ON THE STATIC METRIC EXTENSION PROBLEM

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Abstract. The subject of this Master thesis under the guidance of M. Eichmair is the following theorem of J. Corvino and R. Schoen [5]: *Minimal mass extensions are static.* We revisit their proof giving full details. On the way, we fill in two gaps in their argument and strengthen their result.

Contents

1. Preliminaries

In this section, we introduce the basic notions before stating and proving the main theorem in Section 2. In this thesis, all manifolds and submanifolds are taken to be smooth and orientable.

Definition 1.1. A complete Riemannian 3-manifold (M, g) is called *asymptotically flat* if there exists a compact set $K \subset M$ and a diffeomorphism

$x = (x^1, x^2, x^3) : \mathbb{R}^3 \setminus \bar{B}_1(0) \to M \setminus K$ such that

$$\begin{aligned} g_{ij} - \delta_{ij} &= O(|x|^{-p}) && \text{for all } i, j \in \{1, 2, 3\} \\ R(g) &= O(|x|^{-q}) \end{aligned}$$

as $|x| \to \infty$ in the chart $\mathbb{R}^3 \setminus \bar{B}_1(0)$ with corresponding estimates for all higher order derivatives. Here, $p > \frac{1}{2}$ and $q > 3$, *cf.* [5, p. 161]. For $R > 1$, we define the open set $B_R \subset M$ to be the union of K and the image of $B_R(0) \setminus \bar{B}_1(0)$ under this diffeomorphism.

Remark 1.2. The definition of asymptotic flatness can be extended to multiple ends in a straightforward manner, *cf.* [5, p. 161]. For simplicity we assumed the appropriate decay of all higher derivatives of the metric and the scalar curvature.

Definition 1.3. Let (M, g) be an asymptotically flat Riemannian 3-manifold. The ADM-mass of (M, g), $m_{\text{ADM}}(M, g)$, is defined as (*cf.* [5, p. 161] and [6]):

$$m_{\text{ADM}}(M, g) = \frac{1}{16\pi} \lim_{r \to \infty} \int_{|x|=r} \sum_{i,j=1}^{3} \left(\frac{\partial g_{ij}}{\partial x^i} - \frac{\partial g_{ii}}{\partial x^j} \right) \nu^j.$$

Here, ν denotes the Euclidean unit normal.[1]

Definition 1.4. A surface in a Riemannian 3-manifold is said to be *minimal* if its mean curvature vanishes.

Definition 1.5. An orientable minimal surface Σ in a Riemannian 3-manifold (M, g) is called *stable* if it satisfies the *stability inequality*

$$\int_\Sigma |\nabla f|^2 \geq \int_\Sigma \left(\text{Ric}(\nu, \nu) + |h|^2 \right) f^2$$

for all $f \in C_c^\infty(\Sigma)$ whose support is disjoint from $\partial\Sigma$. Here ν and h denote a unit normal vector field of Σ and the second fundamental form of Σ, respectively.

Recall that $\Sigma \subset M$ is minimal if it is a critical point of the area functional, i.e. if

$$\left. \frac{d}{dt} \right|_{t=0} \text{area}\,(\Sigma_t) = 0$$

[1]This definition of the mass seems to depend on the choice of the chart $x : \mathbb{R}^3 \setminus \bar{B}_1(0) \to M \setminus K$. This is in fact not the case, as shown in [1] and [4].

for every smooth family of surfaces $\{\Sigma_t\}_{|t|<\epsilon}$ with $\Sigma_0 = \Sigma$. It is stable if it is a stable critical point, i.e. for every such variation

$$\left.\frac{d^2}{dt^2}\right|_{t=0} \mathrm{area}\,(\Sigma_t) \geq 0.$$

Definition 1.6. Let (M, g) be a connected asymptotically flat Riemannian 3-manifold with non-negative scalar curvature and minimal closed boundary ∂M (possibly empty) such that there are no minimal closed surfaces in M other than ∂M. Let Ω be a smooth bounded open subset of (M, g) such that $M \setminus \Omega$ is connected and let h be the restriction of g to Ω. We say that (M, g) is an *admissible extension of the Riemannian manifold* (Ω, h). The Bartnik quasi-local mass of (Ω, h), $m_{\mathrm{B}}(\Omega, h)$, is defined as

$$m_{\mathrm{B}}(\Omega, h) = \inf\{m_{\mathrm{ADM}}(M, g) : (M, g) \text{ is an admissible extension of } (\Omega, h)\}.$$

An admissible extension realizing this infimum is called *minimal mass extension.*

Our definition of admissible extensions allows Ω to contain closed minimal surfaces. For simplicity of exposition we assume that $\partial M \subset \bar{\Omega}$. The existence of a minimal mass extension to a given (Ω, h) is an open problem, *cf.* Bartnik's conjecture in [3].

Definition 1.7. A closed surface is called *outer-minimizing* if every closed surface which encloses it has greater or equal area.

Definition 1.8. A minimal surface is called *outermost* if it is not separated from infinity by any other closed minimal surface.

Remark 1.9. An outermost minimal surface is in particular an outer-minimizing closed stable minimal surface, *cf.* [7] and Chapter 4 of [8].

Theorem 1.10. *Let (M, g) be an asymptotically flat Riemannian 3-manifold with minimal boundary ∂M. Then there is a closed outermost minimal surface in M enclosing ∂M.*

Proof. We refer to [7] and Chapter 4 of [8]. □

Staticity as introduced in the definition below is an obstruction to deformations of the scalar curvature in arbitrary directions. The following two lemmas are Lemma 2.1 and Lemma 2.2 in [5].

Lemma 1.11. *For $k \geq 0$, the scalar curvature map $R : \mathcal{M}^{k+2,\alpha}(M) \to C^{k,\alpha}(M)$ is a smooth map of Banach manifolds, where $\mathcal{M}^{k+2,\alpha}(M)$ denotes* *the open set of Riemannian $C^{k+2,\alpha}(M)$-metrics. Furthermore, the linearization L_g of the scalar curvature operator is given by*

$$L_g h = -\triangle_g \left(tr_g(h)\right) + \operatorname{div}\left(\operatorname{div}(h)\right) - g(h, \operatorname{Ric}(g)).$$

Lemma 1.12. *The formal adjoint L_g^* of L_g is given by*

$$L_g^* f = -\left(\triangle_g f\right) g + \operatorname{Hess}(f) - f\operatorname{Ric}(g).$$

In other words,

$$\int_M g(h, L_g^* f) = \int_M (L_g h) f$$

for all $f \in C_c^\infty(M)$ and smooth $(0,2)$-tensors h on M.

