcal{P}(L)$. The restrictions of c_P to $\mathfrak{a}_L^*$ coincide for all $P \subset Q$. If we denote their common value by c_Q, then

$$c_L = \lim_{\Lambda \to 0} \sum_{Q \in \mathcal{P}(L)} \frac{c_Q(\Lambda)}{\theta_Q(\Lambda)}$$

is defined.

Corollary 1. *Let $L \in \mathcal{L}(M)$ be of co-rank m in G. Then for strongly regular $\lambda_0 \in \mathfrak{a}^*_{P_0,+}$ we have*

$$c_L(\mathcal{F}; P_0) = \frac{(-1)^m}{n!} \sum_{\underline{\beta} \in \mathfrak{B}_{P_0} : \underline{\beta} \supset (\Delta_0^L)^\vee} \operatorname{vol}(\underline{\beta}) \partial^{\mathbf{1}_{\underline{\beta} \setminus (\Delta_0^L)^\vee}}_{\mathcal{G}^{\text{lex}}_{\underline{\beta}}}(\mathcal{F}). \tag{3}$$

3. Galleries in root hyperplane arrangements

For the proof of Theorem 1 we need to consider the concept of a gallery in more detail. Before doing so, we first recall some standard facts and notation about hyperplane arrangements in general and root arrangements in particular. For background about hyperplane arrangements and their duality with zonotopes we refer to [OT92] and [Zie95, Ch. 7], respectively.

Let V be a finite-dimensional real vector space and V^* its dual space. A finite set of non-zero vectors $S = \{v_1, \dots, v_m\} \subset V$ such that no two of them are linearly dependent defines a hyperplane arrangement A in the dual space

V^* consisting of the hyperplanes $H_i = \{w \in V^* : \langle w, v_i \rangle = 0\}$. The rank of A is by definition the dimension of the linear span of S. There are two natural combinatorial objects associated to A. First, there is the intersection lattice of A, namely the set of all intersections $\cap_{i \in S} H_i \subset V^*$, $S \subset \{1, \dots, n\}$, with respect to opposite inclusion. Second, there is a partition of V^* into a cone decomposition given by the connected components of $X \setminus \cup_{i: X \not\subset H_i} (X \cap H_i)$ (which are open polyhedral cones in X) where X ranges over the intersection lattice. The set of all cones forms a lattice with $C_1 \le C_2$ if and only if $\overline{C_1} \supset C_2$. There is a natural lattice map from the lattice of cones to the intersection lattice, which associates to each cone C the vector space spanned by it. Dually, we consider the zonotope (i.e. Minkowski sum of line segments) $Z = \sum_{i=1}^m [-1, 1] v_i \subset V$, a convex polytope in the space V' spanned by the vectors $v_1, \dots, v_m$. There is a simple, but useful, duality between the cone decomposition of A and the face lattice of Z. Under this duality a cone C of A is mapped to the face

$$F = \{v \in Z : \langle c, \cdot \rangle \text{ attains its maximum value on } Z \text{ at } v\},$$

where $c \in C$ is arbitrary. This defines a lattice isomorphism, and under this duality the dimensions of C and F satisfy $\dim C + \dim F = \dim V$. In particular, the vertices of Z correspond to the open cones inside V^*, and Z itself to the vector subspace $(V')^\perp = \cap_{i=1}^m H_i$ contained in the closure of all cones. For any element X of the intersection lattice of A, the cones contained in X as open subsets correspond under this bijection to the faces F of Z parallel to $U = X^\perp$, i.e. such that $F - F$ spans U. These faces are all translates of the zonotope $Z_U = \sum_{i: v_i \in U} [-1, 1] v_i \subset Z$. The zonotope Z is determined up to combinatorial equivalence (but not affine equivalence) by the hyperplane arrangement A.

Specializing to the case of root systems, the co-roots $\alpha^\vee \in \mathfrak{a}_M$ define a (simplicial) hyperplane arrangement $\mathcal{A}$ in the space $\mathfrak{a}_M^*$. Its chambers correspond to the set $\mathcal{P}(M)$, while in general, cones in the induced cone decomposition of $\mathfrak{a}_M^*$ correspond to parabolic subgroups of G containing M. The map $L \mapsto \mathfrak{a}_L^*$ defines a lattice isomorphism between $\mathcal{L}(M)$ and the intersection lattice of $\mathcal{A}$. The map from the cone lattice to the intersection lattice corresponds to the canonical map $\mathcal{P}(M) \to \mathcal{L}(M)$ defined by taking the unique Levi subgroup containing M. Dually, we have the root zonotope $\mathcal{Z} = \sum_{\alpha \in \Sigma_0} [-1, 1] \alpha^\vee$ in the space $\mathfrak{a}_M$, the face lattice of which is in bijection with the lattice of parabolics of G containing M. For example, when $G = \mathrm{GL}(n)$ and M is a maximal torus, the root zonotope is the well-known permutahedron (cf. [Zie95, p. 17-18, 200]).

Let us recall the behavior of these objects under changes of the groups G and M. If $L \in \mathcal{L}(M)$ the notation pertaining to L will be used with a superscript L. Henceforth, we will use this convention repeatedly without further comment. For example $\mathcal{P}^L(M)$ and $\mathfrak{a}_M^L$ denote the set of parabolic subgroups of L containing M and the $\mathbb{R}$-vector space spanned by the F-rational co-characters of $Z_M \cap L^{\mathrm{der}}$, respectively. For $L \in \mathcal{L}(M)$ we denote by $\mathrm{rk}_M L$ the co-rank of M in L, i.e. $\dim \mathfrak{a}_M^L$. For simplicity we write Σ_0^L for $\Sigma_{P_0^L}$ and view it as a subset of Σ_0. The cone decomposition in $\mathfrak{a}_L^* \subset \mathfrak{a}_M^*$ given by the hyperplanes of $\mathcal{A} = \mathcal{A}_M$ not containing $\mathfrak{a}_L^*$ (i.e. the roots $\alpha \in \Sigma_0 \setminus \Sigma_0^L$) is the cone decomposition associated to L inside G. On the other hand, we have the projection from $\mathfrak{a}_M^*$ to $(\mathfrak{a}_M^L)^*$, which maps the maximal cones of $\mathcal{A}$ to the maximal cones of the arrangement $\mathcal{A}^L$ associated to M inside L. This corresponds to the map $P \mapsto P^L := P \cap L$ from $\mathcal{P}(M)$ to $\mathcal{P}^L(M)$. The first process might be also dually be viewed as projecting the zonotope $\mathcal{Z}_M$ modulo the vector subspace $\mathfrak{a}_M^L$. The faces of $\mathcal{Z}_M$ parallel to $\mathfrak{a}_M^L$ correspond bijectively to the vertices of the zonotope $\mathcal{Z}_L \subset \mathfrak{a}_L$. For a fixed vertex (a fixed chamber of $\mathcal{A}_L$), the corresponding face of $\mathcal{Z}$ is just a translate of the zonotope $\mathcal{Z}_M^L \subset \mathfrak{a}_M^L$. In order to facilitate the understanding of the analogous case of galleries considered below, we explicate this fact as follows: if $\mu \in \mathfrak{a}_L^*$ is regular, then by considering vectors $\lambda \in \mathfrak{a}_M^*$ in general position and sufficiently close to μ, we obtain all chambers of $\mathcal{A}_M$ containing the chamber of $\mathcal{A}_L$ associated to μ. Furthermore, the projection $\mathfrak{a}_M^* \to (\mathfrak{a}_M^L)^*$ induces a bijection between the set of these chambers and the set of all chambers of $\mathcal{A}_M^L$. On the level of parabolic subgroups, for any $Q \in \mathcal{P}(L)$ the map $P \mapsto P^L$ from $\{P \in \mathcal{P}(M) : P \subset Q\}$ to $\mathcal{P}^L(M)$ is a bijection whose inverse is $P \mapsto Q(P) = PU_Q$.

There is a dual construction for intertwining families. Namely, given a (G, M)-intertwining family $\mathcal{F}$, $L \in \mathcal{L}(M)$ and $Q \in \mathcal{P}(L)$ we consider the family $\mathcal{F}^Q$ given by $\mathcal{F}_P^Q = \mathcal{F}_{Q(P)}$, $P \in \mathcal{P}^L(M)$, and

$$\mathcal{F}^Q_{P_2|P_1} = \mathcal{F}_{Q(P_2)|Q(P_1)}, \quad P_1, P_2 \in \mathcal{P}^L(M).$$

Clearly, this is an intertwining family with respect to (L, M), which is called the restriction of $\mathcal{F}$.

We come back to the hyperplane arrangement $\mathcal{A} = \mathcal{A}_M$ associated to a fixed Levi subgroup M, and consider the galleries from P_0 to $\overline{P_0}$ in this arrangement. It is instructive to think of a gallery in terms of a continuous path in $\mathfrak{a}_M^*$ from $\mathfrak{a}_{P_0,+}^*$ to $\mathfrak{a}_{\overline{P_0},+}^* = \mathfrak{a}_{P_0,-}^*$ which intersects each root hyperplane once, and no two of them at the same time. The order in which the path intersects the root hyperplanes describes the ordering of the gallery. The order of the chambers of the hyperplane arrangement traced by the path describes the sequence $P_0, \dots, P_N$. In the dual picture, a gallery corresponds to a path of minimal length from the vertex of the zonotope $\mathcal{Z}$ associated to P_0 to its opposite vertex. In the special case when P_0 is a *minimal* parabolic, there is yet another description of the galleries of the root system Σ_0, namely as reduced decompositions of the longest element of the Weyl group.

For the sake of completeness we record the following combinatorial characterization of the orderings of Σ_0 arising from galleries. (For the case where P_0 is a minimal parabolic, the statement can be found in [Zhe87].)

Definition 3. We say that an ordering $\alpha_1, \dots, \alpha_N$ of Σ_0 satisfies the *braiding property* if

$$\alpha_j^\vee = \alpha_i^\vee + \alpha_k^\vee \implies \text{either } i < j < k \text{ or } i > j > k.$$

(The analogous property with roots instead of co-roots is equivalent.)

Lemma 1. *An ordering of Σ_0 is obtained from a gallery if and only if it satisfies the braiding property.*

Proof. Suppose that $P_0|^{\alpha_1}P_1|\dots|^{\alpha_N}P_N = \overline{P_0}$ is a gallery. Then $\Sigma_{P_i} = \Sigma_{P_0} \setminus \{\alpha_1,\dots,\alpha_i\} \cup \{-\alpha_1,\dots,-\alpha_i\}$, $\alpha_i \in \Delta_{P_{i-1}}$ and $-\alpha_i \in \Delta_{P_i}$, $i = 1,\dots,N$. If we had $\alpha_j^\vee = \alpha_i^\vee + \alpha_k^\vee$ with $j < i,k$, then this would contradict that $\alpha_j \in \Delta_{P_{j-1}}$. Similarly, if $j > i,k$, this contradicts $-\alpha_j \in \Delta_{P_j}$. Therefore, the braiding condition is satisfied. Conversely, suppose that $\alpha_1,\dots,\alpha_N$ is an ordering of Σ_{P_0} which satisfies the braiding property. It follows that $\alpha_1 \in \Delta_{P_0}$. Therefore, we can take P_1 to be the parabolic adjacent to P_0 along α_1. Suppose that $P_1,\dots,P_i$ are defined with $\Sigma_{P_i} = \Sigma_{P_0} \setminus \{\alpha_1,\dots,\alpha_i\} \cup \{-\alpha_1,\dots,-\alpha_i\}$. We claim that $\alpha_{i+1} \in \Delta_{P_i}$. Otherwise, either $\alpha_{i+1}^\vee = \alpha_l^\vee + \alpha_k^\vee$, $k,l > i$, or $\alpha_{i+1}^\vee = \alpha_k^\vee - \alpha_j^\vee$, $k > i$, $j \le i$, i.e. $\alpha_j^\vee + \alpha_{i+1}^\vee = \alpha_k^\vee$. Both possibilities violate the braiding property. Thus, we can take P_{i+1} to be the parabolic adjacent to P_i along α_{i+1}. In this way we obtain a gallery $P_0|^{\alpha_1}P_1|\dots|^{\alpha_N}P_N = \overline{P_0}$. □

In the following, we will work exclusively with a certain subset of the set of all galleries determined by the (non-canonical) choice of $\lambda_0 \in \mathfrak{a}^*_{P_0,+}$. Namely, we consider the galleries induced by the straight paths in direction λ_0, i.e. $\mu(t) = t\lambda_0 + \mu$ (t decreasing from $+\infty$ to $-\infty$) for any $\mu \in \mathfrak{a}^*_M$ such that the path $\mu(t)$ does not cross any two different root hyperplanes at the same time. This restriction means that the numbers $t_\alpha(\mu) = -\frac{\langle\mu,\alpha^\vee\rangle}{\langle\lambda_0,\alpha^\vee\rangle}$, $\alpha \in \Sigma_0$ are all distinct. To put it otherwise, μ lies outside the union of the hyperplane arrangement $\mathcal{A}_{\lambda_0}$ in $\mathfrak{a}^*_M$ defined by

$$\mathcal{A}_{\lambda_0} = \{\mathcal{H}_L := \mathfrak{a}^*_L + \mathbb{R}\lambda_0 : L \in \mathcal{L}_2(M)\},$$

where $\mathcal{L}_2(M) = \{L \in \mathcal{L}(M) : \mathrm{rk}_M L = 2\}$. Clearly $\cap_L \mathcal{H}_L$ is the line spanned by λ_0. Thus, $\mathcal{A}_{\lambda_0}$ has rank $n-1$. For $\mu \in \mathfrak{a}^*_M \setminus \cup\mathcal{A}_{\lambda_0}$ denote by $\mathcal{G}(\mu)$ the gallery determined by the path $\mu(t)$. Its ordering is given by the ordering of the $t_\alpha(\mu)$'s as real numbers. Let

$$\mathfrak{X} = \mathfrak{X}^G_{\lambda_0} = \{\mathcal{G}(\mu) : \mu \text{ in general position}\}$$

be the set of galleries obtained this way. Note that $\mathcal{G}(\mu)$ depends only on the connected component of μ in the complement of $\cup\mathcal{A}_{\lambda_0}$. On the other hand, if $\mathcal{G}(\mu) = \mathcal{G}(\mu')$, then μ and μ' lie on the same side of any hyperplane $\mathcal{H}_L$. Therefore, under the map $\mu \mapsto \mathcal{G}(\mu)$, the galleries in $\mathfrak{X}_{\lambda_0}$ correspond bijectively to the chambers of $\mathcal{A}_{\lambda_0}$ in $\mathfrak{a}^*_M$, that is to the vertices of the (combinatorially unique) dual zonotope $\mathcal{Z}_{\lambda_0}$ in $\mathbb{H} = \{v \in \mathfrak{a}_M : \langle\lambda_0, v\rangle = 0\}$. Such a zonotope $\mathcal{Z}_{\lambda_0}$ can be constructed as the fiber polytope, in the sense of [BS92], of the linear map $z \mapsto \langle\lambda_0, z\rangle$ from $\mathcal{Z}$ to a line segment. In other words, $\mathcal{Z}_{\lambda_0}$ is the monotone path polytope of $\mathcal{Z}$ in direction λ_0. This follows from [ibid., Lemma 2.3 and Theorem 4.1] by considering the root zonotope $\mathcal{Z}$ as the image of an N-dimensional hypercube.

