Linearized Gravity

Camilla Adams

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1 Introduction

In this chapter we will derive an approximation of Einstein's equations in a scenario with weak gravity. Additionally we will assume the existence of a global inertial coordinate system. Let us now specify what we mean by weak gravity: We will assume that the spacetime metric $g_{\mu\nu}$ is nearly flat and thus can be expressed as

$$g_{\mu\nu} = \eta_{\mu\nu} + \gamma_{\mu\nu}$$

where $\eta_{\mu\nu}$ denotes the Minkowski metric and $\gamma_{\mu\nu}$ the deviation from the Minkowski metric with components $\gamma_{\mu\nu} \ll 1$ in a global inertial coordinate system. γ will also be called perturbation.
Linearised gravity describes the approximation of Einstein's equations that we obtain when we plug in $g_{\mu\nu} = \eta_{\mu\nu} + \gamma_{\mu\nu}$ and only consider the terms that are linear in γ.
Even though we made quite strong assumptions, the approximation of general relativity by using linearized gravity is found to hold quite well in nature, except for phenomena involving high gravity like gravitational collapse and black holes or when dealing with the large scale structure of the universe.

2 Notational conventions

To make life easier, we will use the Einstein summation convention. Furthermore we will use ∂ to denote the derivative operator associated with η and we will also use η to raise and lower indices rather than g, to avoid having γ hidden in a raised or lowered index. The only exception will be, that $g^{\mu\nu}$ shall denote the inverse of $g_{\mu\nu}$ rather than $\eta^{\mu\alpha}\eta^{\nu\beta}g_{\alpha\beta}$.

Lemma 1. *In the linear approximation we have*

$$g^{\mu\nu} := (g^{-1})^{\mu\nu} = \eta^{\mu\nu} - \gamma^{\mu\nu}.$$

Proof.

$$\begin{aligned} g_{ab}(\eta^{bc} - \gamma^{bc}) &= (\eta_{ab} + \gamma_{ab})(\eta^{bc} - \gamma^{bc}) \\ &= \underbrace{\eta_{ab}\eta^{bc}}_{\delta_a^c} - \eta_{ab}\gamma^{bc} + \eta^{bc}\gamma_{ab} - \gamma_{ab}\gamma^{bc} \\ &= \delta_a^c - \underbrace{\eta_{ab}\eta^{bi}}_{\delta_a^i}\eta^{cj}\gamma_{ij} + \eta^{bc}\gamma_{ab} - \gamma_{ab}\gamma^{bc} \\ &= \delta_a^c \underbrace{-\eta^{cj}\gamma_{aj} + \eta^{cb}\gamma_{ab}}_{0} - \gamma_{ab}\gamma^{bc} \\ &= \delta_a^c \end{aligned}$$

in the linear approximation.
$\Rightarrow$ The inverse of $g_{\mu\nu}$ is indeed given by $\eta^{\mu\nu} - \gamma^{\mu\nu}$. $\square$

3 The linearized Einstein equation

Now we want to derive the linearized Einstein equation.
As a first step, we will determine the Christoffel symbols with respect to $g_{ab} = \eta_{ab} + \gamma_{ab}$ in the linear approximation:

$$\begin{aligned} \Gamma^c_{ab} &= \frac{1}{2} g^{cd}(\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab}) \\ &= \frac{1}{2}(\eta^{cd} - \gamma^{cd})(\partial_a(\eta_{bd} + \gamma_{bd}) + \partial_b(\eta_{ad} + \gamma_{ad}) - \partial_d(\eta_{ab} + \gamma_{ab})) \\ &= \frac{1}{2}(\eta^{cd} - \gamma^{cd})(\underbrace{\partial_a\eta_{bd}}_{0} + \partial_a\gamma_{bd} + \underbrace{\partial_b\eta_{ad}}_{0} + \partial_b\gamma_{ad} - \underbrace{\partial_d\eta_{ab}}_{0} - \partial_d\gamma_{ab}) \\ &= \frac{1}{2}(\eta^{cd}(\partial_a\gamma_{bd} + \partial_b\gamma_{ad} - \partial_d\gamma_{ab}) - \underbrace{\gamma^{cd}(\partial_a\gamma_{bd} + \partial_b\gamma_{ad} - \partial_d\gamma_{ab})}_{\text{terms quadratic in } \gamma}) \\ &= \frac{1}{2}\eta^{cd}(\partial_a\gamma_{bd} + \partial_b\gamma_{ad} - \partial_d\gamma_{ab}) \end{aligned}$$