Definition 1.13. Given a Riemannian 3-manifold (M, g) and an open subset $U \subset M$, we say that the metric g is *static on U* if the kernel of L_g^* is non-trivial on U, i.e. there exists a non-trivial $f \in C^\infty(U)$ such that $L_g^* f = 0$ on U.

2. Static extensions

We prove the following theorem from [5, p. 164].

Theorem 2.1. *Let (M, g) be a minimal mass extension of (Ω, h). Then g is static on $M \setminus \bar{\Omega}$.*

The proof given in [5] lacks the verification of two sub-statements. We fill these gaps with the proofs of Proposition 2.6 and Claim 2.21 given below. Furthermore, our construction stays in the class of smooth metrics. We note a direct corollary:

Corollary. *Let (M, g) be a minimal mass extension of (Ω, h). Then the scalar curvature of g vanishes on $M \setminus \Omega$.*

Proof. We combine Theorem 2.1 with Proposition A.4 from the appendix. □

2.1. **Overview of the proof.**

The basic strategy was outlined in [5, p. 164]. Here we give many more details and also strengthen the result some.

We will prove the theorem by contradiction. Assume that the minimal mass extension (M, g) of a smooth bounded open set Ω is non-static on $M \setminus \bar{\Omega}$.

We reduce to the case where the scalar curvature $R(g)$ does not vanish identically on $M \setminus \bar{\Omega}$. Indeed, in the case of $R(g) \equiv 0$ on $M \setminus \bar{\Omega}$, we locally *push up* the scalar curvature in a compact subset of $M \setminus \bar{\Omega}$. This deformation can be chosen such that the new metric is close to g and also non-static on $M \setminus \bar{\Omega}$, *cf.* Section 2.2. Note that this new Riemannian manifold has the same ADM-mass as (M, g) because we only changed the metric in a compact set.

In the case when $R(g) > 0$ at a point in $M \setminus \bar{\Omega}$, we construct a function $u \in C^\infty(M \setminus \Omega)$ close to 1 with the following properties (*cf.* Section 2.3)

(1) $0 < u < 1$ in $M \setminus \bar{\Omega}$;
(2) $u = 1$ on $\partial(M \setminus \Omega)$ and $u \to 1$ as $|x| \to \infty$;
(3) $u^4 g$ is asymptotically flat;
(4) $R(u^4 g) \geq 0$.

Furthermore, u is harmonic outside a compact set and has the following expansion as $|x| \to \infty$:

$$u = 1 + \frac{A}{|x|} + O(|x|^{-2}), \qquad \frac{\partial u}{\partial x^i} = -\frac{A x^i}{|x|^3} + O(|x|^{-3})$$

We show that the constant $A \in \mathbb{R}$ is negative, *cf.* Proposition 2.6. This is asserted but left unverified in [5]. The next lemma follows as in [14, p. 49] by explicit calculation.

Lemma 2.2. *Let (M, g) be an asymptotically flat Riemannian 3-manifold and let u be a smooth function on M such that there is an expansion*

$$u = 1 + \frac{A}{|x|} + O(|x|^{-2}), \qquad \frac{\partial u}{\partial x^i} = -\frac{A x^i}{|x|^3} + O(|x|^{-3})$$

as $|x| \to \infty$. Then

$$m_{\mathrm{ADM}}(M, u^4 g) = m_{\mathrm{ADM}}(M, g) + 2A.$$

Therefore, if we extend u as 1 on Ω, we have a Riemannian manifold $(M, u^4 g)$ with ADM-mass $m_{\mathrm{ADM}}(M, u^4 g) < m_{\mathrm{ADM}}(M, g)$. However, this metric is in fact not smooth across $\partial(M \setminus \Omega)$. Moreover, there may be closed minimal surfaces other than ∂M.[2]

[2] In general, closed minimal surfaces can appear due to conformal deformations. Consider for example the spatial Schwarzschild manifold $\left(\mathbb{R}^3 \setminus \{0\}, \left(1 + \frac{m}{2|x|}\right)^4 \delta_{ij}\right)$. The surface $\{r = m/2\}$ is an outer-minimizing closed stable minimal surface.

We now construct an admissible smooth metric on M that is equal to $u^4 g$ outside a compact set. Let $\chi : M \to [0,1] \subset \mathbb{R}$ be a smooth function such that χ is equal to 1 near $\partial(M \setminus \Omega)$ and vanishes outside a large compact set. Define a new smooth metric $\bar{g}$ on M by

$$\bar{g} = (\chi + (1-\chi)u)^4 g.$$

We can arrange for u so that $\bar{g}$ is as close to g as we like. The scalar curvature of $\bar{g}$ may be negative in some places but not much. Using that g is non-static, it follows that $\bar{g}$ is non-static, *cf.* details in Section 2.5. We locally deform $\bar{g}$ to a smooth metric $\hat{g}$ with everywhere non-negative scalar curvature, *cf.* Section 2.5. It has the same ADM-mass as $(M, u^4 g)$. In fact, we can arrange for u so that $\hat{g}$ is close to g, *cf.* the argument in Section 2.7.

The following theorem is proven in Section 2.6.

Theorem 2.3. *Let (M,g) be asymptotically flat and be such that there are no closed minimal surfaces in M other than the components of ∂M. Let U be an open neighborhood of ∂M in M. There exists $\epsilon > 0$ such that if $\bar{g}$ is a Riemannian metric on M with $g = \bar{g}$ on U such that $|g - \bar{g}|_{C^2(M)} < \epsilon$ and*

$$|(g-\bar{g})_{ab}| + |x||\partial_k (g-\bar{g})_{ab}| + |x|^2 |\partial^2_{k\ell}(g-\bar{g})_{ab}| < \epsilon$$

in the chart $\mathbb{R}^3 \setminus \bar{B}_1(0)$, then there are no closed minimal surfaces in $(M, \bar{g})$ other than the components of ∂M.