Definition 4. We say that $\lambda_0 \in \mathfrak{a}^*_{P_0,+}$ is *strongly regular* if for any $L \in \mathcal{L}(M)$ and $L' \in \mathcal{L}_2(M)$, $\mathfrak{a}^*_L$ is not contained in $\mathcal{H}_{L'}$ unless $L' \subset L$. Equivalently, λ_0 does not lie on the hyperplanes $\mathfrak{a}_L^* + \mathfrak{a}_{L'}^*$ where $L \in \mathcal{L}(M)$, $L' \in \mathcal{L}_2(M)$, $L' \not\subset L$ and $\mathfrak{a}_L^* + \mathfrak{a}_{L'}^* \neq \mathfrak{a}_M^*$.

Assume that λ_0 is strongly regular. In particular, the hyperplanes $\mathcal{H}_L$, $L \in \mathcal{L}_2(M)$ are all distinct. If $x, x' \in \mathfrak{X}$ correspond to two components which are separated by a single hyperplane $\mathcal{H}_L$, $L \in \mathcal{L}_2(M)$, we write $x \overset{L}{\longleftrightarrow} x'$. In this way, the set $\mathfrak{X}$ carries naturally the structure of a labeled graph with the labels of the edges given by the Levi subgroups in $\mathcal{L}_2(M)$. This graph is nothing but the one-skeleton of the zonotope $\mathcal{Z}_{\lambda_0}$ and the labels correspond to the directions of the edges.

For root systems of type A_n, the hyperplane arrangement $\mathcal{A}_{\lambda_0}$ and the zonotope $\mathcal{Z}_{\lambda_0}$ were studied in [Law97]. The dependence of $\mathcal{A}_{\lambda_0}$ and $\mathcal{Z}_{\lambda_0}$ on λ_0 is in general rather subtle (even for λ_0 in general position). As for the intersection lattice of $\mathcal{A}_{\lambda_0}$ we refer the reader to [FL] and the literature cited therein.

The case $n = 2$ is easy to understand. Given an ordered basis (v_1, v_2) of a two-dimensional space V, denote by $v_1 \circlearrowleft v_2$ the associated orientation on V.

Lemma 2. *Suppose that $n = 2$.*

1. *Let $\mathcal{G}$ be a gallery with ordering $\alpha_1, \dots, \alpha_N$. Then the orientations $\alpha_i^\vee \circlearrowleft \alpha_j^\vee$, $i < j$, coincide. We call this orientation on $\mathfrak{a}_M$ the orientation of $\mathcal{G}$.*
2. *A gallery is determined by its orientation.*
3. *For $\mathcal{G} = \mathcal{G}(\mu)$, the dual orientation of $\mathcal{G}$ on $\mathfrak{a}_M^*$ is given by $\mu \circlearrowleft \lambda_0$.*

Thus, there are exactly two galleries, both of the form $\mathcal{G}(\mu)$, corresponding to the two half-planes formed by the complement of the line $\mathbb{R}\lambda_0$ in $\mathfrak{a}_M^*$. The labeled graph $\mathfrak{X}^G$ has exactly two vertices, which are joined by an edge with the label G.

We note that the set $\mathfrak{Y}$ of all galleries from P_0 to $\overline{P_0}$ can also be given the structure of a connected labeled graph in a way that is compatible with the corresponding structure of $\mathfrak{X}$. Namely, two galleries $y, y' \in \mathfrak{Y}$ are joined by an edge with label $L \in \mathcal{L}_2(M)$ if the orderings of y and y' are

$$\alpha_1, \dots, \alpha_{i-1}, \alpha_i, \dots, \alpha_j, \alpha_{j+1}, \dots, \alpha_N$$

and

$$\alpha_1, \dots, \alpha_{i-1}, \alpha_j, \dots, \alpha_i, \alpha_{j+1}, \dots, \alpha_N$$

with $\{\alpha_i, \dots, \alpha_j\} = \Sigma_0^L$ for some $1 \le i < j \le N$. The inclusion $\mathfrak{X}_{\lambda_0} \hookrightarrow \mathfrak{Y}$ is then an embedding of the labeled graph $\mathfrak{X}_{\lambda_0}$ as a complete labeled subgraph of $\mathfrak{Y}$. In general, the sets $\mathfrak{X}_{\lambda_0}$ for any λ_0 and even their union $\cup_{\lambda_0}\mathfrak{X}_{\lambda_0}$ are proper subsets of $\mathfrak{Y}$ (the latter is the case for $G = \mathrm{GL}(n)$, $n \ge 9$, and M a maximal torus [FZ01]). The set $\mathfrak{Y}$ can also be given a natural topological structure that extends its graph structure ([Bau80, BKS94]), but for $n > 2$ the result is more complicated than a polytope, and we do not know if our proof strategy can be adapted to this setting.

If in the construction of $\mathcal{A}_{\lambda_0}$ we omit the hyperplanes $\mathcal{H}_L$ associated to Levi subgroups $L \in \mathcal{L}_2(M)$ with reducible root system Σ^L, we obtain a smaller subarrangement $\mathcal{A}'_{\lambda_0}$, which still carries all the information necessary for our purposes.

The chambers of the modified arrangement correspond to certain sets of galleries, namely the equivalence classes under the equivalence relation generated by $x \sim x'$ if $x \overset{L}{\longleftrightarrow} x'$ for a Levi subgroup $L \in \mathcal{L}_2(M)$ with reducible root system. In the case of $G = \mathrm{GL}(n)$ and M a maximal torus, these arrangements are the discriminantal arrangements of Manin-Schechtman [MS89] with $k = 2$. The set $\mathfrak{Y}'$ of all equivalence classes of galleries is the higher Bruhat order $B(n, 2)$, and the corresponding embedding $\mathfrak{X}'_{\lambda_0} \hookrightarrow \mathfrak{Y}'$ is described by Bayer-Brandt [BB97] and Felsner-Ziegler [FZ01].

Example. Let $G = \mathrm{GL}(4)$ and M be a maximal torus. There are six positive roots, enumerated $\alpha_1, \dots, \alpha_6$ with $\alpha_1, \alpha_2, \alpha_3$ simple, $\alpha_4 = \alpha_1 + \alpha_2$, $\alpha_5 = \alpha_2 + \alpha_3$, $\alpha_6 = \alpha_1 + \alpha_2 + \alpha_3$. The labeled graph $\mathfrak{Y}$ of galleries in $\mathcal{A}$ (equivalently, reduced decompositions of the longest element of the Weyl group) is the following.

$$\begin{tikzcd}
 & 146352 \arrow[r, dash, "253"] \arrow[d, dash, "364"] & 146253 \arrow[r, dash, "26"] & 142653 \arrow[r, dash, "142"] & 241653 \arrow[d, dash, "165"] & \\
 & 136452 \arrow[dl, dash, "13"'] \arrow[dr, dash, "45"] & & & 245613 \arrow[dl, dash, "45"'] \arrow[dr, dash, "13"] & \\
316452 \arrow[dr, dash, "45"'] & & 136542 \arrow[dl, dash, "13"] & 254613 \arrow[dr, dash, "13"'] & & 245631 \arrow[dl, dash, "45"] \\
 & 316542 \arrow[d, dash, "165"] & & & 254631 \arrow[d, dash, "364"] & \\
 & 356142 \arrow[r, dash, "142"] & 356241 \arrow[r, dash, "26"] & 352641 \arrow[r, dash, "253"] & 253641 &
\end{tikzcd}$$

Here each gallery is given by the induced ordering of the positive roots, and the Levi subgroups $L \in \mathcal{L}_2(M)$ are described by enumerating their positive roots. One checks that the strong regularity condition on $\lambda_0 \in \mathfrak{a}^*_{P_0,+}$ is that $\langle \lambda_0, \alpha_1^\vee \rangle \neq \langle \lambda_0, \alpha_3^\vee \rangle$. In this case, the rank two arrangement $\mathcal{A}_{\lambda_0}$ consists of seven hyperplanes (corresponding to the four Levi subgroups of type $\mathrm{GL}(3)$ and the three subgroups of type $\mathrm{GL}(2) \times \mathrm{GL}(2)$), and therefore, the graph $\mathfrak{X}_{\lambda_0}$ is a 14-gon. More precisely, for $\langle \lambda_0, \alpha_1^\vee \rangle > \langle \lambda_0, \alpha_3^\vee \rangle$ (resp. $\langle \lambda_0, \alpha_1^\vee \rangle < \langle \lambda_0, \alpha_3^\vee \rangle$) the graph $\mathfrak{X}_{\lambda_0}$ is obtained from $\mathfrak{Y}$ by deleting the galleries labeled 136542 and 245631 (resp. 316452 and 254613). In the degenerate case $\langle \lambda_0, \alpha_1^\vee \rangle = \langle \lambda_0, \alpha_3^\vee \rangle$ the edges 13 and 45 collide and we get a dodecagon.

If we identify vertices joined by edges of type $A_1 \times A_1$, i.e. labeled by 13, 26 or 45, the graph reduces to an octagon.

$$\begin{tikzcd}
146352 \arrow[r, dash, "253"] \arrow[d, dash, "364"] & 14(62)53 \arrow[r, dash, "142"] & 241653 \arrow[r, dash, "165"] & 2(45)6(13) \arrow[d, dash, "364"] \\
(13)6(45)2 \arrow[r, dash, "165"] & 356142 \arrow[r, dash, "142"] & 35(62)41 \arrow[r, dash, "253"] & 253641
\end{tikzcd}$$

Here parentheses around a pair of indices indicate that the two roots may occur in any order. Note that in this case we have $\mathfrak{X}'_{\lambda_0} = \mathfrak{Y}'$, even for degenerate λ_0.

Similarly, for the root system of type C_3, we label the positive roots as α_1, α_2, α_3 (simple), $\alpha_4 = \alpha_1 + \alpha_2$, $\alpha_5 = \alpha_2 + \alpha_3$, $\alpha_6 = \alpha_1 + \alpha_2 + \alpha_3$, $\alpha_7 = 2\alpha_2 + \alpha_3$, $\alpha_8 = \alpha_1 + 2\alpha_2 + \alpha_3$, $\alpha_9 = 2\alpha_1 + 2\alpha_2 + \alpha_3$. The roots α_3, α_7, α_9 are the long roots. The sets Σ_0^L, $L \in \mathcal{L}_2(M)$, are 13, 124, 2357, 156, 1789, 268, 29, 3469, 38, 458, 47, 59, 67. The labeled graph $\mathfrak{Y}'$ is given by the following 14-gon.

$$
\begin{tikzcd}
14(29)8(67)53 \arrow[r, dash, "286"] \arrow[d, dash, "142"] & 149682753 \arrow[r, dash, "2753"] & 1496(38)572 \arrow[r, dash, "3694"] & (31)6948572 \arrow[d, dash, "485"] \\
24198(67)53 \arrow[d, dash, "1987"] & & & (31)6(59)8(74)2 \arrow[d, dash, "165"] \\
2(47)891653 \arrow[d, dash, "165"] & & & 356198742 \arrow[d, dash, "1987"] \\
2(47)8(95)6(13) \arrow[d, dash, "485"] & & & 35(76)89142 \arrow[d, dash, "142"] \\
2758496(13) \arrow[r, dash, "3694"] & 275(83)6941 \arrow[r, dash, "2753"] & 357286941 \arrow[r, dash, "286"] & 35(76)8(29)41
\end{tikzcd}
$$

Once again we have $\mathfrak{Y}' = \mathfrak{X}'_{\lambda_0}$ for all λ_0 (even in the degenerate case), because any two Levi subgroups in $\mathcal{L}_2(M)$ which are not of type $A_1 \times A_1$ have a common root.

For a description of the case GL(5) (where one still has $\mathfrak{X}'_{\lambda_0} = \mathfrak{Y}'$) and some information about the case GL(6) (where $\mathfrak{X}'_{\lambda_0}$ is always a proper subset) see [Bau80], [Law97] and [FZ01].

We can now explain how Theorem 1 fits into this framework and at the same time formulate a somewhat more general statement suitable for the induction process. For a fixed $\lambda_0 \in \mathfrak{a}^*_{P_0,+}$ and a basis $\underline{\beta} = (\beta_1^\vee, \dots, \beta_n^\vee) \in \mathfrak{B}_{P_0}$ we define an associated linear map

$$\mu_{\underline{\beta}} : \mathbb{R}^n \to \mathfrak{a}^*_M$$

by the linear equations

$$\left\langle \mu_{\underline{\beta}}(\underline{\xi}), \beta_i^\vee \right\rangle = \xi_i \left\langle \lambda_0, \beta_i^\vee \right\rangle, \quad i = 1, \dots, n.$$

That is, $\mu_{\underline{\beta}}$ is the dual of the coordinate map with respect to the basis $\frac{\beta_i^\vee}{\langle \lambda_0, \beta_i^\vee \rangle}$ of $\mathfrak{a}_M$. If $\underline{\xi}$ is in general position (i.e. away from the finitely many hyperplanes $\mu_{\underline{\beta}}^{-1}(\mathcal{H}_L) \subset \mathbb{R}^n$, $L \in \mathcal{L}_2(M)$, $\underline{\beta} \in \mathfrak{B}_{P_0}$), then $\mathcal{G}(\mu_{\underline{\beta}}(\underline{\xi}))$ is well defined and in its ordering β_i precedes β_j if and only if $\xi_i < \xi_j$. Also, if σ is a permutation of $\{1, \dots, n\}$ then

$$\mu_{\sigma\underline{\beta}}(\sigma\underline{\xi}) = \mu_{\underline{\beta}}(\underline{\xi}) \tag{4}$$

where $(\sigma\underline{\xi})_i = \xi_{\sigma^{-1}(i)}$, and similarly for $\sigma\underline{\beta}$.

The generalization of Theorem 1 alluded to above is

$$c_M(\mathcal{F}) = \frac{(-1)^n}{n!} \sum_{\underline{\beta}\in\mathfrak{B}_{P_0}} \operatorname{vol}(\underline{\beta})\partial^{\mathbf{1}_{\underline{\beta}}}_{\mathcal{G}(\mu_{\underline{\beta}}(\underline{\xi}))}(\mathcal{F}) \tag{5}$$

for any $\underline{\xi}$ in general position (and λ_0 strongly regular). Note that $\mathcal{G}^{\mathrm{lex}}_{\underline{\beta}} = \mathcal{G}(\mu_{\underline{\beta}}(\underline{\xi}))$ for $\xi_1 \ll \xi_2 \ll \cdots \ll \xi_n$, which shows how to obtain the original statement as a special case.

Assume now that λ_0 is strongly regular. Let $L \in \mathcal{L}(M)$ and consider the faces of $\mathcal{Z}_{\lambda_0}$ which are parallel to $\mathfrak{a}^L_M \cap \mathbb{H}$. By the condition on λ_0 these are the translates of $\mathcal{Z}^L_{\lambda_0^L}$ where λ_0^L is the projection of λ_0 to $(\mathfrak{a}^L_M)^*$. Note that λ_0^L lies in the positive Weyl chamber with respect to P_0^L and is strongly regular with respect to L. The construction of these faces is analogous to the case of the root zonotopes, but unlike it, we do not get all faces of $\mathcal{Z}_{\lambda_0}$ this way for $n \geq 4$. To explicate it, let ν be a regular element of $\mathfrak{a}_L^*$ which does not lie on any hyperplanes $\mathcal{H}_{L'}$, $L' \in \mathcal{L}_2(M)$, $L' \not\subset L$. (The existence of ν is assured by the assumption on λ_0.) Consider the galleries $\mathcal{G}^G(\mu)$ for all $\mu \in \mathfrak{a}_M^*$ sufficiently close to ν and in general position. By the assumptions on μ and ν, the numbers $t_\alpha = t_\alpha(\mu)$ are all distinct and the relative positions of the t_α for $\alpha \notin \Sigma_0^L$ do not depend on μ. On the other hand, the t_α for $\alpha \in \Sigma_0^L$ are smaller in absolute value than the rest of the t_α's, so that they comprise a contiguous segment. They depend only on the projection μ^L of μ to $(\mathfrak{a}^L_M)^*$ and their ordering matches that of the gallery $\mathcal{G}^L(\mu^L)$ corresponding to the projection $\mu^L(t) = t\lambda_0^L + \mu^L$.

