Initial value formulation of GR

Fatin Nofal

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1 Introduction

The initial value formulation allows us to specify initial data on a three-dimensional hypersurface and then evolve this data forward in time. We will outline the key principles of the initial value formulation, starting with the constraints that initial data must satisfy, and explore the fundamental theorems that guarantee local existence of solutions.

2 Theorem

Let $(\phi_0)_1, ..., (\phi_0)_n$ be any solution of the quasilinear hyperbolic system:

$g^{ab}(x, \phi_j, \nabla_c \phi_j) \nabla_a \nabla_b \phi_i = F_i(x, \phi_j, \nabla_c \phi_j)$ for $i, j = 1, ..., n$ (1)

for n unknown functions $\phi_1, ..., \phi_n$ on a manifold M.

Let $(g_0)^{ab} = g^{ab}(x, (\phi_0)_j, \nabla_c (\phi_0)_j)$ suppose $(M, (g_0)_{ab}$ is globally hyperbolic. Let Σ ba a smooth spacelike Cauchy surface of $(M, (g_0)_{ab}$. Then for initial data on Σ sufficiently close to the initial data for $(\phi_0)_1, ..., (\phi_0)_n$ there exists an open neighborhood O of Σ such that (1) has a unique solution, $\phi_1, ..., \phi_n$ in O and $(O, g_{ab}(x, \phi_j, \nabla_c \phi_j)$ is globally hyperbolic.

3 Initial value formulation of GR

In general relativity initial data should consists of a triple (Σ, h_{ab}, Kab):

Σ is a three-dimenional manifold.

h_{ab} is a Riemannian metric on Σ.

K_{ab} is a symmetric tensor field on Σ.

Consider now the Einstein's equation in vacuum

$0 = G_{ab} = R_{ab} - \frac{1}{2} R g_{ab}$

The metric tensor has 10 independent components $g_{\mu\nu}$ due to its symmetry, while the Einstein field equation provide 10 equations, in the context of the initial value formulation we use the **4 constraint equations** to set up initial conditions, and the **6 evolution equations** to evolve these initial conditions forward in time. Therefore,there is only 6 equations for 10 unknown components.

To solve the Einstein's equation there should be only 6 function in the 10 metric components $g_{\mu\nu}$, and this is the case by introducing a choice of coordinates.

Consider harmonic coordinates x^μ, satisfy

$$H^\mu \equiv \nabla_a \nabla^a x^\mu = 0$$

Writing out the coordinate basis expression for H^μ:

$$\begin{aligned}
H^\mu &= \sum_\alpha \frac{1}{\sqrt{|g|}} \partial_\alpha \left(\sqrt{|g|} \sum_\beta g^{\alpha\beta} \partial_\mu x^\beta \right) \\
&= \sum_\alpha \frac{1}{\sqrt{|g|}} \partial_\alpha \left(\sqrt{|g|} g^{\alpha\mu} \right) \\
&= \sum_\alpha \left[\partial_\alpha g^{\alpha\mu} + \frac{1}{2} g^{\alpha\mu} \left(\frac{1}{g} \partial_\alpha g \right) \right] \\
&= \sum_{\alpha,\rho,\sigma} \left[-g^{\alpha\rho} g^{\mu\sigma} \partial_\alpha g_{\rho\sigma} + \frac{1}{2} g^{\alpha\mu} g^{\rho\sigma} \partial_\alpha g_{\rho\sigma} \right] (2)
\end{aligned}$$

where we used the following equations:

$$\begin{aligned}
&\nabla_a T^a = \sum_\mu \frac{1}{\sqrt{|g|}} \partial_\mu (\sqrt{|g|} T^\mu) \\
&\frac{1}{g} \partial_\alpha g = \sum_{\rho,\sigma} g^{\rho\sigma} \partial_\alpha g_{\rho\sigma} \\
&\partial_\alpha g^{\alpha\mu} = \sum_{\rho,\alpha} -g^{\alpha\rho} g^{\mu\rho} \partial_\alpha g_{\rho\sigma} \\
&g = \det g_{\alpha\beta}
\end{aligned}$$

Now we can compute:

$$\begin{aligned}
&\partial_\nu H^\alpha \\
&= \sum_{\sigma,\beta,\rho} -\partial_\nu (g^{\sigma\beta} g^{\alpha\rho}) \partial_\sigma g_{\beta\rho} - g^{\sigma\beta} g^{\alpha\rho} \partial_\sigma \partial_\nu g_{\beta\rho} \\
&+ \frac{1}{2} \partial_\nu (g^{\sigma\alpha} g^{\beta\rho}) \partial_\sigma g_{\beta\rho} + \frac{1}{2} g^{\sigma\alpha} g^{\beta\rho} \partial_\sigma \partial_\nu g_{\beta\rho} \\
&= \sum_{\sigma,\beta,\rho} -g^{\sigma\beta} g^{\alpha\rho} \partial_\sigma \partial_\nu g_{\beta\rho} + \frac{1}{2} g^{\sigma\alpha} g^{\beta\rho} \partial_\sigma \partial_\nu g_{\beta\rho} \\
&+ (\text{lower order terms}) (3)
\end{aligned}$$

Using (2) and (3), we see that most of second derivative terms in $R_{\mu\nu}$ can be reexpressed in terms of H^μ:

$$R_{\mu\nu} = \sum_{\alpha,\beta} -\frac{1}{2} g^{\alpha\beta} \partial_\alpha \partial_\beta g_{\mu\nu} - \sum_\alpha \left(\frac{1}{2} g_{\alpha\mu} \partial_\nu H^\alpha + \frac{1}{2} g_{\alpha\nu} \partial_\mu H^\alpha \right) + \widetilde{F}_{\mu\nu}(g, \partial g)$$

$\widetilde{F}$ is a non-linear function of the metric components and their first derivatives.