Because of the above and the expansion of u as $|x| \to \infty$, the metric $\hat{g}$ satisfies the assertions of this theorem with respect to g. Recall that (M,g) contains no closed minimal surfaces other than the components of ∂M. By the above theorem $\hat{g}$ has no other closed minimal surfaces than the components of ∂M.

In conclusion, $(M, \hat{g})$ is an admissible manifold with $m_{\mathrm{ADM}}(M, \hat{g}) < m_{\mathrm{ADM}}(M, g)$, *cf.* Section 2.7. This is a contradiction to the minimality of (M,g).

2.2. **Reduction to the case of non-trivial scalar curvature.** Let (M,g) be a non-static admissible extension of a smooth bounded open subset Ω as defined above. Assume further that $R(g) \equiv 0$ on $M \setminus \bar{\Omega}$.

We apply Proposition A.2 from the appendix to $(M \setminus \bar{\Omega}, g)$ to obtain a smooth bounded open set $U \subset M \setminus \bar{\Omega}$ such that g is non-static on U. We use Theorem A.1 from the appendix to *push up* the scalar curvature inside U. We denote the new metric on M by $\tilde{g}$. The ADM-mass is not changed by this local deformation.

The condition that g is non-static on U is an open condition, *cf.* Theorem 2.12. We can arrange $\tilde{g}$ to be close to g on U by choosing $R(\tilde{g})$ to be sufficiently small in U, *cf.* the statement of Theorem A.1 in the appendix. It follows that $\tilde{g}$ is non-static on U. In fact, $\tilde{g}$ is non-static on $M \setminus \bar{\Omega}$. To see this, assume there exists a non-trivial function f in the kernel of $L^*_{\tilde{g}}$ on $M \setminus \bar{\Omega}$. Restrict this function to U. By the above, the kernel of $L^*_{\tilde{g}}$ is trivial on U, so f must vanish on U. However, we can apply a unique continuation argument to f to conclude that $f \equiv 0$ on $M \setminus \bar{\Omega}$, *cf.* the proof of Proposition 2.3 in [5]. This is a contradiction which shows that $\tilde{g}$ is non-static on $M \setminus \bar{\Omega}$.

2.3. **Construction of the conformal deformation.** Let (M, g) be an admissible extension of a smooth bounded open subset Ω as defined earlier. Denote the scalar curvature of g by $R(g)$ and assume it is non-negative and not identically 0. Let $\epsilon > 0$ small and $\chi \in C^\infty(M)$ be a smooth function on $M \setminus \Omega$ such that $\epsilon \leq \chi \leq 1$ on $M \setminus \Omega$ and such that $\chi \equiv 1$ outside a compact set. Consider the following boundary value problem:

$$(\star) \qquad \begin{aligned} \triangle_g u - \frac{1}{8} R(g) u &= -\frac{1}{8} R(g)(1-\chi) && \text{in } M \setminus \bar{\Omega} \\ u &= 1 && \text{on } \partial(M \setminus \Omega) \\ u &\to 1 && \text{as } |x| \to \infty. \end{aligned}$$

Remark 2.4. Given $\epsilon' > 0$ and an integer $k \geq 0$, one can find $\chi = \chi_{\epsilon'}$ such that

$$|R(g)\chi_{\epsilon'}|_{C^k(M \setminus \Omega)} < \epsilon'.$$

Proposition 2.5. *The above system has a unique solution u.*

Proof. We refer to [5] and references therein. □

Note that u is non-constant. Indeed, if u was constant it would be equal to 1 by the boundary conditions. However, $u \equiv 1$ does not satisfy the differential equation as the support of $R(g)$ is non-empty and $\chi > 0$. We note that on the one hand,

$$\triangle_g u - \frac{1}{8}R(g)u \leq 0 \qquad \text{on } M \setminus \bar{\Omega}.$$

The strong maximum principle, *cf.* [13, p. 35], implies that u cannot achieve a non-positive minimum in $M \setminus \bar{\Omega}$. This shows that $u > 0$.

On the other hand, the function $v = u - 1$ satisfies

$$\triangle_g v - \frac{1}{8}R(g)v = \frac{1}{8}R(g)\chi \geq 0 \qquad \text{on } M \setminus \bar{\Omega}.$$

Here the strong maximum principle implies that v cannot achieve a non-negative maximum in $M \setminus \bar{\Omega}$. This shows that $u - 1 = v < 0$.

We conclude that $0 < u < 1$ in $M \setminus \bar{\Omega}$.

As derived in [14, p. 64-71], the function u has the following expansion as $|x| \to \infty$

$$u = 1 + \frac{A}{|x|} + \omega(x)$$

where $A \in \mathbb{R}$ is a constant and the function ω and its derivatives have the following expansion as $|x| \to \infty$

$$\omega(x) = O(|x|^{-2}), \qquad \frac{\partial \omega}{\partial x^i}(x) = O(|x|^{-3}), \qquad \frac{\partial^2 \omega}{\partial x^i \partial x^j}(x) = O(|x|^{-4}).$$

By the above conclusion, $A \leq 0$.

Proposition 2.6. *The constant A in the above expansion of u is negative.*

Proof. The following choice of E_δ and integral calculations already appear in [14]. For any $\delta > 0$ define the set

$$E_\delta = \{x \in M \setminus \Omega : u(x) < 1 - \delta\}.$$

The set E_δ is bounded for all $\delta > 0$. By Sard's theorem there exists $\delta > 0$ arbitrarily small such that ∂E_δ is smooth, which we will choose tacitly from now on. Let $R > 0$ large such that $B_{R/2}$ contains E_δ and let W be the outermost connected component of $B_R \setminus \bar{E}_\delta$, i.e. the connected component that satisfies $S_R \subset \partial W$. Both W and R clearly depend on δ.