Thus $\mathcal{G}^L(\mu) \mapsto \mathcal{G}^G(\mu+\nu)$ (for $\mu \in (\mathfrak{a}^L_M)^*$ close to the origin and in general position) defines an embedding of labeled graphs

$$\psi^G_{L;\nu} : \mathfrak{X}^L_{\lambda_0^L} \to \mathfrak{X}^G_{\lambda_0},$$

in the sense that if $x \overset{L'}{\longleftrightarrow} y$ in $\mathfrak{X}^L_{\lambda_0^L}$ then $\psi^G_{L;\nu}(x) \overset{L'}{\longleftrightarrow} \psi^G_{L;\nu}(y)$ in $\mathfrak{X}^G_{\lambda_0}$. Let $Q \in \mathcal{P}(L)$ be such that $\nu \in \mathfrak{a}^*_{Q,+}$. Then all the galleries in $\psi^G_{L;\nu}(\mathfrak{X}^L_{\lambda_0^L})$ begin with the same sequence from P_0 to $Q(P_0^L)$ and end with the same sequence from $Q(\overline{P_0^L})$ to $\overline{P_0}$. They only differ in the sequence between $Q(P_0^L)$ and $Q(\overline{P_0^L})$, which is dictated by the input gallery in $\mathfrak{X}^L_{\lambda_0^L}$.

For $L \in \mathcal{L}_2(M)$ this embedding yields for each $\nu \in \mathfrak{a}_L^*$ in general position an edge of $\mathfrak{X}^G_{\lambda_0}$ with label L, and by varying ν we obtain all edges with this label. Likewise, we can obtain all vertices by taking any $L \in \mathcal{L}(M)$ with $\operatorname{rk}_M L = 1$ (in which case $\mathfrak{X}^L$ is a point) and varying ν.

Henceforth we will implicitly assume that λ_0 is strongly regular.

4. A combinatorial variant of the problem

We now turn to a combinatorial variant of Theorem 1. We proceed in this way since it is not convenient to work with intertwining families directly. Instead we introduce an algebraic setup which controls their relations. Ultimately, this will reduce the problem to a question in commutative algebra, which is settled in [FL].

Consider first the following abstract situation. Let W be a finite-dimensional vector space over $\mathbb{C}$, $W^\vee$ its dual space, and $\mathfrak{s} = \mathrm{Sym}(W)$ and $\mathfrak{s}^\vee = \mathrm{Sym}(W^\vee)$ the corresponding symmetric algebras, which can be considered as the algebras of polynomials on $W^\vee$ and W, respectively. Both $\mathfrak{s}$ and $\mathfrak{s}^\vee$ are naturally graded objects. Let X be a finite index set and let $S = \mathfrak{s}^X = \bigoplus_{n=0}^\infty S_n$ and $S^\vee = (\mathfrak{s}^\vee)^X = \bigoplus_{n=0}^\infty S_n^\vee$ be the graded modules of functions from X to $\mathfrak{s}$ and $\mathfrak{s}^\vee$, respectively.

The natural perfect duality $W \times W^\vee \to \mathbb{C}$ induces a duality between $\mathfrak{s}$ and $\mathfrak{s}^\vee$, and in turn a duality between S and $S^\vee$ given by a bilinear form $(\cdot,\cdot) : S \times S^\vee \to \mathbb{C}$ satisfying $(S_n, S_m^\vee) = 0$ for all $n \neq m$. Thus, $(V^\perp)^\perp = V$ for all graded vector subspaces V of S. The natural diagonal action of $\mathfrak{s}$ on S translates into a dual diagonal action of $\mathfrak{s}$ on $S^\vee$, which lowers degree and can be understood as applying the elements of $\mathfrak{s}$ as differential operators to the components. In particular, for $w \in W$ and $s^\vee \in S^\vee$ we write $\partial_w s^\vee$ for the element of $S^\vee$ satisfying $(s, \partial_w s^\vee) = (ws, s^\vee)$ for all $s \in S$. In the following, we will always consider S and $S^\vee$ as $\mathfrak{s}$-modules in this way. We also think of $S^\vee$ as a free $\mathfrak{s}^\vee$-module in the obvious way. Note that $R \subset S^\vee$ is a graded $\mathfrak{s}$-submodule of $S^\vee$ if and only if $T = R^\perp$ is a graded $\mathfrak{s}$-submodule of S, in which case

$$T = \{s = (s_x) \in S : \sum_x s_x u_x = 0 \text{ for all } u = (u_x) \in R\}.$$

In this situation we can also think of T as the graded dual of $S^\vee/R$, i.e. $T = \bigoplus_{n=0}^\infty (S_n^\vee / R_n)^*$.

Lemma 3. *Let R be a graded $\mathfrak{s}$-submodule of $S^\vee$ and suppose that $T = R^\perp \subset S$ is generated as an $\mathfrak{s}$-module by its homogeneous elements of degree $< K$. Then an element u of $S_k^\vee$, $k \geq K$, belongs to R if (and only if) $\partial_w u \in R$ for all $w \in W$.*

Proof. Suppose that $u \in S_k^\vee$ and $\partial_w u \in R$ for all $w \in W$. Then $\partial_w(\sum_x s_x u_x) = \sum s_x \partial_w u_x = 0$ for all $s = (s_x)_{x \in X} \in T$ and $w \in W$. Therefore $\sum s_x u_x$ is a constant for all $s \in T$. However, if $s \in T_l$, $l < K$, then $\deg(\sum s_x u_x) = k - l \geq K - l > 0$ and therefore $\sum s_x u_x = 0$. We conclude that $\sum s_x u_x = 0$ for a set of generators s of T and therefore $u \in T^\perp = R$. $\square$

To connect this discussion to the previous setup, let W be the vector space $\oplus_{\alpha \in \Sigma_{P_0}} \mathbb{C}\alpha^\vee$ with the canonical projection $\mathrm{pr} : W \to \mathfrak{a}_M$. We write pr^* both for the dual map $\mathfrak{a}_M^* \to W^\vee$ and for the induced map $\mathrm{Sym}(\mathfrak{a}_M^*) \to \mathfrak{s}^\vee$. In order to distinguish $\alpha^\vee$ as an element of W and $\mathfrak{a}_M$ we write $(\Theta_{\alpha^\vee})_{\alpha \in \Sigma_{P_0}}$ for the basis of W where $\Theta_{\alpha^\vee} = \alpha^\vee$ in the α-component and 0 otherwise. Denote by $(\varpi_{\alpha^\vee})$ the corresponding dual basis of $W^\vee$. Set $\varpi_{\underline{\alpha}} = \prod_{i=1}^m \varpi_{\alpha_i^\vee}$ for any (multi-)set $\underline{\alpha} = \{\alpha_1^\vee, \dots, \alpha_m^\vee\} \subset \Sigma_0^\vee$. The index set X will be the set $\mathfrak{X} = \mathfrak{X}_{\lambda_0}^G$ of all galleries obtained from straight paths in direction λ_0. We denote the resulting modules S and $S^\vee$ by $\mathcal{S}$ and $\mathcal{S}^\vee$. We also denote by $\mathbf{1}_{\mathcal{G}}$ the element of $\mathcal{S}^\vee$ with component 1 at the index $\mathcal{G} \in \mathfrak{X}$ and all other components zero.

We define $s\mathcal{F} \in \operatorname{End}(\mathcal{F}_{P_0})$ for an element $s \in \mathcal{S}^\vee$ and an intertwining family $\mathcal{F}$ by linear extension in the first variable of

$$\left(\prod_{\alpha \in \Sigma_0} \varpi_{\alpha^\vee}^{m(\alpha^\vee)} \mathbf{1}_{\mathcal{G}} \right) \mathcal{F} := \partial_{\mathcal{G}}^m(\mathcal{F}).$$

Thus, the module $\mathcal{S}^\vee$ is a means to keep track of all possible logarithmic differential operators which can be applied to intertwining families. The action of $\mathfrak{s}$ on $\mathcal{S}^\vee$ has a simple description in this framework, which we include here since it may serve as motivation for our inductive strategy. For $\alpha \in \Sigma_{P_0}$ and an intertwining family $\mathcal{F}$, we consider the new intertwining family $\mathcal{F}^\alpha$ defined by $\mathcal{F}_P^\alpha = \mathcal{F}_P$ and

$$\mathcal{F}_{Q|P}^\alpha(\lambda) = (1 + \langle \lambda, \alpha^\vee \rangle)^{\mathbf{1}_{\Sigma_P}(\alpha) - \mathbf{1}_{\Sigma_Q}(\alpha)} \mathcal{F}_{Q|P}(\lambda).$$

Then

$$s\mathcal{F}^\alpha = s\mathcal{F} + (\partial_{\Theta_{\alpha^\vee}} s)\mathcal{F} \tag{6}$$

for any $s \in \mathcal{S}^\vee$.

Because of the commuting relations satisfied by intertwining families, there exist many elements $s \in \mathcal{S}^\vee$ annihilating all families. We denote the space of these elements by $\mathcal{K}$. Equation (6) implies that $\mathcal{K}$ is actually an $\mathfrak{s}$-module (but not an $\mathfrak{s}^\vee$-module). While it may be difficult to describe the module $\mathcal{K}$ completely, it is not difficult to construct certain elements in $\mathcal{K}$ explicitly.

First, for any gallery $\mathcal{G}$ and any $D \in \operatorname{Sym}(\mathfrak{a}_M^*)$, acting as a differential operator on functions on $\mathfrak{a}_M^*$, we have

$$\mathcal{F}_{\overline{P_0}|P_0}(0)^{-1} \left[D\mathcal{F}_{\overline{P_0}|P_0} \right](0) = \left(\operatorname{pr}^*(D) \mathbf{1}_{\mathcal{G}} \right) \mathcal{F}.$$

In particular, the right-hand side is independent of $\mathcal{G}$, and it follows that for any $x, x' \in \mathfrak{X}$ we have $\operatorname{Sym}(\operatorname{pr}^*(\mathfrak{a}_M^*)) \left(\mathbf{1}_x - \mathbf{1}_{x'} \right) \subset \mathcal{K}$.

Let $L \in \mathcal{L}(M)$ and identify W^L with a subspace of W. For any $\nu \in \mathfrak{a}_L^*$ in general position we construct a map $\phi_{L;\nu}^G$ from $(\mathcal{S}^L)^\vee$ to $(\mathcal{S}^G)^\vee$. Recall the embedding

$$\psi_{L;\nu}^G : \mathfrak{X}_{\lambda_0^L}^L \to \mathfrak{X}_{\lambda_0}^G$$

of labeled graphs defined in §3 above. Define

$$\phi_{L;\nu}^G : (\mathcal{S}^L)^\vee = \operatorname{Sym}((W^L)^\vee)^{\mathfrak{X}^L} \to \operatorname{Sym}(W^\vee)^{\mathfrak{X}^G} = (\mathcal{S}^G)^\vee$$

by

$$\phi_{L;\nu}^G(s\mathbf{1}_{\mathcal{G}}) = \pi_L^*(s) \mathbf{1}_{\psi_{L;\nu}^G(\mathcal{G})},$$

where $\pi_L : W \to W^L$ is the linear transformation defined by

$$\Theta_{\alpha^\vee}^G \mapsto \begin{cases} \Theta_{\alpha^\vee}^L & \text{if } \alpha \in \Sigma_0^L, \\ 0 & \text{otherwise.} \end{cases}$$

By the description of the image of $\psi_{L;\nu}^G$, it follows that if $\nu \in \mathfrak{a}_{Q,+}^*$, $s \in (\mathcal{S}^L)^\vee$, and $A \in \operatorname{Sym}((W^L)^\perp) \subset \mathfrak{s}^\vee$ then $A\phi_{L;\nu}^G(s)\mathcal{F}$ can be written as the composition of $s\mathcal{F}^Q$ with certain operators (on the left and on the right). Thus, for all $\nu \in \mathfrak{a}_L^*$ in general position we have

$$\mathrm{Sym}((W^L)^\perp)\phi_{L;\nu}^G(\mathcal{K}_{\lambda_0^L}^L) \subset \mathcal{K}_{\lambda_0}^G.$$

Let

$$\mathrm{Rel}_L = \mathrm{Ker}\ \mathrm{pr}|_{W^L} = \{\sum_{\alpha\in\Sigma_0^L} c_\alpha \Theta_{\alpha^\vee} : \sum_\alpha c_\alpha \alpha^\vee = 0\}$$

be the space of relations associated to L. Its annihilator $\mathrm{Rel}_L^\perp \subset W^\vee$ can be expressed as

$$\mathrm{Rel}_L^\perp = (W^L)^\perp + \pi_L^*\,\mathrm{pr}_L^*((\mathfrak{a}_M^L)^*) = (W^L)^\perp + \mathrm{pr}^*(\mathfrak{a}_M^*).$$

Consequently, for any galleries x, $x' \in \psi_{L;\nu}^G(\mathfrak{X}_{\lambda_0^L}^L)$ we have

$$\mathrm{Sym}(\mathrm{Rel}_L^\perp)\,(\mathbf{1}_x - \mathbf{1}_{x'}) \subset \mathcal{K}.$$

We are interested in the sum of all these subspaces of $\mathcal{K}$, for which it is evidently enough to consider only Levi subgroups $L \in \mathcal{L}_2(M)$. For any two galleries x, $x' \in \mathfrak{X}$ such that $x \overset{L}{\longleftrightarrow} x'$ for $L \in \mathcal{L}_2(M)$, define

$$\mathcal{R}_{x,x'} = \mathrm{Sym}(\mathrm{Rel}_L^\perp)\,(\mathbf{1}_x - \mathbf{1}_{x'})\,,$$

and set

$$\mathcal{R} = \sum_{x \overset{L}{\longleftrightarrow} x'} \mathcal{R}_{x,x'}.$$

Then $\mathcal{R} \subset \mathcal{K}$. Clearly, each vector space $\mathcal{R}_{x,x'}$, and therefore also their sum $\mathcal{R}$, is a graded $\mathfrak{s}$-submodule of $\mathcal{S}^\vee$. Also, for any $L \in \mathcal{L}(M)$ and $\nu \in \mathfrak{a}_L^*$ in general position we have the compatibility relation

$$\mathrm{Sym}((W^L)^\perp)\phi_{L;\nu}^G(\mathcal{R}_{\lambda_0^L}^L) \subset \mathcal{R}_{\lambda_0}^G. \tag{7}$$

The module $\mathcal{R}$ corresponds to the relations for intertwining families given by elementary Coxeter moves. Its annihilator $\mathcal{T} = \mathcal{R}^\perp \subset \mathcal{S}$ can be described as

$$\mathcal{T} = \{s = (s_x)_{x\in\mathfrak{X}} \in \mathcal{S} : s_x - s_{x'} \in (\mathrm{Rel}_L) \text{ for all } x \overset{L}{\longleftrightarrow} x'\},$$

where (Rel_L) denotes the ideal of $\mathfrak{s}$ generated by the vector space $\mathrm{Rel}_L \subset W$. It is clear that $\mathcal{T}$ is in fact an $\mathfrak{s}$-algebra, although we will have no use of this fact. The key fact which we will use is [FL, Corollary 3]. Applying it to the hyperplane $\mathbb{H}$ of $\mathfrak{a}_M$ we infer

(8) $\mathcal{T}$ is generated as a $\mathfrak{s}$-module by its homogeneous elements of degree smaller than n.