Next we will determine the Ricci tensor to linear order:

$$\begin{aligned} R_{ab} &= \partial_c\Gamma^c_{ab} - \partial_a\Gamma^c_{cb} + \underbrace{\Gamma^d_{ab}\Gamma^c_{dc} - \Gamma^d_{cb}\Gamma^c_{da}}_{\text{terms quadratic in } \gamma} \\ &= \partial_c\left(\frac{1}{2}\eta^{cd}(\partial_a\gamma_{bd} + \partial_b\gamma_{ad} - \partial_d\gamma_{ab})\right) - \partial_a\left(\frac{1}{2}\eta^{cd}(\partial_c\gamma_{bd} + \partial_b\gamma_{cd} - \partial_d\gamma_{cb})\right) \\ &= \frac{1}{2}\eta^{cd}(\partial_c\partial_a\gamma_{bd} + \partial_c\partial_b\gamma_{ad} - \partial_c\partial_d\gamma_{ab}) - \frac{1}{2}\eta^{cd}(\partial_a\partial_c\gamma_{bd} + \partial_a\partial_b\gamma_{cd} - \partial_a\partial_d\gamma_{cb}) \\ &= \frac{1}{2}(\partial^d\partial_b\gamma_{ad} - \partial^d\partial_d\gamma_{ab} - \partial_a\partial_b\gamma^c_c + \partial_a\partial^c\gamma_{cb}) \\ &= \frac{1}{2}(\partial^c\partial_b\gamma_{ac} + \partial^c\partial_a\gamma_{bc} - \partial^c\partial_c\gamma_{ab} - \partial_a\partial_b\gamma^c_c) \end{aligned}$$

As a last step before we can write down the linearized Einstein equation, we need to determine the scalar curvature:

$$\begin{aligned}
R &= g^{ij}R_{ij} \\
&= (\eta^{ij} - \gamma^{ij})\frac{1}{2}(\partial^c\partial_j\gamma_{ic} + \partial^c\partial_i\gamma_{jc} - \partial^c\partial_c\gamma_{ij} - \partial_i\partial_j\gamma^c_c) \\
&= \frac{1}{2}\eta^{ij}(\partial^c\partial_j\gamma_{ic} + \partial^c\partial_i\gamma_{jc} - \partial^c\partial_c\gamma_{ij} - \partial_i\partial_j\gamma^c_c) \\
&- \frac{1}{2}\underbrace{\gamma^{ij}(\partial^c\partial_j\gamma_{ic} + \partial^c\partial_i\gamma_{jc} - \partial^c\partial_c\gamma_{ij} - \partial_i\partial_j\gamma^c_c)}_{\text{terms quadratic in } \gamma} \\
&= \frac{1}{2}(\partial^c\partial^i\gamma_{ic} + \partial^c\partial^j\gamma_{jc} - \partial^c\partial_c\gamma^i_i - \partial^j\partial_j\gamma^c_c) \\
&= \partial^c\partial^i\gamma_{ci} - \partial^c\partial_c\gamma^i_i
\end{aligned}$$