Claim 2.7. *There exists a small $\delta_0 > 0$ such that for all $\delta \leq \delta_0$*

$$\partial W \cap \partial(M \setminus \Omega) = \emptyset.$$

Proof of Claim. Let $R' > 0$ such that $\bar{\Omega} \subset B_{R'}$. As $u < 1$ in $M \setminus \bar{\Omega}$, $u_0 \doteq \max_{S_{R'}} u < 1$. Pick $\delta_0 < \frac{1-u_0}{2}$. This implies that $S_{R'}$ is contained in the open set E_δ for all $\delta \leq \delta_0$. In particular, for $\delta \leq \delta_0$ it follows that W is disjoint from $B_{R'}$. Therefore W clearly does not share any boundary with Ω, i.e. $\partial W \cap \partial \Omega = \emptyset$. □

Thus, for $\delta > 0$ small enough, ∂W is the disjoint union of S_R and a subset of ∂E_δ. We integrate the differential equation over the set W and apply the divergence theorem to obtain

$$\int_W -\frac{1}{8}R(g)(1-\chi) + \frac{1}{8}R(g)u = \int_W \triangle_g u = \int_{S_R} \nabla_\nu u + \int_{\partial W \cap \partial E_\delta} \nabla_\nu u.$$

Here, ν denotes the unit normal vector to the respective boundary pointing out of W. We use the known expansion of u and its derivatives as $|x| \to \infty$ and the asymptotic flatness of g to calculate

$$\begin{aligned}\lim_{R\to\infty} \int_{S_R} \nabla_\nu u &= \lim_{R\to\infty} \int_{S_R} \left(\frac{x^i}{|x|}\partial_i + \sum_i O\left(|x|^{-1}\right)\partial_i \right) \left(1 + \frac{A}{|x|} + O\left(|x|^{-2}\right) \right) \\ &= \lim_{R\to\infty} \int_{S_R} \left(-\frac{A}{|x|^2} + O\left(|x|^{-3}\right) \right) \\ &= -4\pi A.\end{aligned}$$

We let $R \to \infty$ and rewrite the above[3] as

$$4\pi A = \int_{\partial W \cap \partial E_\delta} \nabla_\nu u - \int_W \frac{1}{8}R(g)u + \int_W \left(\frac{1}{8}R(g)(1-\chi) \right).$$

The third term on the right-hand side vanishes for $\delta > 0$ sufficiently small by the fact that $1-\chi$ has compact support in $M \setminus \Omega$.[4]. The second term is non-positive as $R(g) \geq 0$ and $u > 0$. The first term is negative by the strong maximum principle. Indeed, consider the function $w = u - (1-\delta)$ and notice that it satisfies

$$\begin{aligned}\triangle_g w - \frac{1}{8}R(g)w &= \frac{1}{8}R(g)(\chi - \delta) && \text{in } E_\delta \\ w &= 0 && \text{on } \partial E_\delta.\end{aligned}$$

[3] By the asymptotic flatness of g and our choice of χ, the functions $\frac{1}{8}R(g)$ and $-\frac{1}{8}R(g)(1-\chi)$ are integrable and the limit is well-defined, *cf.* [14].

[4] Note that the sets $(E_\delta)_{\delta>0}$ constitute an exhaustion of $M \setminus \Omega$. So for $\delta > 0$ small enough, the support of $1-\chi$ is contained in E_δ and therefore does not intersect W.

Note that E_δ contains the support of $R(g)$ for small δ. The function $\chi - \delta$ is positive on the set E_δ for δ sufficiently small by our choice of χ. This, together with the boundary conditions, implies that w is non-constant. Consequently for small δ the maximum principle implies that w cannot achieve a non-negative maximum in the interior of E_δ. Specifically, the maximum is attained strictly on the boundary. We apply Hopf's Lemma (*cf.* [13, p. 34]) and conclude the negativity of the first term. □

2.4. **Bounding the conformal factor.** Let $\epsilon > 0$ and let $\chi \in C^\infty(M \setminus \Omega)$ be a function such that $\epsilon \leq \chi \leq 1$ as before. Let $v \in C^\infty(M \setminus \Omega)$ denote a solution to the boundary value problem

$$\begin{aligned} \triangle_g v - \frac{1}{8}R(g)v &= \frac{1}{8}R(g)\chi && \text{in } M \setminus \bar{\Omega}, \\ v &= 0 && \text{on } \partial(M \setminus \Omega), \\ v &\to 0 && \text{as } |x| \to \infty. \end{aligned}$$

The following is a straightforward consequence of Lemma 3.1 in [14]:

Lemma 2.8. *There is a constant $C > 0$ depending on $M \setminus \Omega$ and the exact asymptotic decay of g so that for any function ξ with compact support on $M \setminus \Omega$, we have the inequality*

$$\left(\int_{M \setminus \Omega} \xi^6 \right)^{1/3} \leq C \int_{M \setminus \Omega} |\nabla \xi|^2.$$

Remark 2.9. It is not required that $\xi = 0$ on $\partial(M \setminus \Omega)$.

Again let $\delta > 0$ small such that $E_\delta = \{x \in M \setminus \Omega : v(x) < -\delta\}$ is a smooth subset. The function $v + \delta$ satisfies

$$\begin{aligned} \triangle_g (v+\delta) - \frac{1}{8}R(g)(v+\delta) &= \frac{1}{8}R(g)(\chi - \delta) && \text{in } E_\delta \\ v + \delta &= 0 && \text{on } \partial E_\delta. \end{aligned}$$

By partial integration we estimate

$$\begin{aligned} \int_{E_\delta} |\nabla v|^2 &= -\int_{E_\delta} (v+\delta)\triangle_g (v+\delta) + \int_{\partial E_\delta} (v+\delta)\nabla_\nu v \\ &= -\int_{E_\delta} (v+\delta) \left[\frac{1}{8}R(g)(\chi-\delta) + \frac{1}{8}R(g)(v+\delta) \right] \end{aligned}$$

$$\leq \int_{E_\delta} \left(-(v+\delta)\frac{1}{8}R(g)(\chi-\delta) \right)$$
$$\leq \left(\int_{E_\delta} \left(\frac{1}{8}R(g)(\chi-\delta) \right)^{6/5} \right)^{5/6} \left(\int_{E_\delta} (v+\delta)^6 \right)^{1/6}.$$

We apply the previous lemma to the function

$$f(x) = \begin{cases} v+\delta & x \in E_\delta \\ 0 & \text{else.} \end{cases}$$

It follows that

$$\left(\int_{E_\delta} (v+\delta)^6 \right)^{1/3} \leq C \int_{E_\delta} |\nabla v|^2.$$

Plugging this into the current calculation, we obtain

$$\left(\int_{E_\delta} |\nabla v|^2 \right)^{1/2} \leq C \left(\int_{E_\delta} \left(\frac{1}{8}R(g)(\chi-\delta) \right)^{6/5} \right)^{5/6}.$$