Note that the elements of $\mathcal{T}$ are constant on the equivalence classes in $\mathfrak{X}'$, since $\mathrm{Rel}_L = 0$ for $L \in \mathcal{L}_2(M)$ with reducible root system Σ^L. In studying $\mathcal{T}$ it is therefore possible to consider the smaller subarrangement $\mathcal{A}'_{\lambda_0}$ instead of $\mathcal{A}_{\lambda_0}$. Although this observation is useful for working out examples, it is not needed in the proof.

Example. In rank two, there are just two galleries $\mathcal{G}_1$ and $\mathcal{G}_2$, and the module $\mathcal{T} = \{s_1 \mathbf{1}_{\mathcal{G}_1} + s_2 \mathbf{1}_{\mathcal{G}_2} : s_1 - s_2 \in (\mathrm{Rel}_L)\}$ is clearly generated by the elements $\mathbf{1}_{\mathcal{G}_1} + \mathbf{1}_{\mathcal{G}_2}$ (of degree zero) and $r\mathbf{1}_{\mathcal{G}_1}$ for $r \in \mathrm{Rel}_L$ (of degree one).

In rank three, the graph $\mathfrak{X}_{\lambda_0}$ is necessarily a circuit. We number its vertices as $\mathcal{G}_0, \dots, \mathcal{G}_{m-1}$ in such a way that for any $0 \le i \le m-1$ we have $\mathcal{G}_i \overset{L_i}{\longleftrightarrow} \mathcal{G}_{i+1}$ with $L_i \in \mathcal{L}_2(M)$ (where we set $\mathcal{G}_m := \mathcal{G}_0$). Let $W_i = \mathrm{Rel}_{L_i} \subset W$. Then it is an exercise in commutative algebra to verify that the module $\mathcal{T} = \{\sum_i s_i \mathbf{1}_{\mathcal{G}_i} : s_{i+1} - s_i \in (W_i),\ i = 0, \dots, m-1\}$ (where we again identify s_m and s_0) is generated by the elements $\sum_i \mathbf{1}_{\mathcal{G}_i}$ (of degree zero), $\sum_{j<i} w_j \mathbf{1}_{\mathcal{G}_i}$ for $w_j \in W_j$ with $\sum_j w_j = 0$ (of degree one) and $w_i w_j \sum_{i<k\le j} \mathbf{1}_{\mathcal{G}_k}$ for $i < j$, $w_i \in W_i$, $w_j \in W_j$ (of degree two).

We leave open the question of whether in fact $\mathcal{R} = \mathcal{K}$. The question is subtle because it is not clear how to construct many examples of intertwining families. However, the lack of an affirmative answer is only a bookkeeping difficulty. (For $n = 2$ it is in fact not difficult to show that $\mathcal{R} = \mathcal{K}$. We omit the proof of this statement, since it is unnecessary for the proof of our main result.)

The main identity (5) for $c_M(\mathcal{F})$ can now be reduced to an identity of elements of the quotient space $\mathcal{S}_n^\vee / \mathcal{R}_n$. First, it is clear that the right hand side of (5) can be written as $\mathfrak{d}_{\underline{\xi}} \mathcal{F}$, where

$$\mathfrak{d}_{\underline{\xi}} = \frac{(-1)^n}{n!} \sum_{\underline{\beta} \in \mathfrak{B}_{P_0}} \mathrm{vol}(\underline{\beta}) \varpi_{\underline{\beta}} \mathbf{1}_{\mathcal{G}(\mu_{\underline{\beta}}(\underline{\xi}))} \in \mathcal{S}_n^\vee.$$

It is also not hard to express $c_M(\mathcal{F})$ directly in the form $\mathfrak{c}\mathcal{F}$ for a (different) element $\mathfrak{c} = \mathfrak{c}_{\eta;(\mu_P)_P} \in \mathcal{S}_n^\vee$. For any $P \in \mathcal{P}(M)$ let $\mathrm{pr}_{P_0;P} = \mathrm{pr} \circ \pi_{P_0;P} : W \to \mathfrak{a}_M$ where $\pi_{P_0;P} : W \to \oplus_{\alpha \in \Sigma_{P_0;P}} \mathbb{C}\alpha^\vee$ is the projection map. The dual map $\mathfrak{a}_M^* \to W^\vee$ is then given by

$$\mathrm{pr}_{P_0;P}^*(\eta) = \sum_{\alpha \in \Sigma_{P_0;P}} \langle \eta, \alpha^\vee \rangle \, \varpi_{\alpha^\vee}.$$

We also denote by $\mathrm{pr}_{P_0;P}^*$ the induced map $\mathrm{Sym}(\mathfrak{a}_M^*) \to \mathfrak{s}^\vee$.

Lemma 4. *We have $c_M(\mathcal{F}) = \mathfrak{c}_{\eta;(\mu_P)_P} \mathcal{F}$ for all intertwining families $\mathcal{F}$, where*

$$\mathfrak{c}_{\eta;(\mu_P)_P} = \frac{1}{n!} \sum_{P \in \mathcal{P}(M)} \frac{\mathrm{pr}_{P_0;P}^*(\eta^n)}{\prod_{\alpha \in \Delta_P} \langle \eta, \alpha^\vee \rangle} \mathbf{1}_{\mathcal{G}(\mu_P)} \in \mathcal{S}_n^\vee.$$

Here $\eta \in \mathfrak{a}_M^*$ with $\langle \eta, \alpha^\vee \rangle \neq 0$ for all $\alpha \in \Sigma_0$, and for each $P \in \mathcal{P}(M)$, $\mu_P \in \mathfrak{a}_{P,+}^*$ is in general position.

Proof. Following Arthur, we evaluate (1) by setting $\lambda = t\eta$ and taking the limit as $t \to 0$ using de L'Hôpital's rule. That is,

$$c_M(\mathcal{F}) = \frac{1}{n!} \sum_{P \in \mathcal{P}(M)} \frac{\mathcal{F}_{P|P_0}(0)^{-1} \left[(\frac{\partial}{\partial \eta})^n \mathcal{F}_{P|P_0} \right](0)}{\theta_P(\eta)}$$

The gallery $\mathcal{G}(\mu_P)$ contains a sub-gallery from P_0 to P, obtained by restricting the path $t\lambda_0 + \mu_P$ to $t \geq 0$. Therefore, for any $D \in \mathrm{Sym}(\mathfrak{a}_M^*)$, acting as a differential operator on functions on $\mathfrak{a}_M^*$, we have

$$\mathcal{F}_{P|P_0}(0)^{-1}\left[D\mathcal{F}_{P|P_0}\right](0) = \left(\mathrm{pr}^*_{P_0;P}(D)\mathbf{1}_{\mathcal{G}(\mu_P)}\right)\mathcal{F}.$$

The Lemma follows. □

Theorem 1 asserts that $\mathfrak{c}_{\eta;(\mu_P)_P} - \mathfrak{d}_{\underline{\xi}} \in \mathcal{K}$. We will show the following stronger algebraic statement.

Theorem 2. *Let $\mathfrak{c}_{\eta;(\mu_P)_P} \in \mathcal{S}_n^\vee$ and $\mathfrak{d}_{\underline{\xi}} \in \mathcal{S}_n^\vee$ be as above. Then*

$$\mathfrak{c}_{\eta;(\mu_P)_P} - \mathfrak{d}_{\underline{\xi}} \in \mathcal{R}_n.$$

Note that this implies in particular that modulo $\mathcal{R}_n$, $\mathfrak{d}_{\underline{\xi}}$ and $\mathfrak{c}_{\eta;(\mu_P)_P}$ are independent of all choices. In fact, in the proof we first check this independence.

Lemma 5. *The image of $\mathfrak{c}_{\eta;(\mu_P)_P}$ in $\mathcal{S}_n^\vee/\mathcal{R}_n$ is independent of the choices of μ_P, $P \in \mathcal{P}(M)$, and $\eta \in \mathfrak{a}_M^*$. Moreover,*

$$\frac{1}{n!}\sum_{P\in\mathcal{P}(M)} \frac{\mathrm{pr}^*_{P_0;P}(\eta^k)}{\prod_{\alpha\in\Delta_P}\langle\eta,\alpha^\vee\rangle}\mathbf{1}_{\mathcal{G}(\mu_P)} \in \mathcal{R}_k$$

for all $k < n$.

Proof. First, note that the class modulo $\mathcal{R}_n$ of the term $\mathrm{pr}^*_{P_0;P}(\eta^n)\mathbf{1}_{\mathcal{G}(\mu_P)}$ corresponding to $P \in \mathcal{P}(M)$ does not depend on the choice of $\mu_P \in \mathfrak{a}^*_{P,+}$. Indeed, since $\langle\lambda_0, \alpha^\vee\rangle > 0$ for all $\alpha \in \Sigma_0$, if μ_P moves in the chamber associated to P, it can only cross hyperplanes of $\mathcal{A}_{\lambda_0}$ associated to $L \in \mathcal{L}_2(M)$ with either $\Sigma_0^L \subset \Sigma_0 \cap \Sigma_P$ or $\Sigma_0^L \subset \Sigma_0 \cap \Sigma_{\overline{P}}$. The difference

$$\mathrm{pr}^*_{P_0;P}(\eta^n)\left(\mathbf{1}_{\mathcal{G}(\mu_P)} - \mathbf{1}_{\mathcal{G}(\mu'_P)}\right) = \Big(\sum_{\alpha\in\Sigma_{P_0;P}}\langle\eta,\alpha^\vee\rangle\,\varpi_{\alpha^\vee}\Big)^n\left(\mathbf{1}_{\mathcal{G}(\mu_P)} - \mathbf{1}_{\mathcal{G}(\mu'_P)}\right)$$

for vectors μ_P and μ'_P separated by the single hyperplane $\mathcal{H}_L$ lies in the relation space $\mathcal{R}_{\mathcal{G}(\mu_P),\mathcal{G}(\mu'_P)}$. Namely, in the first case no root of L occurs in this expression, and in the second case the roots of L form a subset of $\Sigma_{P_0;P}$. Therefore the factor $\mathrm{pr}^*_{P_0;P}(\eta^n)$ lies in $\mathrm{Sym}(\mathrm{Rel}_L^\perp)$ in both cases.

The class $\overline{\mathfrak{c}_\eta}$ of $\mathfrak{c}_\eta$ in $\mathcal{S}_n^\vee/\mathcal{R}_n$ is now a priori a rational function of η of homogeneous degree zero with values in a finite-dimensional vector space and at most simple singularities along the hyperplanes $\langle\eta,\alpha^\vee\rangle = 0$, $\alpha \in \Sigma_{P_0}$. We show that for any pair $P|^\alpha P'$ of adjacent parabolics the poles along $\langle\eta,\alpha^\vee\rangle = 0$ in the contribution from P and P' to $\overline{\mathfrak{c}_\eta}$ cancel. Indeed, we have

$$\mathrm{pr}^*_{P_0;P'}(\eta^n) = \sum_{k=0}^n \binom{n}{k}\langle\eta,\alpha^\vee\rangle^k\,\varpi^k_{\alpha^\vee}\,\mathrm{pr}^*_{P_0;P}(\eta^{n-k}).$$

On the other hand, we can take $\mu_P \in \mathfrak{a}^*_{P,+}$ and $\mu_{P'} \in \mathfrak{a}^*_{P',+}$ very close to each other so that $\mathcal{G}(\mu_{P'}) = \mathcal{G}(\mu_P)$. Then the total contribution

$$\left(\frac{\mathrm{pr}^*_{P_0;P}(\eta^n)}{\prod_{\gamma \in \Delta_P} \langle \eta, \gamma^\vee \rangle} + \frac{\mathrm{pr}^*_{P_0;P'}(\eta^n)}{\prod_{\gamma \in \Delta_{P'}} \langle \eta, \gamma^\vee \rangle} \right) \mathbf{1}_{\mathcal{G}(\mu_P)}$$

of P and P' is regular along the hyperplane $\langle \eta, \alpha^\vee \rangle = 0$. Therefore, $\overline{\mathfrak{c}_\eta}$ is in fact a polynomial in η. Since it has homogeneous degree zero, it is constant.

The second part follows in a similar vein, except that now the function is regular of homogeneous degree $k - n < 0$, and is therefore zero. □

The same independence property is true for the right hand side of the formula. In fact, for the inductive process we need a somewhat more precise statement.

Proposition 1. *The class $\overline{\mathfrak{d}_{\underline{\xi}}}$ of $\mathfrak{d}_{\underline{\xi}}$ in $\mathcal{S}^\vee_n / \mathcal{R}_n$ is independent of the choice of $\underline{\xi}$. More generally, let $I \subset \{1, \dots, n\}$ and $\underline{\beta}_0 = (\alpha_i^\vee)_{i \in I}$ with $\alpha_i \in \Sigma_0$ and set*

$$\mathfrak{B}_0(\underline{\beta}_0) = \{(\beta_1^\vee, \dots, \beta_n^\vee) \in \mathfrak{B}_0 : \beta_i = \alpha_i \text{ for all } i \in I\}.$$

Then the image of

$$\mathfrak{d}_{\underline{\beta}_0, \underline{\xi}} = \frac{(-1)^n}{n!} \sum_{\underline{\beta} \in \mathfrak{B}_0(\underline{\beta}_0)} \mathrm{vol}(\underline{\beta}) \varpi_{\underline{\beta} \setminus \underline{\beta}_0} \mathbf{1}_{\mathcal{G}(\mu_{\underline{\beta}}(\underline{\xi}))} \tag{9}$$

in $\mathcal{S}^\vee / \mathcal{R}$ depends only on $|I|$ and the set underlying $\underline{\beta}_0$, and not on $\underline{\xi}$, I or the ordering of $\underline{\beta}_0$.