Now we can plug everything into the Einstein equation $G_{ab} = 8\pi T_{ab}$:

$$\begin{aligned}
G_{ab} &= R_{ab} - \frac{1}{2}\eta_{ab}R \\
&= \frac{1}{2}(\partial^c\partial_b\gamma_{ac} + \partial^c\partial_a\gamma_{bc} - \partial^c\partial_c\gamma_{ab} - \partial_a\partial_b\gamma^c_c) - \frac{1}{2}\eta_{ab}(\partial^c\partial^d\gamma_{cd} - \partial^c\partial_c\gamma^i_i) \\
&= \frac{1}{2}(\partial^c\partial_b\gamma_{ac} + \partial^c\partial_a\gamma_{bc} - \partial^c\partial_c\gamma_{ab} - \partial_a\partial_b\gamma^c_c - \eta_{ab}\partial^c\partial^d\gamma_{cd} + \eta_{ab}\partial^c\partial_c\gamma^i_i) \\
&= \frac{1}{2}(\partial^c\partial_b\gamma_{ac} + \partial^c\partial_a\gamma_{bc} - \partial^c\partial_c\gamma_{ab} - \frac{1}{2}\partial_a\partial_b\gamma^i_i - \frac{1}{2}\partial_b\partial_a\gamma^i_i - \eta_{ab}\partial^c\partial^d\gamma_{cd} \\
&+ \frac{1}{2}\eta_{ab}\partial^c\partial_c\gamma^i_i + \frac{1}{2}\eta_{ab}\partial^c\partial_c\gamma^i_i) \\
&= \frac{1}{2}(\partial^c\partial_b\gamma_{ac} + \partial^c\partial_a\gamma_{bc} - \partial^c\partial_c\gamma_{ab} - \frac{1}{2}\eta_{ac}\partial^c\partial_b\gamma^i_i - \frac{1}{2}\eta_{bc}\partial^c\partial_a\gamma^i_i - \eta_{ab}\partial^c\partial^d\gamma_{cd} \\
&+ \frac{1}{2}\eta_{ab}\partial^c\partial_c\gamma^i_i + \frac{1}{2}\eta_{ab}\eta_{cd}\partial^c\partial^d\gamma^i_i) \\
&= \frac{1}{2}(\partial^c\partial_b\gamma_{ac} - \frac{1}{2}\eta_{ac}\partial^c\partial_b\gamma^i_i + \partial^c\partial_a\gamma_{bc} - \frac{1}{2}\eta_{bc}\partial^c\partial_a\gamma^i_i - \partial^c\partial_c\gamma_{ab} + \frac{1}{2}\eta_{ab}\partial^c\partial_c\gamma^i_i \\
&- \eta_{ab}\partial^c\partial^d\gamma_{cd} + \frac{1}{2}\eta_{ab}\eta_{cd}\partial^c\partial^d\gamma^i_i) \\
&= \frac{1}{2}(\partial^c\partial_b(\gamma_{ac} - \frac{1}{2}\eta_{ac}\gamma^i_i) + \partial^c\partial_a(\gamma_{bc} - \frac{1}{2}\eta_{bc}\gamma^i_i) - \partial^c\partial_c(\gamma_{ab} - \frac{1}{2}\eta_{ab}\gamma^i_i) \\
&- \eta_{ab}\partial^c\partial^d(\gamma_{cd} - \frac{1}{2}\eta_{cd}\gamma^i_i)) \\
&= \frac{1}{2}(\partial^c\partial_b\overline{\gamma}_{ac} + \partial^c\partial_a\overline{\gamma}_{bc} - \partial^c\partial_c\overline{\gamma}_{ab} - \eta_{ab}\partial^c\partial^d\overline{\gamma}_{cd})
\end{aligned}$$

with $\overline{\gamma}_{ab} := \gamma_{ab} - \frac{1}{2}\eta_{ab}\gamma^i_i$. Hence we get

$$G_{ab} = \frac{1}{2}(\partial^c\partial_b\overline{\gamma}_{ac} + \partial^c\partial_a\overline{\gamma}_{bc} - \partial^c\partial_c\overline{\gamma}_{ab} - \eta_{ab}\partial^c\partial^d\overline{\gamma}_{cd}) = 8\pi T_{ab}$$

as the linearized Einstein equation.