All of the integrals

$$\int_{M\setminus\Omega} v^6 \qquad \int_{M\setminus\Omega} |\nabla v|^2 \qquad \int_{M\setminus\Omega} (R(g)\chi)^{6/5}$$

are finite, since g is asymptotically flat and v decays appropriately[5]. Therefore we can take the limit $\delta \to 0$ (by Sard's theorem there exists a sequence within the set of regular values of v) in the above estimates and get

$$\left(\int_{M\setminus\Omega} |\nabla v|^2 \right)^{1/2} \leq C \left(\int_{M\setminus\Omega} \left(\frac{1}{8}R(g)\chi \right)^{6/5} \right)^{5/6},$$
$$\left(\int_{M\setminus\Omega} v^6 \right)^{1/6} \leq \tilde{C} \left(\int_{M\setminus\Omega} \left(\frac{1}{8}R(g)\chi \right)^{6/5} \right)^{5/6}.$$

The right hand side of both estimates can be controlled by $|R(g)\chi|_{C^0(M\setminus\Omega)}$.

Apply the interior and boundary elliptic estimates to v, *cf.* Theorem 2 in [15, p. 314] and the proof of Theorem 5 in [15, p. 323].[6] Consequently

[5]This is clear as $v = u - 1$ where u denotes the function in Section 2.4.

[6]Note that we use the boundary estimates only on balls centered on the boundary $\partial(M\setminus\Omega)$ on which v vanishes. Inside $M \setminus \Omega$, we only apply interior estimates so that no boundary term appears in our estimate. Given a compact set $V \subset M \setminus \Omega$, we cover it by open balls contained in $M \setminus \bar{\Omega}$ and open balls centered on $\partial(M \setminus \Omega)$. On each ball, we apply

we see by employing the Theorem 6 (Sobolev embedding) from [15, p. 270] with $k = 2$, $n = 3$ and $p = 2$ that for any compact $V \subset M \setminus \Omega$

$$|v|_{C^{0,1/2}(V)} \leq C' |R(g)\chi|_{C^0(M\setminus\Omega)},$$

where C' depends on $M \setminus \Omega$ and V.

Furthermore, applying the higher regularity interior and boundary Schauder estimates as stated in [13, p. 141-142], it follows as above that for any compact $V \subset M \setminus \Omega$

$$|v|_{C^{k+2,\alpha}(V)} \leq C'' |R(g)\chi|_{C^{k,\alpha}(M\setminus\Omega)}$$

where C'' depends on V and $\alpha \in [0, 1)$.

Remark 2.10. By appropriately choosing $\chi = \chi_\epsilon$, we obtain a $u = u_\epsilon \in C^\infty(M \setminus \Omega)$ such that $u_\epsilon \to 1$ smoothly as $\epsilon \to 0$.

2.5. **Gluing and the subsequent scalar curvature deformation.** Let (M, g) be a non-static minimal mass extension of a given smooth bounded open subset Ω as defined earlier. Assume $R(g)$ non-negative and not identically 0 on $M \setminus \Omega$. Let $u \in C^\infty(M \setminus \Omega)$ be the non-trivial solution to the boundary value problem $(\star)$ in Section 2.3.

Let $\chi : M \to [0, 1] \subset \mathbb{R}$ be a smooth function such that χ is equal to 1 near $\partial(M \setminus \Omega)$ and vanishes outside a large smooth bounded open subset of $M \setminus \bar{\Omega}$. Define a new smooth metric $\bar{g}$ on M by

$$\bar{g} = (\chi + (1 - \chi)u)^4 g.$$

Let $V = \mathrm{supp}(R(g) - R(\bar{g})) \subset M \setminus \bar{\Omega}$.

Claim 2.11. *There exists a smooth bounded open set $U \subset M \setminus \bar{\Omega}$ such that $V \subset U$ and g is non-static on U.*

Proof of Claim. This follows directly from the proof of Proposition A.2 in the appendix. In this proof an exhaustion $\{\Omega_k\}$ of $M \setminus \bar{\Omega}$ by smooth bounded open sets is used. It is shown that there exists a finite integer k_0 such that the kernel of L_g^* is trivial on Ω_k for all $k \geq k_0$. As the Ω_k constitute an

the estimate to obtain an estimate over V. Note that because g is asymptotically flat, the constant in the estimate on the interior ball $B_r(x)$ converges to the Euclidean value as $|x| \to \infty$. This Euclidean value only depends on r. Consequently we can uniformly estimate this constant on $M \setminus \Omega$.

exhaustion, there exists k large enough such that Ω_k contains the bounded set $\mathrm{supp}(R(g) - R(\bar{g}))$ and such that g is non-static on Ω_k.

We want to deform the scalar curvature of $\bar{g}$ inside U to be equal to $R(g)$. To do so, it is necessary to show that the range of surjectivity of the scalar curvature map at $\bar{g}$ includes $R(g)$, i.e. that we can deform the scalar curvature of $\bar{g}$ enough to in fact reach $R(g)$. This is *a priori* not clear, *cf.* the discussion in the beginning of Section 5 in [5].

The following theorem is analogous to Theorem 3 in [5], *cf.* Remark 2.6 in [16] for a discussion regarding the uniformity statement. For any positive measurable function ρ, let L^2_ρ be the set of functions f such that $|f|\rho^{1/2} \in L^2$; define H^2_ρ analogously.

Theorem 2.12. *Let g be a non-static Riemannian metric on a smooth bounded open set U. Then there is a constant $C = C(n, g, U, \rho)$, uniform for metrics C^∞-near g, so that for $f \in H^2_{\mathrm{loc}}(U)$,*

$$\|f\|_{H^2_\rho(U)} \leq C \|L^*_g(f)\|_{L^2_\rho(U)}.$$

Consequently, there exists a *uniform* lower bound on the surjectivity radius of the scalar curvature map at $\tilde{g}$ for all $\tilde{g}$ smoothly close to g, *cf.* [5]. By Remark 2.10 and the definition of $\bar{g}$, we can arrange that $\bar{g}$ is sufficiently close to g such that

(1) g and $\bar{g}$ share the same uniform constant from the previous theorem
(2) $R(g)$ is within the lower bound of the surjectivity radius of the scalar curvature map at $\bar{g}$ implied by this constant.