Proof. By (4) it suffices to show that (9) is independent of $\underline{\xi}$. Clearly, (9) depends only on the connected component of $\underline{\xi}$ in the complement of

$$\bigcup_{\underline{\beta} \in \mathfrak{B}_0(\underline{\beta}_0), H \in \mathcal{A}_{\lambda_0}} \mu_{\underline{\beta}}^{-1}(H).$$

Suppose that $\underline{\xi}^{(i)}$, $i = 1, 2$, are separated by a single wall $\mathcal{H}$. Let $\underline{\beta} \in \mathfrak{B}$ and set $\mathcal{G}_i(\underline{\beta}) = \mathcal{G}(\mu_{\underline{\beta}}(\underline{\xi}^{(i)}))$, $i = 1, 2$. Then $\mathcal{G}_1(\underline{\beta}) = \mathcal{G}_2(\underline{\beta})$ unless $\mu_{\underline{\beta}}(\mathcal{H}) = \mathcal{H}_L$ for some $L \in \mathcal{L}_2(M)$ in which case $\mathcal{G}_1(\underline{\beta}) \overset{L}{\longleftrightarrow} \mathcal{G}_2(\underline{\beta})$. Thus,

$$\mathfrak{d}_{\underline{\beta}_0, \underline{\xi}^{(1)}} - \mathfrak{d}_{\underline{\beta}_0, \underline{\xi}^{(2)}} = \frac{(-1)^n}{n!} \sum_{\underline{\beta} \in \mathcal{C}} \mathrm{vol}(\underline{\beta}) \varpi_{\underline{\beta} \setminus \underline{\beta}_0} (\mathbf{1}_{\mathcal{G}_1(\underline{\beta})} - \mathbf{1}_{\mathcal{G}_2(\underline{\beta})}), \tag{10}$$

where

$$\mathcal{C} = \{\underline{\beta} \in \mathfrak{B}_0(\underline{\beta}_0) : \mu_{\underline{\beta}}(\mathcal{H}) \in \mathcal{A}_{\lambda_0}\}.$$

If $\underline{\beta} \in \mathcal{C}$, we denote by $L(\underline{\beta})$ the Levi $L \in \mathcal{L}_2(M)$ such that $\mu_{\underline{\beta}}(\mathcal{H}) = \mathcal{H}_L$. Since $\underline{\beta} \setminus \underline{\beta}_0$ is linearly independent, we have $|\mathfrak{a}_M^{L(\underline{\beta})} \cap \underline{\beta} \setminus \underline{\beta}_0| \le 2$.

Consider first $\underline{\beta} \in \mathcal{C}$ such that $\mathfrak{a}_M^{L(\underline{\beta})} \cap \underline{\beta} \subset \underline{\beta}_0$. Then $\varpi_{\underline{\beta} \setminus \underline{\beta}_0} \in \mathrm{Sym}(\mathrm{Rel}^\perp_{L(\underline{\beta})})$ and therefore

$$\mathrm{vol}(\underline{\beta}) \varpi_{\underline{\beta} \setminus \underline{\beta}_0} (\mathbf{1}_{\mathcal{G}_1(\underline{\beta})} - \mathbf{1}_{\mathcal{G}_2(\underline{\beta})}) \in \mathcal{R}_{\mathcal{G}_1(\underline{\beta}), \mathcal{G}_2(\underline{\beta})}.$$

Next, suppose that $\mathfrak{a}_M^{L(\underline{\beta})} \cap \underline{\beta} = \{\beta_i^\vee, \beta_j^\vee\}$ with $i, j \notin I$. Then $\mathcal{H}$ is the hyperplane $\xi_i = \xi_j$. Let $\underline{\beta}'$ be the basis obtained from $\underline{\beta}$ by interchanging $\beta_i^\vee$ and $\beta_j^\vee$. Then $\underline{\beta}' \in \mathcal{C}$, $\mathcal{G}_i(\underline{\beta}') = \mathcal{G}_{3-i}(\underline{\beta})$ and $L(\underline{\beta}) = L(\underline{\beta}')$, and therefore the contributions of $\underline{\beta}$ and $\underline{\beta}'$ to (10) cancel.

Finally, we have to consider the contribution from

$$\mathcal{C}_1 = \{\underline{\beta} \in \mathcal{C} : |\mathfrak{a}_M^{L(\underline{\beta})} \cap \underline{\beta} \setminus \underline{\beta}_0| = 1\}.$$

We may write $\mathcal{C}_1$ as the disjoint union over $L \in \mathcal{L}_2(M)$ of

$$\mathcal{C}_1^L = \{\underline{\beta} \in \mathcal{C}_1 : L(\underline{\beta}) = L\}.$$

We can partition each $\mathcal{C}_1^L$ further into sets of the form

$$\mathcal{C}_{L,i,\underline{\beta}'} = \{\iota_{\beta^\vee,i}(\underline{\beta}') := (\beta_1^\vee, \dots, \beta_{i-1}^\vee, \beta^\vee, \beta_{i+1}^\vee, \dots, \beta_n^\vee) : \beta \in \Sigma_{P_0}^L\} \cap \mathfrak{B}_{P_0}$$

for some $i \notin I$ and $\underline{\beta}' = (\beta_j^\vee)_{j \neq i}$. Fix i and $\underline{\beta}'$ such that $\mathcal{C}_{L,i,\underline{\beta}'} \subset \mathcal{C}_1^L$. It remains to show that

$$\sum_{\beta \in \Sigma_{P_0}^L : \beta^\vee \notin \operatorname{span}(\{\beta_j^\vee\}_{j \neq i})} \operatorname{vol}(\iota_{\beta^\vee,i}(\underline{\beta}')) \varpi_{\underline{\beta}' \setminus \underline{\beta}_0} \varpi_{\beta^\vee} (\mathbf{1}_{\mathcal{G}_1(\iota_{\beta^\vee,i}(\underline{\beta}))} - \mathbf{1}_{\mathcal{G}_2(\iota_{\beta^\vee,i}(\underline{\beta}))}) \in \mathcal{R}. \tag{11}$$

Note that $\mathcal{G}_1(\mathcal{C}_{L,i,\underline{\beta}'}) \cup \mathcal{G}_2(\mathcal{C}_{L,i,\underline{\beta}'})$ consists of two vertices $\mathcal{G}_1', \mathcal{G}_2' \in \mathfrak{X}$ such that $\mathcal{G}_1' \overset{L}{\longleftrightarrow} \mathcal{G}_2'$. For any n-tuple $\underline{\beta}$ of vectors in $\mathfrak{a}_M$ let $M(\underline{\beta})$ be the transition matrix from $\Delta_0^\vee$ to $\underline{\beta}$. We claim that $\mathcal{G}_1(\iota_{\beta^\vee,i}(\underline{\beta}')) = \mathcal{G}_k'$ where k is determined by the sign of the determinant of $M(\iota_{\beta^\vee,i}(\underline{\beta}'))$.

Indeed, let $0 \neq v_1 \in \mathfrak{a}_M^L \cap \operatorname{span}(\{\beta_j^\vee\}_{j \neq i})$. The sign of $\det M(\iota_{w,i}(\underline{\beta}'))$, $w \in \mathfrak{a}_M^L$, is determined by $w \circlearrowleft v_1$. On the other hand, $\mathcal{G}_1(\iota_{\beta^\vee,i}(\underline{\beta}'))$ is determined by its projection to L. By Lemma 2 the orientation of the latter differs from $\beta^\vee \circlearrowleft v_1$ by the sign of

$$\begin{vmatrix} \left\langle \mu_{\iota_{\beta^\vee,i}(\underline{\beta}')}(\underline{\xi}^{(1)}), \beta^\vee \right\rangle & \langle \lambda_0, \beta^\vee \rangle \\ \left\langle \mu_{\iota_{\beta^\vee,i}(\underline{\beta}')}(\underline{\xi}^{(1)}), v_1 \right\rangle & \langle \lambda_0, v_1 \rangle \end{vmatrix} = \langle \lambda_0, \beta^\vee \rangle \, (\xi_i^{(1)} \langle \lambda_0, v_1 \rangle - \left\langle \mu_{\iota_{\beta^\vee,i}(\underline{\beta}')}(\underline{\xi}^{(1)}), v_1 \right\rangle),$$

which is independent of β since $\langle \lambda_0, \beta^\vee \rangle > 0$ and $v_1 \in \operatorname{span}(\{\beta_j^\vee\}_{j \neq i})$.

Therefore, up to a sign, (11) is equal to

$$\sum_{\beta \in \Sigma_{P_0}^L} \det M(\iota_{\beta^\vee,i}(\underline{\beta}')) \varpi_{\underline{\beta}' \setminus \underline{\beta}_0} \varpi_{\beta^\vee} (\mathbf{1}_{\mathcal{G}_1'} - \mathbf{1}_{\mathcal{G}_2'}).$$

This belongs to $\mathcal{R}_{\mathcal{G}_1', \mathcal{G}_2'}$, since $\beta^\vee \mapsto \det M(\iota_{\beta^\vee,i}(\underline{\beta}'))$ is the restriction of a linear functional on $\mathfrak{a}_M^L$. □

5. The induction argument

We now turn to the inductive proof of Theorems 1 and 2. For $\alpha \in \Sigma_0$ consider the set $\mathcal{L}^{\hat{\alpha}}(M)$ of Levi subgroups $L \in \mathcal{L}(M)$ of co-rank one in G such that $\alpha \notin \Sigma_0^L$. For $L \in \mathcal{L}^{\hat{\alpha}}(M)$ we write $\mathcal{P}(L) = \{Q_L, \overline{Q_L}\}$ where $\alpha \notin \Sigma^{Q_L}$ and denote by $\varpi_L \in \mathfrak{a}_L^*$ the vector satisfying $\langle \varpi_L, \beta^\vee \rangle = 1$ where $\Delta_{Q_L} = \{\beta\}$. The set $\{Q_L : L \in \mathcal{L}^{\hat\alpha}(M)\}$ consists of the maximal parabolic subgroups Q of G containing M such that $\alpha \notin \Sigma^Q$.

The motivation for our proof strategy is that the difference $c_M(\mathcal{F}^\alpha) - c_M(\mathcal{F})$ for an intertwining family $\mathcal{F}$ can be expressed in terms of limits c_M^L for the groups $L \in \mathcal{L}^{\hat\alpha}(M)$. In our algebraic setup, this translates into the consideration of $\partial_{\Theta_{\alpha^\vee}} \mathfrak{c}$ and $\partial_{\Theta_{\alpha^\vee}} \mathfrak{d}$. In virtue of Lemma 5 and Proposotion 1 we may and will suppres the dependence of $\mathfrak{c}$ and $\mathfrak{d}$ on their auxiliary parameters from the notation.

Proposition 2. *For $\alpha \in \Sigma_0$ we have*

$$\partial_{\Theta_{\alpha^\vee}} \mathfrak{c}^G \equiv \sum_{L \in \mathcal{L}^{\hat\alpha}(M)} \langle \varpi_L, \alpha^\vee \rangle \, \phi^G_{L;\varpi_L}(\mathfrak{c}^L) \pmod{\mathcal{R}_{n-1}}. \tag{12}$$

Proof. For any $P \in \mathcal{P}(M)$ and $\beta \in \Delta_P$ let P_β be the maximal parabolic subgroup of G containing P defined by β, i.e. such that $\Delta_P \setminus \Sigma_P^{P_\beta} = \{\beta\}$. We first claim that

$$\partial_{\Theta_{\alpha^\vee}} \operatorname{pr}^*_{P_0;P}(\eta^n) = n \sum_{\beta \in \Delta_P : \alpha \notin \Sigma^{P_\beta}} \left\langle \varpi_{P_\beta}, \alpha^\vee \right\rangle \langle \eta, \beta^\vee \rangle \operatorname{pr}^*_{P_0;P}(\eta^{n-1}). \tag{13}$$

Both sides are zero if $\alpha \in \Sigma_P$. Otherwise, the condition $\alpha \notin \Sigma^{P_\beta}$ is equivalent to $\alpha \notin \Sigma^{L_\beta}$, where $L_\beta \in \mathcal{L}(M)$ is the Levi subgroup of P_β. Since ϖ_{P_β}, $\beta \in \Delta_P$, is the basis of $\mathfrak{a}_M^*$ dual to Δ_P, we have

$$\sum_{\beta \in \Delta_P : \alpha \notin \Sigma^{P_\beta}} \left\langle \varpi_{P_\beta}, \alpha^\vee \right\rangle \beta^\vee = \alpha^\vee.$$

The equality (13) reduces in this case to the chain rule identity

$$\partial_{\Theta_{\alpha^\vee}} \operatorname{pr}^*_{P_0;P}(\eta^n) = n \, \langle \eta, \alpha^\vee \rangle \operatorname{pr}^*_{P_0;P}(\eta^{n-1}).$$

Using (13) and the relation

$$\theta_P(\eta) = \theta^{L_\beta}_{P^{L_\beta}}(\eta) \, \langle \eta, \beta^\vee \rangle$$

for $\beta \in \Delta_P$, we may now write the left-hand side of (12) as the sum over $L \in \mathcal{L}^{\hat\alpha}(M)$ of

$$\frac{1}{(n-1)!} \sum_{P \in \mathcal{P}(M), \beta \in \Delta_P : P_\beta = Q_L} \langle \varpi_L, \alpha^\vee \rangle \, \frac{\operatorname{pr}^*_{P_0;P}(\eta^{n-1})}{\theta^L_{P^L}(\eta)} \mathbf{1}_{\mathcal{G}(\mu_P)}.$$

Note that the sum is over $P \in \mathcal{P}(M)$ such that $P \subset Q_L$. It remains to show that for any $L \in \mathcal{L}^{\hat\alpha}(M)$ we have

$$\phi^G_{L;\varpi_L}(\mathfrak{c}^L) \equiv \frac{1}{(n-1)!} \sum_{P \in \mathcal{P}(M),\, P \subset Q_L} \frac{\operatorname{pr}^*_{P_0;P}(\eta^{n-1})}{\theta^L_{P^L}(\eta)} \mathbf{1}_{\mathcal{G}(\mu_P)} \pmod{\mathcal{R}_{n-1}} \tag{14}$$

for any $\eta \in \mathfrak{a}_M^*$ and $\mu_P \in \mathfrak{a}^*_{P,+}$ in general position. To show this, we first observe that

$$\phi^G_{L;\varpi_L}(\mathfrak{c}^L) = \phi^G_{L;\varpi_L}\Big(\frac{1}{(n-1)!} \sum_{P \in \mathcal{P}^L(M)} \frac{\operatorname{pr}^*_{P_0^L;P}(\eta^{n-1})}{\theta_P(\eta)} \mathbf{1}_{\mathcal{G}^L(\mu_P)}\Big)$$

with $\mu_P \in (\mathfrak{a}^L_{P,+})^*$, which we can assume to be close to the origin. Note that this expression depends only on η^L, and is therefore valid for any $\eta \in \mathfrak{a}^*_M$. Also, $\psi^G_{L;\varpi_L}(\mathcal{G}^L(\mu_P)) = \mathcal{G}^G(\varpi_L + \mu_P)$. Therefore,

$$\phi^G_{L;\varpi_L}(\mathfrak{c}^L) = \frac{1}{(n-1)!} \sum_{P \in \mathcal{P}^L(M)} \frac{(\mathrm{pr}_{P_0^L;P}\, \pi_L)^*(\eta^{n-1})}{\theta_P(\eta)} \mathbf{1}_{\mathcal{G}^G(\varpi_L+\mu_P)}$$

$$= \frac{1}{(n-1)!} \sum_{P \in \mathcal{P}^L(M)} \frac{\mathrm{pr}^*_{Q_L(P_0^L);Q_L(P)}(\eta^{n-1})}{\theta_P(\eta)} \mathbf{1}_{\mathcal{G}^G(\varpi_L+\mu_P)}.$$

Since $\varpi_L + \mu_P \in \mathfrak{a}^*_{Q_L(P),+}$, in order to obtain (14) it remains to show that we can replace $\mathrm{pr}^*_{Q_L(P_0^L);Q_L(P)}$ in the expression above by $\mathrm{pr}^*_{P_0;Q_L(P)}$ without changing its value modulo $\mathcal{R}_{n-1}$. We have $\Sigma_{Q_L(P_0^L)} \cap \Sigma_{\overline{Q_L(P)}} \subset \Sigma_0^L \subset \Sigma_0$, and therefore

$$\mathrm{pr}^*_{P_0;Q_L(P)}(\eta^{n-1}) = \sum_{k=0}^{n-1} \binom{n-1}{k} \mathrm{pr}^*_{P_0;Q_L(P_0^L)}(\eta^k)\, \mathrm{pr}^*_{Q_L(P_0^L);Q_L(P)}(\eta^{n-1-k}).$$

We are reduced to showing that modulo $\mathcal{R}_{n-1}$ only the term $k = 0$ contributes. It follows from the second part of Lemma 5 applied to L that

$$\sum_{P \in \mathcal{P}^L(M)} \frac{\mathrm{pr}^*_{P_0^L;P}(\eta^{n-1-k})}{\theta_P(\eta)} \mathbf{1}_{\mathcal{G}^L(\mu_P)} \in \mathcal{R}^L_{n-1-k}$$

for $k > 0$. Using (7), we infer that

$$\mathrm{pr}^*_{P_0;Q_L(P_0^L)}(\eta^k) \sum_{P \in \mathcal{P}^L(M)} \frac{\mathrm{pr}^*_{Q_L(P_0^L);Q_L(P)}(\eta^{n-1-k})}{\theta_P(\eta)} \mathbf{1}_{\mathcal{G}^G(\varpi_L+\mu_P)} \in \mathcal{R}_{n-1}.$$

Thus, only the term $k = 0$ contributes, which yields (14). □

The analogous result is true for the right hand side.