4 Gauge Freedom

In this section we will use the gauge freedom in general relativity to further simplify the Einstein equation. For that purpose we will introduce one-parameter groups of diffeomorphisms and based on this the Lie derivative.

Definition 1. *A one-parameter group of diffeomorphisms is a $\mathcal{C}^\infty$-map $\phi : \mathbb{R} \times M \to M$ such that for fixed $t \in \mathbb{R}$, $\phi_t := \phi(t, \cdot) : M \to M$ is a diffeomorphism and for all $t, s \in \mathbb{R}$ it holds that $\phi_t \circ \phi_s = \phi_{t+s}$.*

Let ϕ_t a one-parameter group of diffeomorphisms. Then for fixed $p \in M$, $\phi_t(p) : \mathbb{R} \to M$ defines a curve in M, that passes through p at $t = 0$. Now we define $X|_p$ as the tangent to this curve at $t = 0$. Then $X : M \to TM$ given by $p \mapsto X|_p$ defines a vector field (TM denotes the tangent bundle).
On the other hand, if a vector field X is given, we can get a corresponding one-parameter group ϕ_t by setting ϕ_t as the flow of X.
Using this, we get a direct correspondance between one-parameter groups of diffeomorphisms and vector fields.

Now we can define the Lie derivative. The Lie derivative is given by the change of a tensor field, along the flow generated by a vector field:

Definition 2. *Let M a manifold, X a vector field on M and T a smooth tensor field on M with components $T^{a_1 \dots a_k}{}_{b_1 \dots b_l}$. Then the Lie derivative of T with respect to X is defined as*

$$\mathcal{L}_X T := \lim_{t \to 0} \left(\frac{\phi^*_{-t} T^{a_1 \dots a_k}{}_{b_1 \dots b_l} - T^{a_1 \dots a_k}{}_{b_1 \dots b_l}}{t} \right),$$

where ϕ_t is the flow generated by X and ϕ^*_{-t} denotes the pullback of T via ϕ_{-t}.

Since the Lie derivative as defined above is not easily used in calculations, we will now state one of its useful properties:

Lemma 2. *Let g_{ab} be a metric tensor field and ξ a vector field. Then we have*

$$\mathcal{L}_\xi g_{ab} = \nabla_a \xi_b + \nabla_b \xi_a$$

with ∇ the Levi-Civita connection corresponding to g_{ab}.

Proof. see Wald, General Relativity, Appendix C □

Now we introduced everything we need for the gauge transformation. The goal will be to find a gauge in which $\partial^b \overline{\gamma}_{ab} = 0$ holds.

Einstein's equations are invariant under diffeomorphisms, meaning if $\phi : M \to M$ is a diffeomorphism and $g_{\mu\nu}$ a solution to Einstein's equations, then $\phi^* g_{\mu\nu}$, with ϕ^* the pullback, is also a solution to Einstein's equations.

We define $g_{ab}(\epsilon) := \eta_{ab} + \epsilon\gamma_{ab}$ for small ϵ. Note that $(M, g_{ab}(\epsilon))$ and $(M, \phi_\epsilon^* g_{ab}(\epsilon))$ represent the same spacetime, for ϕ_ϵ an arbitrary one-parameter group of diffeomorphisms. Then we have

$$\gamma_{ab} = \frac{\mathrm{d}}{\mathrm{d}\epsilon} g_{ab}(\epsilon).$$

If we define

$$\tilde{\gamma}_{ab} := \frac{\mathrm{d}}{\mathrm{d}\epsilon} \phi_\epsilon^* g_{ab}(\epsilon)$$

we easily see that γ_{ab} and $\tilde{\gamma}_{ab}$ describe the same perturbation. In the linear approximation we have