In other words, we can arrange $\bar{g}$ such that the $\epsilon > 0$ in Theorem A.1 from the appendix is sufficiently large so that we can deform $\bar{g}$ inside U to a metric $\hat{g}$ with scalar-curvature equal to $R(g)$ in U. Note that $\hat{g} = g$ near $\partial(M \setminus \Omega)$ and $\hat{g} = u^4 g$ outside a bounded subset of $M \setminus \bar{\Omega}$.

2.6. **The absence of closed minimal surfaces is an open condition.**

Theorem 2.13. *Let (M, g) be asymptotically flat and such that there are no closed minimal surfaces in M other than the components of ∂M. Let U be an open neighborhood of ∂M in M. There exists $\epsilon > 0$ such that if $\bar{g}$ is a Riemannian metric on M with $g = \bar{g}$ on U such that $|g - \bar{g}|_{C^2(M)} < \epsilon$ and*

$$|(g - \bar{g})_{ab}| + |x||\partial_k (g - \bar{g})_{ab}| + |x|^2 |\partial^2_{k\ell}(g - \bar{g})_{ab}| < \epsilon$$

in the chart $\mathbb{R}^3 \setminus \bar{B}_1(0)$*, then there are no closed minimal surfaces in* $(M, \bar{g})$ *other than the components of* ∂M*.*

To prove this theorem, assume by contradiction that there exists a sequence $\epsilon_i \searrow 0$ and a sequence of Riemannian metrics g_i such that for all $i \geq 1$

(1) $g_i = g$ on U and $|g_i - g|_{C^2(M)} < \epsilon_i$
(2) $|(g - g_i)_{ab}| + |x||\partial_k(g - g_i)_{ab}| + |x|^2|\partial^2_{k\ell}(g - g_i)_{ab}| < \epsilon_i$ in the chart $\mathbb{R}^3 \setminus \bar{B}_1(0)$
(3) There exists a closed surface Σ_i which is both minimal with respect to g_i and disjoint from ∂M.

We will show that this implies that there exists a closed minimal surface Σ other than ∂M in (M, g). By the proof of Theorem 2.2, we can assume without loss of generality that each Σ_i is an outer-minimizing closed stable minimal surface.[7]

Claim 2.14. *Given an asymptotically flat Riemannian* 3*-manifold* (M, g)*, there exists a large constant* $R_0 > 0$*, depending on the exact asymptotic decay of* g*, such that every closed minimal surface* $\Sigma \subset M$ *is contained in the coordinate ball of radius* R_0*.*

Proof of Claim. The asymptotic flatness of g implies that there exists a large $R_0 > 0$ such that for all $R \geq R_0$ the coordinate sphere S_R of radius R is mean convex with respect to g. This can be seen by direct calculation employing the following lemma, using that S_R is for large $R > 0$ a level set of $f(x) = |x|$ (in the chart).

Lemma 2.15. *Assume that the the hypersurface* Σ *is given as the regular level set of a function* G *in a chart of* M*. Then its mean curvature can be expressed in this chart as*

$$H_{\Sigma,g} = \left(g^{ab} - \frac{G^a G^b}{|\nabla_g G|^2} \right) \frac{(\nabla^2_g G)_{ab}}{|\nabla_g G|}$$

Here the indices a, b *run over* $1, 2, 3$*. If* $G(x^1, x^2, x^3) = u(x^1, x^2) - x^3$*, where* u *is a smooth function, then the mean curvature in this chart is a quasi-linear elliptic differential operator in* u *of the form*

$$H(u) = a_{ij}(x, u, \partial u)\partial_i \partial_j u + b(x, \partial u).$$

[7] In fact, for all $i \geq 1$, the existence of Σ_i implies that the outermost minimal surface for g_i is not given by ∂M. Note that these surfaces do not converge to ∂M in C^0 as $i \to \infty$ because $g_i = g$ in a fixed open neighbourhood of ∂M for every i.

Proof. The second statement follows by explicit calculation using the expression for level sets. □

Assume there exists a closed minimal surface Σ which is not contained in the ball of radius R_0. By compactness, we can find a radius $R_1 > R_0$ such that the sphere of radius R_1 exactly touches the surface Σ. Let $p \in \Sigma$ be one of the touching points.

There exists a coordinate system with origin at p such that S_R and Σ can locally be written as C^2-graphs over the x^1-x^2-plane and $\frac{\partial}{\partial x^3}$ points towards the interior of B_R at p. Denote the functions u and v. By construction, the following holds locally around p:

$$u \leq v$$
$$u(p) = v(p)$$
$$H(u) > H(v) = 0.$$

Here $H(u)$ denotes the mean curvature of the graph of u with respect to g.

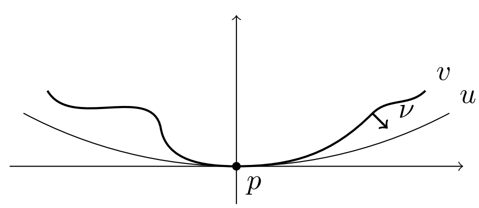

Image: A simplified sketch of the functions u, v and the outward-pointing unit normal ν.

Lemma 2.5 enables us to apply a comparison principle, *cf.* Theorem 10.1 in [13, p. 263]. It follows that $H(u) > H(v)$ and $u \leq v$ near p imply that $u < v$ near p. This is a contradiction to $u(p) = v(p)$, which proves the claim. □

We now return to the assumptions of the proof of Theorem 2.13. The above claim allows us to prove the following:

Claim 2.16. *There exist an $R_0 > 0$ and an i_0 such that Σ_i is contained in a ball of radius R_0 for all $i \geq i_0$.*

Proof of Claim. As in the proof of the previous claim, we let $R_0 > 0$ large such that $H_{S_R,g} > 0$ on S_R for all $R \geq R_0$. Because of the bound on $g_i - g$ and its derivatives as $|x| \to \infty$, there exists i_0 such that $H_{S_R,g_i} > 0$ for all

$R \geq R_0$ and $i \geq i_0$. By a barrier argument similar to the above, we conclude that $\Sigma_i \subset B_{R_0}$ for all $i \geq i_0$. □

Claim 2.17. *We can uniformly bound the area of the Σ_i's with respect to the g-measure.*

Proof of Claim. As all surfaces Σ_i are outer-minimizing, it follows that the enclosing sphere S_{R_0} with radius R_0 from the previous claim has greater area, i.e. $\mu_{g_i}(S_{R_0}) > \mu_{g_i}(\Sigma_i)$. Because $g_i \to g$ as $i \to \infty$, it follows that $\mu_{g_i}(S_{R_0}) \to \mu_g(S_{R_0})$ as $i \to \infty$, so that we can uniformly bound $\mu_{g_i}(\Sigma_i)$ and consequently $\mu_g(\Sigma_i)$ as well. □

The following lemma is Lemma 2.4 in [9].