Proposition 3. *For $\alpha \in \Sigma_0$ we have*

$$\partial_{\Theta_{\alpha^\vee}} \mathfrak{d}^G \equiv \sum_{L \in \mathcal{L}^{\hat{\alpha}}(M)} \langle \varpi_L, \alpha^\vee \rangle\, \phi^G_{L;\varpi_L}(\mathfrak{d}^L) \pmod{\mathcal{R}_{n-1}}. \tag{15}$$

Proof. The left-hand side is given by

$$\frac{(-1)^n}{n!} \sum_{\underline{\beta} \in \mathfrak{B}_{P_0} :\, \alpha^\vee \in \underline{\beta}} \mathrm{vol}(\underline{\beta}) \varpi_{\underline{\beta} \setminus \{\alpha^\vee\}} \mathbf{1}_{\mathcal{G}(\mu_{\underline{\beta}}(\underline{\xi}))}.$$

We write this as the sum over $j = 1, \ldots, n$ of

$$\frac{(-1)^n}{n!} \sum_{\underline{\beta} \in \mathfrak{B}^\alpha_{P_0}} \mathrm{vol}(\iota_j(\underline{\beta})) \varpi_{\underline{\beta}} \mathbf{1}_{\mathcal{G}(\mu_{\iota_j(\underline{\beta})}(\underline{\xi}))}, \tag{16}$$

where $\mathfrak{B}^\alpha_{P_0}$ is the set of $(n-1)$-tuples $\underline{\beta}$ of elements of $\Sigma_0^\vee$ such that $\iota_j(\underline{\beta}) \in \mathfrak{B}_{P_0}$, $\iota_j(\underline{\beta}) = \iota_{\alpha^\vee,j}(\underline{\beta})$ denoting the n-tuple obtained from $\underline{\beta}$ by inserting $\alpha^\vee$ in the

j-th position. By Proposition 1 applied with $I = \{j\}$ and $\alpha_j = \alpha$, for each j (16) does not depend on $\underline{\xi}$ modulo $\mathcal{R}_{n-1}$. In particular, in (16) we can assume that $\xi_j < 0$ and $|\xi_j| \gg |\xi_i|$ for all $i \neq j$. We group together the summands according to the linear span of $\underline{\beta}$. The latter is of the form $\mathfrak{a}_M^L$ for a uniquely determined $L \in \mathcal{L}^{\hat{\alpha}}(M)$. Note that $\underline{\beta} \in \mathfrak{B}_{P_0^L}^L$ (which we abbreviate as $\mathfrak{B}_0^L$) and $\operatorname{vol}(\iota_j(\underline{\beta})) = -\langle \varpi_L, \alpha^\vee \rangle \operatorname{vol}^L(\underline{\beta})$, since the co-root lattice is generated by $\Delta_{Q_L(P_0^L)}^\vee$. Therefore we can write (16) as

$$\frac{1}{n} \sum_{L \in \mathcal{L}^{\hat{\alpha}}(M)} \langle \varpi_L, \alpha^\vee \rangle \frac{(-1)^{n-1}}{(n-1)!} \sum_{\underline{\beta} \in \mathfrak{B}_0^L} \operatorname{vol}^L(\underline{\beta}) \varpi_{\underline{\beta}} \mathbf{1}_{\mathcal{G}(\mu_{\iota_j(\underline{\beta})}(\underline{\xi}))}.$$

It remains to show that

$$\frac{(-1)^{n-1}}{(n-1)!} \sum_{\underline{\beta} \in \mathfrak{B}_0^L} \operatorname{vol}^L(\underline{\beta}) \varpi_{\underline{\beta}} \mathbf{1}_{\mathcal{G}(\mu_{\iota_j(\underline{\beta})}(\underline{\xi}))} = \phi_{L;\varpi_L}^G(\mathfrak{d}_{\underline{\xi}^{(j)}}^L)$$

where $\underline{\xi}^{(j)}$ denotes the $(n-1)$-tuple obtained from $\underline{\xi}$ by deleting ξ_j, which amounts to showing that $\mathcal{G}(\mu_{\iota_j(\underline{\beta})}(\underline{\xi})) = \psi_{L;\varpi_L}^G(\mathcal{G}^L(\mu_{\underline{\beta}}^L(\underline{\xi}^{(j)})))$. This follows from the fact that the projection of $\mu_{\iota_j(\underline{\beta})}(\underline{\xi})$ to $(\mathfrak{a}_M^L)^*$ is $\mu_{\underline{\beta}}^L(\underline{\xi}^{(j)})$, whereas $\left\langle \mu_{\iota_j(\underline{\beta})}(\underline{\xi}), \beta^\vee \right\rangle \gg 0$ if $\Delta_{Q_L} = \{\beta\}$ by the assumption on ξ_j. $\square$

Proof of Theorems 1 and 2. We show that $\mathfrak{e} = \mathfrak{c} - \mathfrak{d} \in \mathcal{R}$ by induction on n. The case $n = 1$ is covered by Remark 1 in §2. Using (12), (15) and the induction hypothesis, we obtain that $\partial_{\Theta_{\alpha^\vee}} \mathfrak{e} \in \mathcal{R}$ for all $\alpha \in \Sigma_{P_0}$. According to (8) the conditions of Lemma 3 are satisfied for $R = \mathcal{R}$ with $K = n$. Applying it to $u = \mathfrak{e}$ we infer that $\mathfrak{e} \in \mathcal{R}$, and a fortiori $\mathfrak{e} \in \mathcal{K}$, as required. $\square$

Proof of Corollary 1. We prove (3) by induction on $k = n - m$, the case $k = 0$ being Theorem 1. Let f be the (G, M)-family given by

$$f_P(\lambda) = \prod_{\alpha \in \Sigma_{\overline{P}} \cap \Delta_0^L} (1 + \langle \lambda, \alpha^\vee \rangle).$$

Using [Art82b, Lemma 7.1] and the product formula [Art81, Corollary 6.5] we have

$$(fc)_M = \sum_{A \subset \Delta_0^L} (-1)^{|A|} c_{M_A}, \tag{17}$$

where $M_A \in \mathcal{L}^L(M)$ is such that $\mathfrak{a}_M^{M_A}$ is spanned by A. On the other hand, we can compute $(fc)_M$ using Theorem 1 applied to the (G, M)-intertwining family

$$\tilde{\mathcal{F}}_{P_2|P_1}(\lambda) = \prod_{\alpha \in \Delta_0^L} (1 + \langle \lambda, \alpha^\vee \rangle)^{\mathbf{1}_{\Sigma_{P_1}}(\alpha) - \mathbf{1}_{\Sigma_{P_2}}(\alpha)} \mathcal{F}_{P_2|P_1}(\lambda).$$

We obtain

$$(fc)_M = \frac{(-1)^n}{n!} \sum_{A \subset \Delta_0^L} \sum_{\underline{\beta} \in \mathfrak{B}_{P_0} : \underline{\beta} \supset A} \mathrm{vol}(\underline{\beta}) \partial_{\mathcal{G}^{\mathrm{lex}}_{\underline{\beta}}}^{\mathbf{1}_{\underline{\beta} \setminus A}} (\mathcal{F}). \tag{18}$$

Comparing (17) and (18) and using the induction hypothesis we obtain (3), and therefore Corollary 1. □

Remark 5. We can rewrite Corollary 1 as the equality

$$c_L(\mathcal{F}; P_0) = \frac{(-1)^m}{m!} \sum_{\substack{\underline{\beta} = (\beta_1^\vee, \dots, \beta_n^\vee) \in \mathfrak{B}_{P_0} : \\ \beta_i = \alpha_i \quad i = 1, \dots, k}} \mathrm{vol}(\underline{\beta}) \partial_{\mathcal{G}^{\mathrm{lex}}_{\underline{\beta}}}^{\mathbf{1}_{\{\beta_{k+1}^\vee, \dots, \beta_n^\vee\}}} (\mathcal{F}),$$

where $\Delta_0^L = \{\alpha_1, \dots, \alpha_k\}$. Indeed, by Proposition 1 the partial sums in (3) pertaining to each of the $\frac{n!}{m!}$ placements of $\alpha_1, \dots, \alpha_k$ in $\underline{\beta}$ are all equal.

6. Absolute convergence of the spectral side

Let now G be a reductive group defined over a number field F and let $\mathbb{A}$ be the ring of adeles of F. We consider the spectral side of Arthur's trace formula whose fine expansion was obtained in [Art82b]. For a test function $f \in C_c^\infty(G(\mathbb{A})^1)$ it is given by an absolutely convergent sum

$$\sum_{\chi \in \mathfrak{X}} J_\chi(f)$$

where χ ranges over cuspidal data of G, that is over $G(F)$-conjugacy classes of pairs (M, π) consisting of a Levi subgroup M defined over F and an irreducible cuspidal representation of $M(\mathbb{A})^1$.

To describe the distributions J_χ in a convenient way we first recall, with some minor modifications, additional notation from [Art82a] and [Art82b]. Fix a maximal F-split torus T_0 and let M_0 be its centralizer, which is a minimal Levi defined over F. Denote by A_0 the connected component of the identity of $T_0(\mathbb{R})$ where $\mathbb{R}$ is embedded in $\mathbb{A}$ through $\mathbb{R} \hookrightarrow F_\infty = \mathbb{R} \otimes_{\mathbb{Q}} F$. We also fix a maximal compact subgroup $K = K_\infty K_f$ of $G(\mathbb{A}) = G(F_\infty) G(\mathbb{A}_f)$ which is admissible with respect to M_0. The Weyl group W_0 of (G, T_0) acts on the set $\mathcal{L}(M_0)$ by conjugation. Let $M \in \mathcal{L}(M_0)$. Denote by $T_M = T_0 \cap Z(M)$ the split part of the center $Z(M)$ of M and let $A_M = A_0 \cap T_M$. We have $M(\mathbb{A}) = M(\mathbb{A})^1 \times A_M$ and identifying A_M/A_G with $\mathfrak{a}_M$ we obtain a homomorphism $H_M : M(\mathbb{A}) \to \mathfrak{a}_M$ which is trivial on $M(\mathbb{A})^1$. For any $P \in \mathcal{P}(M)$ (in which case we write $\mathfrak{a}_P = \mathfrak{a}_M$), let $\mathcal{A}^2(P)$ be the space of automorphic forms ϕ on $N_P(\mathbb{A}) M(F) \backslash G(\mathbb{A})$ such that for all $x \in G(\mathbb{A})$ the function $\phi_x(m) := \delta_P(m)^{-\frac{1}{2}} \phi(mx)$ belongs to $L^2(A_M M(F) \backslash M(\mathbb{A}))$ where δ_P is the modulus function of $P(\mathbb{A})$. The decomposition

$$L^2(M(F) \backslash M(\mathbb{A})^1) = \hat{\bigoplus} L^2(M(F) \backslash M(\mathbb{A}))_\chi$$

according to cuspidal data gives rise to a decomposition $\mathcal{A}^2(P) = \oplus \mathcal{A}^2_\chi(P)$. Let $\bar{\mathcal{A}}^2(P)$ be the Hilbert space completion of $\mathcal{A}^2(P)$ with respect to the inner product

$$(\phi_1, \phi_2) = \int_{A_M M(F) N_P(\mathbb{A}) \backslash G(\mathbb{A})} \phi_1(g) \overline{\phi_2(g)} \, dg.$$

The map H_M uniquely extends to a left $N_P(\mathbb{A})$ and right K-invariant map $H_P : G(\mathbb{A}) \to \mathfrak{a}_M$. We endow $i\mathfrak{a}_M^*$ with the Haar measure which is dual to the one on $\mathfrak{a}_M$ for which the co-root lattice has co-volume one. We denote by $\rho(P, \lambda)$, $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$, the induced representation of $G(\mathbb{A})$ on $\bar{\mathcal{A}}^2(P)$ given by

$$(\rho(P, \lambda, y)\phi)(x) = \phi(xy) e^{\langle \lambda, H_P(xy) \rangle} e^{-\langle \lambda, H_P(x) \rangle}.$$

It is isomorphic to the representation parabolically induced from the representation on $M(\mathbb{A})$ which, via the decomposition $M(\mathbb{A}) = M(\mathbb{A})^1 \times A_M$, is the tensor product of the discrete part of $L^2(M(F)\backslash M(\mathbb{A})^1)$ (i.e. the sum of all irreducible subrepresentations) with the character $e^{\langle \lambda, H_M(\cdot) \rangle}$ of A_M.

Given $M' \in \mathcal{L}(M_0)$, let $W(\mathfrak{a}_M, \mathfrak{a}_{M'})$ be the set of all linear isomorphisms from $\mathfrak{a}_M$ to $\mathfrak{a}_{M'}$ which are restrictions of elements of W_0. The set $W(\mathfrak{a}_M) = W(\mathfrak{a}_M, \mathfrak{a}_M)$ can be identified with the quotient of the stabilizer of M in W_0 by the Weyl group W_0^M of (M, T_0). For $P \in \mathcal{P}(M)$, $Q \in \mathcal{P}(M')$ and $s \in W(\mathfrak{a}_P, \mathfrak{a}_Q) = W(\mathfrak{a}_M, \mathfrak{a}_{M'})$ let

$$M_{Q|P}(s, \lambda) : \mathcal{A}^2(P) \to \mathcal{A}^2(Q), \quad \lambda \in \mathfrak{a}^*_{M,\mathbb{C}},$$

be the intertwining operator [Art82b, §1], which is the meromorphic continuation in λ of

$$\int_{N_Q(\mathbb{A}) \cap w_s N_P(\mathbb{A}) w_s^{-1} \backslash N_Q(\mathbb{A})} \phi(w_s^{-1} n x) e^{\left\langle \lambda, H_P(w_s^{-1} n x) \right\rangle} e^{-\left\langle s\lambda, H_Q(x) \right\rangle} \, dn,$$

where $\phi \in \mathcal{A}^2(P)$ and $x \in G(\mathbb{A})$. In particular, for $P, Q \in \mathcal{P}(M)$ set

$$M_{Q|P}(\lambda) := M_{Q|P}(1, \lambda).$$

Suppose that $t \in W_0$ and $P \in \mathcal{P}(M)$. Let $tM = w_t M w_t^{-1} \in \mathcal{L}(M_0)$ and $tP = w_t P w_t^{-1} \in \mathcal{P}(tM)$, so that $t \in W(\mathfrak{a}_M, \mathfrak{a}_{tM})$. The map $t : \mathcal{A}^2(P) \to \mathcal{A}^2(tP)$ given by $t\phi(x) = \phi(w_t^{-1} x)$ is an isometry which intertwines $\rho(P, \lambda)$ with $\rho(tP, t\lambda)$. It also satisfies $t\mathcal{A}^2_\chi(P) = \mathcal{A}^2_\chi(tP)$ for all $\chi \in \mathfrak{X}$ and $tM_{Q|P}(s, \lambda) = M_{tQ|tP}(s, t\lambda)t$ for all $s \in W(\mathfrak{a}_P, \mathfrak{a}_Q)$ and $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$ such that $s\lambda = \lambda$ (cf. [Art82b, (1.4), (1.5)]).