$$\tilde{\gamma}_{ab} - \gamma_{ab} = \frac{\mathrm{d}}{\mathrm{d}\epsilon} \left(\phi_\epsilon^* g_{ab}(\epsilon) - g_{ab}(\epsilon)\right) = \mathcal{L}_\xi \eta_{ab}$$

where ξ is the vector field that generates $\phi_{-\epsilon}$. Hence, if $g_{\mu\nu} = \eta_{\mu\nu} + \gamma_{\mu\nu}$ is a solution to Einstein's equation, then $\tilde{g}_{\mu\nu} = \eta_{\mu\nu} + \tilde{\gamma}_{\mu\nu}$ with $\tilde{\gamma}_{\mu\nu} = \gamma_{\mu\nu} + \mathcal{L}_\xi \eta_{\mu\nu}$ is also a solution. Using Lemma 2, we follow

$$\tilde{\gamma}_{\mu\nu} = \gamma_{\mu\nu} + \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = \gamma_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi \mu.$$

Now we want to find a transformation, such that $\partial^b \overline{\tilde{\gamma}}_{ab} = 0$. We compute

$$\begin{aligned}
\partial^b \overline{\tilde{\gamma}}_{ab} &= \partial^b (\overline{\gamma_{ab} + \partial_a \xi_b + \partial_b \xi_a}) \\
&= \partial^b (\gamma_{ab} + \partial_a \xi_b + \partial_b \xi_a - \frac{1}{2}\eta_{ab}(\gamma_c^c + \partial^c \xi_c + \partial_c \xi^c)) \\
&= \partial^b (\overline{\gamma}_{ab} + \partial_a \xi_b + \partial_b \xi_a - \frac{1}{2}\eta_{ab}(\partial^c \xi_c + \partial_c \xi^c)) \\
&= \partial^b \overline{\gamma}_{ab} + \partial^b (\partial_a \xi_b + \partial_b \xi_a - \frac{1}{2}\eta_{ab}(\eta^{cd}\partial_d \xi_c + \eta^{cd}\partial_c \xi_d)) \\
&= \partial^b \overline{\gamma}_{ab} + \partial^b (\partial_a \xi_b + \partial_b \xi_a - \eta_{ab}\eta^{cd}\partial_d \xi_c) \\
&= \partial^b \overline{\gamma}_{ab} + \partial^b (\partial_a \xi_b + \partial_b \xi_a - \eta_{ab}\partial^c \xi_c) \\
&= \partial^b \overline{\gamma}_{ab} + \partial^b \partial_a \xi_b + \partial^b \partial_b \xi_a - \eta_{ab}\partial^b \partial^c \xi_c \\
&= \partial^b \overline{\gamma}_{ab} + \partial_a \partial^b \xi_b + \partial^b \partial_b \xi_a - \partial_a \partial^c \xi_c \\
&= \partial^b \overline{\gamma}_{ab} + \partial^b \partial_b \xi_a.
\end{aligned}$$

Hence we have $\partial^b \overline{\tilde{\gamma}}_{ab} = 0 \Leftrightarrow \partial^b \partial_b \xi_a = -\partial^b \overline{\gamma}_{ab}$. Therefor by solving

$$\partial^b \partial_b \xi_a = -\partial^b \overline{\gamma}_{ab}$$

for ξ we find a gauge that satisfies

$$\partial^b \overline{\tilde{\gamma}}_{ab} = 0.$$

In this gauge, the linearized Einstein equation simplifies to

$$16\pi T_{ab} = \underbrace{\partial^c \partial_b \overline{\gamma}_{ac}}_{0} + \underbrace{\partial^c \partial_a \overline{\gamma}_{bc}}_{0} - \partial^c \partial_c \overline{\gamma}_{ab} - \eta_{ab}\partial^c \underbrace{\partial^d \overline{\gamma}_{cd}}_{0} = -\partial^c \partial_c \overline{\gamma}_{ab}.$$

References

The script and talk are based on chapter 4.4 and Appendix C of the book "General Relativity" by Robert M. Wald, published 1984 by The University of Chicago Press.