Lemma 2.18. *Consider a metric on $B_1(0)$ of the form $g_{ij} = \delta_{ij} + b_{ij}$ where*

$$|x|^{-1}|b_{ij}| + |x|^{-2}|\partial_k b_{ij}| \leq C'$$

for all $|x| \leq 1$. Let $\Sigma \subset B_1(0)$ be an oriented surface and let $h_{\Sigma,g}, h_{\Sigma,\delta}$ and $H_{\Sigma,g}, H_{\Sigma,\delta}$ denote the (1,1)-second fundamental forms and the mean curvature scalars of Σ computed with respect to g and δ. Then

$$|h_{\Sigma,g} - h_{\Sigma,\delta}|_\delta \leq C|x|^2 \left(|x|^{-1}|h_{\Sigma,g}|_g + 1\right)$$
$$|H_{\Sigma,g} - H_{\Sigma,\delta}| \leq C|x|^2 \left(|x|^{-1}|h_{\Sigma,g}|_g + 1\right)$$

for all $|x| \leq r_0$, where r_0 and C depend only on C'.

We recall a local uniform graph representation theorem, *cf.* Theorem E.2.4 in [10], which will imply the local convergence of the surfaces Σ_i to a hypersurface Σ (possibly after passing to a subsequence).

Theorem 2.19. *Let $\{M_j\}$ denote a sequence of n-dimensional immersed submanifolds in $\mathbb{R}^{n+1}$. Consider a sequence of points $x_j \in M_j$ with $x_j \to x_* \in \mathbb{R}^{n+1}$. Suppose that we are given a radius $\rho > 0$ within which no M_j has a boundary, i.e.*

$$\left(\overline{M}_j \setminus M_j\right) \cap B_\rho(x_*) = \emptyset \text{ for all } j$$

and the following uniform curvature estimate applies:

$$\sup_{x \in M_j \cap B_\rho(x_*)} |h|^2 \leq \frac{C_0}{\rho^2} \text{ for all } j.$$

Then there exists a constant $\sigma(\rho, C_0)$ and an affine space $T^n \subset \mathbb{R}^{n+1}$ containing x_* such that, after possibly passing to a subsequence, the submanifolds representing the connected component around x_j of M_j within a given *cylinder*[8],

$$\mathcal{C}_{x_j}\left(M_j \cap C_{\sigma,\frac{2\rho}{3},T^n}(x_*)\right)$$

converge in $C^{1,\alpha}$, $\alpha \in [0,1)$, *as graphs to an* n*-dimensional* $C^{1,1}$*-submanifold* M_* *containing* x_* *and tangent to* T^n *at that point. If we additionally have higher order curvature estimates,*

$$\sup_{x\in M_j\cap B_\rho(x_*)} |\nabla^m h|^2 \leq \frac{C_m}{\rho^{2(m+1)}} \text{ for all } m \geq 0 \text{ and all } j$$

then the convergence is in C^∞ *to a smooth submanifold.*

Remark 2.20. The idea of the proof is the following: First, given a point x in a surface M, one shows that a bound on the second fundamental form of M around x implies that there exists an $r_0 > 0$ such that M can locally be written as a graph over a ball of radius r_0 centered at x in the tangent plane to M at x. Second, for j sufficiently large, one can write the surfaces M_j locally around x_j as graphs over *one* hyperplane. By the uniform curvature bound there exists an $r_1 > 0$ such that locally around each x_j the surfaces M_j can be written as graphs over a ball of radius r_1 in this one hyperplane. It follows that for j large, the domain of each graph representation contains the ball of radius $r_1/8 > 0$ centered at x_*. One applies Arzelà-Ascoli on this ball to obtain the hypersurface M (after possibly going to a subsequence).

We apply the above results to show that there exists a converging subsequence of $\{\Sigma_i\}$.

Let $p_i \in \Sigma_i$ be a sequence of points such that $p_i \to p$ for a point $p \in M$ as $i \to \infty$. We choose normal coordinates in a neighbourhood O of p such that

(1) p is the origin of the coordinate system,
(2) $g_{ab}(x) = \delta_{ab} + O(|x|^2)$ for x small (in the Euclidean norm on the coordinate chart).

Given a stable minimal hypersurface in a Riemannian n-manifold (M,g), for $3 \leq n \leq 5$, minimal surface theory allows us to estimate its second fundamental form pointwise by its area, *cf.* [11]. By Claim 2.17 and the fact that $g_i \to g$ for $i \to \infty$, we have a uniform g-bound for h_{Σ_i} calculated with

[8] Let $C_{R,h,T^n}(x_0)$ denote the cylinder of radius R, height $2h$, whose axis is normal to T^n centered at x_0. For any set W containing x_j, let $\mathcal{C}_{x_j}(W)$ denote the connected component of W which contains x_j.

respect to g_i. As a consequence of Lemma 2.7 and the convergence of the g_i, we can uniformly bound the Euclidean norm of the second fundamental form of each surface Σ_i calculated with respect to the Euclidean norm.

We employ Theorem 2.8 to conclude the existence of a $C^{1,1}$-hypersurface Σ_p and a subsequence $\Sigma_i \to \Sigma_p$ in $C^{1,\alpha}$ for $i \to \infty$ within a non-empty cylinder centered at p whose radius we can bound below.[9]

The local convergence of a subsequence of the minimal surfaces $\Sigma_i \to \Sigma_p$ in $C^{1,1}$ as $i \to \infty$ together with the convergence $g_i \to g$ as $i \to \infty$ implies that $H_{\Sigma_p,g} = 0$ holds weakly on Σ_p. Regularity theory for minimal surfaces, *cf.* [12], shows that Σ_p is in fact smooth and consequently Σ_p is minimal in the classical sense.