Fix $P \in \mathcal{P}(M)$ and $\lambda \in i\mathfrak{a}_M^*$. For $Q \in \mathcal{P}(M)$ and $\Lambda \in i\mathfrak{a}_M^*$ define

$$\mathcal{M}_Q(P, \lambda, \Lambda) = M_{Q|P}(\lambda)^{-1} M_{Q|P}(\lambda + \Lambda).$$

Then

$$\{\mathcal{M}_Q(P, \lambda, \Lambda) \mid \Lambda \in i\mathfrak{a}_M^*, \ Q \in \mathcal{P}(M)\}$$

is a (G, M)-family with values in the space of operators on $\mathcal{A}^2(P)$ [Art82b, p. 1310]. Therefore, for any $L \in \mathcal{L}(M)$ and $Q \in \mathcal{P}(L)$ the restriction $\mathcal{M}_Q(P, \lambda, \cdot)$

of $\mathcal{M}_{Q_1}(P,\lambda,\cdot)$ to $\mathrm{i}\mathfrak{a}_L^*$ does not depend on $Q_1 \in \mathcal{P}(M)$ provided that $Q_1 \subset Q$, and the limit

$$\mathcal{M}_L(P,\lambda) = \lim_{\substack{\Lambda\in\mathrm{i}\mathfrak{a}_L^*\\ \Lambda\to 0}} \sum_{Q\in\mathcal{P}(L)} \frac{\mathcal{M}_Q(P,\lambda,\Lambda)}{\theta_Q(\Lambda)}$$

exists.

Recall that any $s \in W(\mathfrak{a}_M)$ uniquely determines a Levi $L(s) \in \mathcal{L}(M)$ such that $\{H \in \mathfrak{a}_M \mid sH = H\} = \mathfrak{a}_{L(s)}$ (cf. [OT92, Theorem 6.27]). In fact $L(s)$ is the smallest $L \in \mathcal{L}(M)$ such that $s \in W^L$ (cf. [Art05, p. 129]). We set

$$\iota_s = \left|\det(s-1)_{\mathfrak{a}_M^{L(s)}}\right|^{-1}.$$

Note that for any $t \in W_0$ and $s \in W(\mathfrak{a}_M)$ we have $tst^{-1} \in W(\mathfrak{a}_{tM})$, $L(tst^{-1}) = tL(s)$ and

$$\mathcal{M}_{L(tst^{-1})}(tP,t\lambda)M_{tP|tP}(tst^{-1},0)\rho(tP,t\lambda,h)t = t\mathcal{M}_{L(s)}(P,\lambda)M_{P|P}(s,0)\rho(P,\lambda,h)$$

for all $\lambda \in \mathrm{i}\mathfrak{a}_L^*$. Also, for all $Q \in \mathcal{P}(L)$, $P' \in \mathcal{P}(M)$ we have

$$M_{P|P'}(\lambda)\mathcal{M}_Q(P',\lambda,\Lambda) = \mathcal{M}_Q(P,\lambda,\Lambda)M_{P|P'}(\lambda+\Lambda)$$

and therefore

$$M_{P|P'}(\lambda)\mathcal{M}_{L(s)}(P',\lambda)M_{P'|P'}(s,0)\rho(P',\lambda,h) \\ = \mathcal{M}_{L(s)}(P,\lambda)M_{P|P}(s,0)\rho(P,\lambda,h)M_{P|P'}(\lambda).$$

Since the orbit of M under W_0 is of size $\frac{|W_0|}{|W_0^M|}|W(\mathfrak{a}_M)|^{-1}$, we can reformulate [Art82b, Theorems 8.1 and 8.2] (cf. [Art05, p. 137]) as follows. For any bi-K-finite $h \in C_c^\infty(G(\mathbb{A})^1)$ we have

(19)
$$J_\chi(h) = \sum_{[P],s\in W(\mathfrak{a}_P)} \frac{\iota_s}{|W(\mathfrak{a}_P)|} \int_{\mathrm{i}\mathfrak{a}_{L(s)}^*} \operatorname{tr}\left(\mathcal{M}_{L(s)}(P,\lambda)M_{P|P}(s,0)\rho(P,\lambda,h)|_{\bar{\mathcal{A}}_\chi^2(P)}\right)\,d\lambda,$$

where P ranges over parabolic subgroups up to association, and the integral is absolutely convergent with respect to the trace norm $\|\cdot\|_1$. Implicit here is that the operator $\mathcal{M}_{L(s)}(P,\lambda)\rho(P,\lambda,h)$ extends to a trace class operator on $\bar{\mathcal{A}}_\chi^2(P)$. (Note that $M_{P|P}(s,0)$ commutes with $\rho(P,\lambda,h)$ for $\lambda \in \mathrm{i}\mathfrak{a}_{L(s)}^*$.) Our goal is to rewrite the integrand on the right-hand side of (19) in terms of first-order derivatives of co-rank one intertwining operators. Summing over χ we will obtain a refined spectral expansion which is valid for a larger class of test functions and for which the expression is absolutely convergent with respect to the trace norm.

Fix an open subgroup K_0 of K_f. The space $K_0\backslash G(\mathbb{A})^1/K_0$ is a discrete union of countably many copies of $G(\mathbb{A})^1 \cap G(F_\infty)$ and in particular, it is a differentiable manifold. Let $C^\infty(G(\mathbb{A})^1, K_0)$ be the space of smooth functions on $K_0\backslash G(\mathbb{A})^1/K_0$, viewed as bi-$K_0$-invariant functions on $G(\mathbb{A})^1$. Let $U(\mathfrak{g}_\mathbb{C}^1)$ be the universal enveloping algebra of the complexified Lie algebra of $G(F_\infty) \cap G(\mathbb{A})^1$. Let $\mathcal{C}(G(\mathbb{A})^1, K_0)$ be the topological vector space of $h \in C^\infty(G(\mathbb{A})^1, K_0)$ such that $|X * h * Y|_{L^1(G(\mathbb{A})^1)} < \infty$ for all $X, Y \in U(\mathfrak{g}^1_{\mathbb{C}})$. For any $h \in \mathcal{C}(G(\mathbb{A})^1, K_0)$, the image of $\rho(P, \lambda, h)$ lies in the smooth and K_0-invariant part of $\bar{\mathcal{A}}^2(P)$.

Let $\Pi(M(\mathbb{A}))$ be the set of equivalence classes of irreducible unitary representations of $M(\mathbb{A})$. For $\pi \in \Pi(M(\mathbb{A}))$ let $\mathcal{A}^2_\pi(P)$ be the subspace of all $\phi \in \mathcal{A}^2(P)$ such that for each $x \in G(\mathbb{A})$, ϕ_x transforms under $M(\mathbb{A})$ according to the representation π. In particular, $\mathcal{A}^2_\pi(P) = 0$ unless π is trivial on A_M. We have a canonical isomorphism of $G(\mathbb{A}_f) \times (\mathfrak{g}_{\mathbb{C}}, K_\infty)$-modules

$$j_P : \operatorname{Hom}(\pi, L^2(A_M M(F)\backslash M(\mathbb{A}))) \otimes \operatorname{Ind}(\pi) \to \mathcal{A}^2_\pi(P).$$

Let $\mathcal{A}^2_\pi(P)_{K_0}$ be the subspace of K_0-invariant functions in $\mathcal{A}^2_\pi(P)$. Its closure $\bar{\mathcal{A}}^2_\pi(P)_{K_0}$ in $\bar{\mathcal{A}}^2(P)$ is a unitary representation of $G(F_\infty)$ which is isomorphic to finitely many copies of $I(\pi_\infty)$. The smooth part $\bar{\mathcal{A}}^2_\pi(P)^\infty_{K_0}$ of $\bar{\mathcal{A}}^2_\pi(P)_{K_0}$ is a Fréchet space with respect to the semi-norms

$$\sup_{k \in K} \|(X\phi)_k\|_{L^2(M(F)\backslash M(\mathbb{A})^1)}, \qquad X \in U(\mathfrak{g}_{\mathbb{C}}).$$

Let

$$\Pi_{\mathrm{disc}}(M(\mathbb{A}); K_0) = \{\pi \in \Pi(M(\mathbb{A})) : \mathcal{A}^2_\pi(P)_{K_0} \neq 0\}.$$

Recall the local normalized intertwining operators $R_{Q|P}(\pi_v, \lambda)$ of [Art82b, §6]. If $R_{Q|P}(\pi, \lambda) = \otimes_v R_{Q|P}(\pi_v, \lambda)$ then the intertwining operator on $\mathcal{A}^2_\pi(P)$ admits a factorization

$$M_{Q|P}(\lambda) \circ j_P = r_{Q|P}(\pi, \lambda) \cdot j_Q \circ (\operatorname{Id} \otimes R_{Q|P}(\pi, \lambda))$$

where $r_{Q|P}(\pi, \lambda)$ is the global normalizing factor (cf. [Mül02, (2.17)]).

Let $\mathcal{F}$ be the intertwining family defined by $\mathcal{F}_P = \mathcal{A}^2(P)$ and $\mathcal{F}_{Q|P}(\lambda) = M_{Q|P}(\lambda)$, $Q, P \in \mathcal{P}(M)$. Technically, the spaces $\mathcal{F}_P$ are not finite-dimensional. However we may restrict $\mathcal{F}$ to any K- and $\mathfrak{z}$-type (where $\mathfrak{z}$ is the center of $U(\mathfrak{g}_{\mathbb{C}})$). Therefore, Corollary 1 applies.

Suppose that $P|^\alpha Q$. Then $M_{Q|P}(\lambda)$ depends only on $s = \langle \lambda, \alpha^\vee \rangle$ and we denote its restriction to $\mathcal{A}^2_\pi(P)$ by $M_{Q|P}(\pi, s)$. By the discussion following [Lap07, Corollary 2] there exists a discrete set $X_\pi \subset \mathrm{i}\mathbb{R}$ such that for all $\phi \in \bar{\mathcal{A}}^2_\pi(P)^\infty_{K_0}$ the function $M_{Q|P}(\pi, \cdot)\phi$ is holomorphic (with values in $\bar{\mathcal{A}}^2_\pi(P)^\infty_{K_0}$) for $s \in \mathrm{i}\mathbb{R} \setminus X_\pi$. (In fact, it is possible to take $X_\pi = \emptyset$ but we will not need to use this fact.)

The main technical statement is the following Theorem.

Theorem 3. *Fix K_0 and let $M \in \mathcal{L}(M_0)$, $L \in \mathcal{L}(M)$, $P \in \mathcal{P}(M)$, $\mathcal{G}$ a gallery with respect to P and $\underline{\beta} \subset \Sigma^\vee_P$ such that $\underline{\beta} \cup (\Delta^L_M)^\vee$ forms a basis of $\mathfrak{a}_M$. Then the semi-norm*

$$\int_{\mathrm{i}\mathfrak{a}^*_L} \|\partial^{\mathbf{1}_{\underline{\beta}}}_{\mathcal{G}}(\mathcal{F})(\lambda)\rho(P, \lambda, h)\|_1 \, d\lambda$$

on $\mathcal{C}(G(\mathbb{A})^1, K_0)$ is continuous.

Note that by the above, for almost all $\lambda \in \mathrm{i}\mathfrak{a}^*_L$, the operator $\partial^{\mathbf{1}_{\underline{\beta}}}_{\mathcal{G}}(\mathcal{F})(\lambda)$ is defined on $\oplus_{\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A}); K_0)} \bar{\mathcal{A}}^2_\pi(P)^\infty_{K_0}$ (algebraic direct sum), and therefore, $\partial^{\mathbf{1}_{\underline{\beta}}}_{\mathcal{G}}(\mathcal{F})(\lambda)\rho(P, \lambda, h)$ is defined on the dense subspace $\oplus_\pi \bar{\mathcal{A}}^2_\pi(P)$ (algebraic direct sum) of $\bar{\mathcal{A}}^2(P)$. Implicit in Theorem 3 is that for almost all λ, $\partial_{\mathcal{G}}^{\mathbf{1}_{\underline{\beta}}}(\mathcal{F})(\lambda)\rho(P,\lambda,h)$ extends to a trace class operator on $\bar{\mathcal{A}}^2(P)$.

Remark 6. The case $P = G$ essentially amounts to the assertion that $\rho(G,h)$ is of trace class. This is the trace-class conjecture of Selberg which was settled in [Mül89] for K-finite test functions and in general in [Mül98] and independently by Ji [Ji98].

Let $\mathcal{C}(G(\mathbb{A})^1)$ be the inductive limit of $\mathcal{C}(G(\mathbb{A})^1, K_0)$ over the open subgroups K_0 of K_f. Using (19), Theorem 3 and Corollary 1 we get the following consequence which is our main result. (The passage from bi-K-finite functions to compactly supported functions is explained in [Art82b, p. 1326] using [Art82a, Proposition 2.3].)

Corollary 2. *For any $h \in C_c^\infty(G(\mathbb{A})^1)$ the spectral side of Arthur's trace formula is given by*

$$\sum_{[P]} \frac{1}{|W(\mathfrak{a}_P)|} \sum_{s \in W(\mathfrak{a}_P)} \iota_s \int_{\mathrm{i}\mathfrak{a}^*_{L(s)}} \operatorname{tr}(\mathcal{M}_L(P,\lambda) M_{P|P}(s,0)\rho(P,\lambda,h)) \, d\lambda.$$

Choosing strongly regular $\lambda_0(P) \in \mathfrak{a}^*_{P,+}$ for each associate class $[P]$, we can also write it as

$$\sum_{[P]} \frac{1}{n(P)!} \frac{1}{|W(\mathfrak{a}_P)|} \sum_{s \in W(\mathfrak{a}_P)} (-1)^{\mathrm{rk}_L G} \cdot \iota_s \sum_{\underline{\beta} \in \mathfrak{B}_P : \underline{\beta} \supset \Delta^\vee_{P L(s)}} [\mathbb{Z}(\Delta_P^\vee) : \mathbb{Z}(\underline{\beta})]$$
$$\int_{\mathrm{i}\mathfrak{a}^*_{L(s)}} \operatorname{tr}(\partial_{\mathcal{G}^{\mathrm{lex}}_{\underline{\beta}}}^{\mathbf{1}_{\underline{\beta} \setminus \Delta^\vee_{P L(s)}}} (\mathcal{F})(\lambda) M_{P|P}(s,0)\rho(P,\lambda,h)) \, d\lambda.$$

In both expressions the integrals are absolutely convergent with respect to the trace norm and define a distribution on $\mathcal{C}(G(\mathbb{A})^1)$.

We note that in the case $G = \mathrm{GL}(n)$ the absolute convergence of the spectral side was established by a different method in [MS04].