Recall the above: There exist $a, b > 0$ such that for any i and any point $p_i \in \Sigma_i$ there exists a cylinder $C_{a,b}(p_i)$ of radius a and height b centered at p_i such that $\mathcal{C}_{p_i}(\Sigma_i \cap C_{a,b}(p_i))$ can be written as graph over a ball of radius a. This implies a lower area bound of the area of Σ_i covered by this graph. Above we showed that the area of each Σ_i is uniformly bounded. Consequently there exists an integer $N' \geq 1$ and an atlas for each Σ_i consisting of at most N' charts, and each of these charts is graphical.

By a diagonal sequence argument we conclude the existence of a $C^{1,1}$-surface Σ and a subsequence $\Sigma_{i_k} \to \Sigma$ in $C^{1,\alpha}$ as $k \to \infty$, for $\alpha \in [0,1)$. The surface Σ is minimal by the above, which yields a contradiction.

2.7. **The construction yields an admissible manifold.**

Claim 2.21. *$(M, \hat{g})$ is an admissible extension.*

Proof of Claim. First, the metric $\hat{g}$ is asymptotically flat by the asymptotic flatness of g and the expansion of the function u and its derivatives as $|x| \to \infty$ given in Section 2.3. Second, $R(\hat{g}) \geq 0$ by the construction of $\hat{g}$. Note that ∂M is minimal in $\hat{g}$ because $\hat{g}$ is equal to g near ∂M. Third, following Remark 2.10, we can arrange for u such that $\bar{g}$ is arbitrarily close to

[9]The lower bound depends on the injectivity radius of the exponential map at p and the exact growth of $|g - \delta|_\delta$ around p (as in the assumptions of Lemma 1.5). The radius and the growth can be bounded uniformly over B_{R_0} from below and above, respectively. In particular, we can bound the radius from below independent of the point p.

g. This implies that $R(\bar{g})$ can be constructed arbitrarily close to $R(g)$. Using the dependence in the statement of Theorem A.1, we can therefore arrange for $\hat{g}$ arbitrarily close to g on M. Recall that in the case of vanishing scalar curvature, the initial scalar curvature deformation can be made arbitrarily small. This, together with the asymptotic expansion of u and its derivatives given in Section 2.3, implies that $\hat{g}$ satisfies the assertions of Theorem 2.3. Note that (M, g) contains by definition no closed minimal surfaces other than ∂M. We conclude that $(M, \hat{g})$ must also contain no closed minimal surfaces other than ∂M, and is subsequently an admissible manifold. □

Appendix A. Results about non-static metrics

The following local scalar curvature deformation theorem is a special case of Theorem 1.2 in [16]:

Theorem A.1. *Let $k \geq 4$. Let $\left(\overline{\Omega}, g\right)$ be a compact $C^{k,\alpha}$ Riemannian manifold of dimension $n \geq 2$ with boundary, and let Ω be the manifold interior of $\overline{\Omega}$. Assume that g is non-static on $\overline{\Omega}$. Let $\Omega_0 \subset \Omega$ be a non-empty open set that is compactly contained in Ω. There exist $\epsilon, C > 0$ such that for any $\sigma \in C^{k-4}\left(\overline{\Omega}\right)$ with support in Ω_0 and with $\|\sigma\|_{C^{k-4,\alpha}} < \epsilon$, there is a $C^{k-2,\alpha}$-metric γ on $\overline{\Omega}$ so that* $\mathrm{supp}(\gamma - g)$ *is compactly contained in Ω, such that $\|\gamma - g\|_{C^{k-2,\alpha}} \leq C\|\sigma\|_{C^{k-4,\alpha}}$, and such that $R(\gamma) = R(g) + \sigma$. If g and σ are smooth, we can arrange for γ to be smooth as well.*

We prove the following proposition by following the steps from Proposition 2.3, Corollary 2.4 and Proposition 3.2 in [5] or, similarly, Proposition 2.1 and the subsequent remark in [16].

Proposition A.2. *Given a non-static Riemannian metric g on a 3-manifold M without boundary, there exists a non-empty smooth bounded open set $U \subset M$ such that g is non-static on U.*

Proof. We prove the proposition by contradiction. At first, we show that the kernel of L_g^* is finite-dimensional on any set Ω. Let $c(t)$ be a geodesic starting from some $x_0 \in \Omega$ and $f \in C^\infty(\Omega)$ be a non-trivial element of the kernel of L_g^* on Ω. Define $h(t) = f(c(t))$. This function satisfies

$$
\begin{aligned}
h''(t) &= \mathrm{Hess}_{c(t)}(f)(c'(t), c'(t)) \\
&= \left[\left(\mathrm{Ric}(g) - \frac{R(g)}{n-1}g\right)(c'(t), c'(t))\right] h(t).
\end{aligned}
$$

This is a linear second-order ODE for $h(t)$. The initial value space of pairs $(f(x_0), \nabla f(x_0))$ for this ODE is finite-dimensional, which implies that the dimension of $\ker(L_g^*, \Omega)$ must be finite-dimensional. Indeed, trivial initial data implies that f vanishes in a neighborhood around x_0. As f satisfies an elliptic equation, a unique continuation argument shows that f vanishes identically on Ω, *cf.* the proof of Proposition 2.3 in [5].
Let $\{\Omega_k\}$ be a sequence of bounded smooth open sets constituting an exhaustion of M. By inclusion,

$$\ker(L_g^*, \Omega_{k+1}) \subset \ker(L_g^*, \Omega_k)$$

This, together with the fact that the kernel over Ω_1 is finite-dimensional, implies that the sequence of kernels must *stabilize*, i.e. become constant. Either it stabilizes at $\{0\}$, which establishes the proposition, or it stabilizes at a non-trivial set W. In the latter case, let $f \in W$ be non-trivial. Then, by definition, f would be a non-trivial element of $\ker(L_g^*, \Omega_k)$ for all $k \geq k_0$ for a finite $k_0 \geq 1$. This implies that $f \in \ker(L_g^*, M)$, as $\{\Omega_k\}$ is an exhaustion of M. This is a contradiction to the assertion of the proposition. □

Remark A.3. Note that an element $f \in H^2_{\text{loc}}$ in the kernel of L_g^* lies in fact in C^∞ by elliptic regularity, *cf.* Proposition 2.5 in [5].

The proposition below follows directly from Proposition 2.3 in [5].

Proposition A.4. *If the kernel of L_g^* is non-trivial on $M \setminus \Omega$, then the scalar curvature is constant in $M \setminus \Omega$.*

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