We will now prove Theorem 3 completing the analysis of [Mül98] and [Mül02]. Fix M, P, L, $\mathcal{G}$ as above and let m be the co-rank of L in G. Let

$$\Delta = \mathrm{Id} - \Omega + 2\Omega_{K_\infty}$$

where Ω is the Casimir operator of $G(F_\infty) \cap G(\mathbb{A})^1$. For any $k \in \mathbb{N}$ we have

$$\begin{aligned} \|\partial_{\mathcal{G}}^{\mathbf{1}_{\underline{\beta}}}(\mathcal{F})(\lambda)\rho(P,\lambda,h)\|_1 &\le \|\partial_{\mathcal{G}}^{\mathbf{1}_{\underline{\beta}}}(\mathcal{F})(\lambda)\rho(P,\lambda,\Delta^{2k})^{-1}\|_1 \|\rho(P,\lambda,\Delta^{2k}h)\| \\ &\le \|\partial_{\mathcal{G}}^{\mathbf{1}_{\underline{\beta}}}(\mathcal{F})(\lambda)\rho(P,\lambda,\Delta^{2k})^{-1}\|_1 \left|\Delta^{2k} h\right|_{L^1(G(\mathbb{A})^1)}. \end{aligned}$$

It remains to show the convergence, for $k \gg 1$, of

$$\int_{\mathrm{i}\mathfrak{a}^*_L} \|\partial_{\mathcal{G}}^{\mathbf{1}_{\underline{\beta}}}(\mathcal{F})(\lambda)\rho(P,\lambda,\Delta^{2k})^{-1}\|_1 \, d\lambda.$$

For $\tau \in \widehat{K_\infty}$ we denote by $\mathcal{A}^2_\pi(P)_{K_0,\tau}$ the τ-isotypical subspace of $\mathcal{A}^2_\pi(P)_{K_0}$. Then $\mathcal{A}^2_\pi(P)_{K_0,\tau}$ is a finite-dimensional subspace on which $\rho(P,\lambda,\Delta)$ acts by a scalar $\mu(\pi,\lambda,\tau)$ satisfying

$$|\mu(\pi,\lambda,\tau)|^2 \geq \frac{1}{4}(1+\|\lambda\|^2+\lambda_\pi^2+\lambda_\tau^2) \tag{20}$$

where λ_π and λ_τ denote the Casimir eigenvalues of π_∞ and τ, respectively ([Mül02, (6.9)]).

Suppose that in $\mathcal{G}$ we have $P_{k_i}|^{\beta_i}P_{k_i+1}$, $i=1,\dots,m$. We denote the restriction of $M_{P_{k_i+1}|P_{k_i}}(\pi,s)$ to $\mathcal{A}^2_\pi(P)_{K_0,\tau}$ by $M_{P_{k_i+1}|P_{k_i}}(\pi,s)_{K_0,\tau}$. Using the inequality

$$\|A\|_1 \leq \dim V \|A\|$$

for any operator A on a finite-dimensional space V, and the unitarity of $M_{Q|P}(\lambda)$, we reduce to showing the convergence, for sufficiently large k, of

$$\sum_{\tau\in\widehat{K_\infty}}\sum_\pi \dim(\mathcal{A}^2_\pi(P)_{K_0,\tau})\int_{\mathrm{i}\mathfrak{a}_L^*}|\mu(\pi,\lambda,\tau)|^{-2k}\prod_{i=1}^m\|M'_{P_{k_i+1}|P_{k_i}}(\pi,\langle\lambda,\beta_i^\vee\rangle)_{K_0,\tau}\|\;d\lambda.$$

By [Mül98, Corollary 0.3] there exists $k\in\mathbb{N}$ such that

$$\sum_{\tau\in\widehat{K_\infty}}\sum_{\pi\in\Pi_{\mathrm{disc}}(M(\mathbb{A});K_0)}\dim(\mathcal{A}^2_\pi(P)_{K_0,\tau})(1+\lambda_\pi^2+\lambda_\tau^2)^{-k}<\infty.$$

Therefore, using (20), it suffices to show that there exist C, N and N_1 such that

$$\int_{\mathrm{i}\mathfrak{a}_L^*}\prod_{i=1}^m\|M'_{P_{k_i+1}|P_{k_i}}(\pi,\langle\lambda,\beta_i^\vee\rangle)_{K_0,\tau}\|(1+|\lambda|)^{-N}\;d\lambda \leq C(1+\lambda_\tau^2+\lambda_\pi^2)^{N_1}$$

for all $\tau\in\widehat{K_\infty}$ and $\pi\in\Pi_{\mathrm{disc}}(M(\mathbb{A});K_0)$. Using the change of variables $\lambda\mapsto(\langle\lambda,\beta_i^\vee\rangle)_{i=1}^m$, it suffices to establish a similar bound for

$$\int_{\mathrm{i}\mathbb{R}}\|M'_{Q|P}(\pi,x)_{K_0,\tau}\|(1+|x|)^{-N}\;dx$$

for $P|^\alpha Q$, or, what is the same, for

$$\int_{\mathrm{i}\mathbb{R}}\|M_{Q|P}(\pi,x)^{-1}_{K_0,\tau}M'_{Q|P}(\pi,x)_{K_0,\tau}\|(1+|x|)^{-N}\;dx.$$

For $P|^\alpha Q$ the global normalizing factor is given by

$$r_{Q|P}(\pi,\lambda)=r_\alpha(\pi,\langle\lambda,\alpha^\vee\rangle),\quad \lambda\in\mathfrak{a}^*_{P,\mathbb{C}}$$

for a certain single variable meromorphic function $r_\alpha(\pi,\cdot)$. We can therefore write $M_{Q|P}(\pi,x)^{-1}M'_{Q|P}(\pi,x)$ as

$$\frac{r'_\alpha(\pi,x)}{r_\alpha(\pi,x)}\,\mathrm{Id}\;+j_P\circ(\mathrm{Id}\otimes R_{Q|P}(\pi,x)^{-1}R'_{Q|P}(\pi,x))\circ j_P^{-1}.$$

By [Mül02, Theorem 5.3] there exist C, N, N_1 such that

$$\int_{\mathrm{i}\mathbb{R}}\left|\frac{r'_\alpha(\pi,x)}{r_\alpha(\pi,x)}\right|(1+|x|)^{-N}\;dx\leq C(1+\Lambda_\pi^2)^{N_1}$$

for all $\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A}); K_0)$. Here, as in [ibid.],

$$\Lambda_\pi = \min_{\tau \in W_P(\pi_\infty)} (\lambda_\pi^2 + \lambda_\tau^2)^{1/2}$$

where $W_P(\pi_\infty)$ denotes the set of minimal K_∞-types of the induced representation $\mathrm{Ind}_P^G(\pi_\infty)$.

To deal with the term involving the normalized intertwining operator, we may assume that $K_0 = \prod_{v<\infty} K_v$ where K_v is an open compact subgroup of $G(F_v)$, and K_v is hyperspecial for almost all v. Let $\pi = \otimes_v \pi_v$ and let $R_{Q|P}(\pi_v, x)$ be the local normalized intertwining operators. Let $R_{Q|P}(\pi_v, x)_{K_v}$ denote the restriction of $R_{Q|P}(\pi_v, x)$ to the subspace of K_v-invariant vectors, and for $\tau = \prod_{v|\infty} \tau_v \in \widehat{K_\infty}$ let $R_{Q|P}(\pi_v, x)_{\tau_v}$ be the restriction of $R_{Q|P}(\pi_v, x)$ to the τ_v-isotypical subspace. We recall that there exists a finite set of places S, including the archimedean ones, such that

$$R_{Q|P}(\pi_v, x)_{K_v} = \mathrm{Id}, \quad v \notin S, \quad \pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A}); K_0)$$

([Art89]). Thus $R_{Q|P}(\pi, x)^{-1} R'_{Q|P}(\pi, x) = \sum_{v \in S} R_{Q|P}(\pi_v, x)^{-1} R'_{Q|P}(\pi_v, x)$ on $I(\pi)_{K_0}$. Using the unitarity of $R_{Q|P}(\pi_v, x)$ for $x \in \mathrm{i}\mathbb{R}$, we are reduced to the estimation of

$$\int_{\mathrm{i}\mathbb{R}} \|R'_{Q|P}(\pi_v, x)_{K_v}\| (1 + |x|)^{-N} \, dx, \quad v \in S \text{ finite}, \tag{21}$$

and

$$\int_{\mathrm{i}\mathbb{R}} \|R'_{Q|P}(\pi_v, x)_{\tau_v}\| (1 + |x|)^{-N} \, dx \quad v|\infty. \tag{22}$$

Since $\|(a_{i,j})\| \le \sum |a_{i,j}|$, the integrals (21) and(22) are bounded by

$$\sum_{i,j} \int_{\mathrm{i}\mathbb{R}} \left| (R'_{Q|P}(\pi_v, x) e_i, e_j) \right| (1 + |x|)^{-N} \, dx,$$

where e_i is an orthonormal basis for $\mathrm{Ind}(\pi_v)^{K_v}$ (in the p-adic case) or $\mathrm{Ind}(\pi_v)_\tau$ (in the archimedean case). Note that $\dim \mathrm{Ind}(\pi_v)^{K_v}$ is bounded independently of π_v in the p-adic case and $\dim \mathrm{Ind}(\pi_v)_{\tau_v} \le (\deg \tau_v)^2$ for $v|\infty$. Let $\|\tau_v\|$ be the norm of the highest weight of τ_v. By Weyl's dimension formula, $\deg \tau_v$ is bounded polynomially in $\|\tau_v\|$.

Lemma 6. *Let $P_1|^\alpha P_2 \in \mathcal{P}(M)$ and $\pi_v \in \Pi(M(F_v))$.*

(1) *Suppose that v is p-adic and $(\mathrm{Ind}_P^G \pi_v)^{K_v} \neq 0$. Then any matrix coefficient $(R_{P_2|P_1}(s\varpi)\varphi_1, \varphi_2)$ with $\varphi_1, \varphi_2 \in (\mathrm{Ind}\, \pi_v)^{K_v}$ is of the form $f(q^s)$ for some rational function f with $\deg f$ bounded in terms of K_v only.*

(2) *Suppose that v is archimedean and let $\tau \in \widehat{K_v}$. Then any matrix coefficient $f(s) = (R_{P_2|P_1}(s\varpi)\varphi_1, \varphi_2)$ with $\varphi_1, \varphi_2 \in (\mathrm{Ind}\, \pi_v)_\tau$ is a rational function with $\deg f \le c(1 + \|\tau\|)$ where c depends only on G.*

Proof. We argue as in [MS04]. The rationality of f in both cases follows from [Art89]. Suppose first that v is p-adic. In the following, the notation will be relative to F_v. (In particular, M_0 is a minimal Levi defined over F_v and so on.)

Without loss of generality we can assume that M is of co-rank one in G. Write π_v as a Langlands quotient $J_Q^M(\sigma_v,\mu)$ where $Q \in \mathcal{P}^M(M_0)$, σ_v is a tempered representation of M_Q and $\mu \in \mathfrak{a}^*_{Q,+} \subset (\mathfrak{a}^M)^*$. Therefore, π_v is a quotient of $\operatorname{Ind}_R^M(\delta_v,\mu)$ where $R \in \mathcal{P}^M(M_0)$, $R \subset Q$, $\delta_v \in \Pi_2(M_R)$ and $\mu \in \overline{\mathfrak{a}^*_{R,+}}$. Then, as explained in [Art89, p. 30], we have a commutative diagram

$$\begin{array}{ccc} \operatorname{Ind}_{P(R)}(\delta,\mu+s\varpi) & \xrightarrow{R_{\bar{P}(R)|P(R)}(\delta,\mu+s\varpi)} & \operatorname{Ind}_{\bar{P}(R)}(\delta,\mu+s\varpi) \\ \downarrow & & \downarrow \\ \operatorname{Ind}_P(\pi,s) & \xrightarrow{R_{\bar{P}|P}(\pi,s)} & \operatorname{Ind}_{\bar{P}}(\pi,s) \end{array}$$

Therefore, any matrix coefficient of $R_{\bar{P}|P}(\pi_v,s)$ is one for $R_{\bar{P}(R)|P(R)}(\delta_v,\mu+s)$. By factoring the latter into rank one intertwining operators, we are reduced to the case where π is square integrable. However, up to a twist by an unramified character there are only finitely many square-integrable representations such that $(\operatorname{Ind}\pi)^{K_v} \neq 0$.

In the archimedean case $\deg f$ is the number of poles of f since $|f(s)| \leq 1$ on the unitary axis. By [MS04, Proposition A.2] this number is bounded by $c(1+\|\tau\|)$ where c depends on G only. □

To complete the proof we appeal to the following Lemma. We thank Benjamin Weiss for simplifying its proof considerably from an earlier version.

Lemma 7. *Let C be either the imaginary axis or the unit circle. Let $f(z)$ be a scalar valued rational function of degree $\leq m$ such that $|f(z)| \leq 1$ for all $z \in C$. Then*

$$\oint_C |f'(z)| \ |dz| \leq 8m \tag{23}$$

Proof. Assume first that f takes real values on C. Then the left-hand side of (23) is the total variation of f on C, i.e. $\sum_{j=1}^k |f(z_j)-f(z_{j-1})|$ where z_j, $j=1,\dots,k$ are the extrema of f on C and we set $z_0 = z_k$. Since $k \leq 2m$, we get

$$\oint |f'(z)| \ |dz| \leq 4m$$

in this case. The general case follows immediately. □

Remark 7. Let C be as in Lemma 7. Borwein and Erdélyi proved the following stronger inequality ([BE96]). Given $z_0 \in C$, $a_1,\dots,a_m \notin C$ and $f(z)$ such that $|f(z)| \leq 1$ on C and $\prod(z-a_i)f(z)$ is polynomial of degree $\leq m$ we have

$$|f'(z_0)| \leq \max(\left|\left[\prod_{j:|a_j|>1} \frac{1-\bar{a}_j z}{z-a_j}\right]'(z_0)\right|, \left|\left[\prod_{j:|a_j|<1} \frac{1-\bar{a}_j z}{z-a_j}\right]'(z_0)\right|)$$

for the unit circle and

$$|f'(z_0)| \le \max\left(\left| \left[\prod_{j:\operatorname{Re} a_j > 0} \frac{z - \bar{a}_j}{z + a_j} \right]' (z_0) \right|, \left| \left[\prod_{j:\operatorname{Re} a_j < 0} \frac{z - \bar{a}_j}{z + a_j} \right]' (z_0) \right| \right)$$

for the imaginary axis. Estimating the maximum by the sum and integrating over C we obtain Lemma 7 with 8 replaced by 2π which is best possible.

The operators $R_{Q|P}(\pi_v, x)_{K_v}$ are unitary on the imaginary axis, and therefore their matrix coefficients are bounded by 1. By Lemmas 6 and 7 it follows that there exist $C > 0$ and $N_1 \in \mathbb{N}$ such that for all $\pi \in \Pi(M(F_v))$

$$\int_{\mathrm{i}\mathbb{R}} \|R'_{Q|P}(\pi_v, x)_{K_v}\| (1 + |x|)^{-N} \, dx \le C$$

if $v \in S$ is non-archimedean, and

$$\int_{\mathrm{i}\mathbb{R}} \|R'_{Q|P}(\pi_v, x)_{\tau_v}\| (1 + |x|)^{-N} \, dx \le C(1 + \|\tau_v\|)^{N_1}$$

if v is archimedean and $\tau_v \in \widehat{K_v}$. This completes the proof of Theorem 3.

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