Thus the vacuum Einstein equation in harmonic coordinates becomes

$$\begin{aligned} 0 &= R^H_{\mu\nu} + \textstyle\sum_\alpha \left(\frac{1}{2} g_{\alpha\mu}\partial_\nu H^\alpha + \frac{1}{2} g_{\alpha\nu}\partial_\mu H^\alpha\right) \\ &= \textstyle\sum_{\alpha,\beta} -\frac{1}{2} g^{\alpha\beta}\partial_\alpha\partial_\beta g_{\mu\nu} + \widetilde{F}_{\mu\nu}(g, \partial g) \equiv R^H_{\mu\nu} (4) \end{aligned}$$

Einstein's equation is equivalent to the system (4), together with the harmonic coordinate condition $H^\mu \equiv 0$. Thus equation (4) is knowen as the "reduced Einstein equation". The equation is of the form (1) for which the theorem applies.

4 Local existence

Now we can prove local existence of a solution to Einstein's equation for initial data sufficiently near that of flat spacetime.

Let h_{ab} and Kab be given on Σ satisfying the constraint equations.

Plan: we choose initial data set, which is sufficiently near that of flat spacetime then according to the theorem we can solve (4) in a neighborhood of Σ.

Proof: $R^H_{\mu\nu} = 0$ can be solved and suffices to show that $H^\mu \equiv 0$.

Step1: choose the metric components at $t = 0$ as follows:

$$\begin{aligned} g_{00} = -1 \ , \ g_{0\mu} &\text{ for } \mu = 1, 2, 3 \\ g_{\mu\nu} = h_{\mu\nu} &\text{ for } \mu, \nu = 1, 2, 3 \end{aligned}$$

and

$$\partial_t g_{\mu\nu} = K_{\mu\nu} \text{ for } \mu, \nu = 1, 2, 3$$

for the rest time derivatives $\partial_t g_{\mu\nu}$ specify these such that $H^\mu = 0$.

Step2: we prove that if $R^H_{\mu\nu} = 0$ at $t = 0$ with initial data set (Σ, h_{ab}, K_{ab}) then $\partial_t H^\mu = 0$. proof: the vanishing of $R^H_{\mu\nu} = 0$ gives:

$$0 = R_{0\nu} + \textstyle\sum_\alpha \left(\frac{1}{2} g_{\alpha 0}\partial_\nu H^\alpha + \frac{1}{2} g_{\alpha\nu}\partial_0 H^\alpha\right)$$

$R_{0\nu} = 0$ and the spatial derivatives of H^μ vanish ($H^\mu = 0$) so $0 = \partial_t H^\alpha$.

step3:by Bianchi identity, we have

$$\nabla^\mu \left(R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu}\right) = 0$$

if $R^H_{\mu\nu} = 0$ is satisfied then:

$$R_{\mu\nu} = \textstyle\sum_\alpha -\frac{1}{2} g_{\alpha\mu}\partial_\nu H^\alpha - \frac{1}{2} g_{\alpha\nu}\partial_\mu H^\alpha$$

and

$$R = -\partial_\mu H^\mu$$

Hence

$$
\begin{aligned}
0 &= -\sum_{\rho,\mu,\alpha} g^{\rho\mu}\nabla_\rho\left[-\tfrac{1}{2}g_{\alpha\mu}\partial_\nu H^\alpha - \tfrac{1}{2}g_{\alpha\nu}\partial_\mu H^\alpha + \tfrac{1}{2}\partial_\alpha H^\alpha g_{\mu\nu}\right]\\
&= -\sum_{\rho,\mu,\alpha}\left[-\tfrac{1}{2}g^{\rho\mu}g_{\alpha\mu}\partial_\nu\partial_\rho H^\alpha - \tfrac{1}{2}g^{\rho\mu}g_{\alpha\nu}\partial_\mu\partial_\rho H^\alpha + \tfrac{1}{2}g^{\rho\mu}\partial_\alpha H^\alpha g_{\mu\nu}\right] +\\
&\quad \text{(lower order terms linear in } H^\alpha)\\
&= -\sum_{\rho,\mu,\alpha}\tfrac{1}{2}g^{\rho\mu}g_{\alpha\nu}\partial_\mu\partial_\rho H^\alpha + \text{(lower order terms linear in } H^\alpha)
\end{aligned}
$$

We have

$$\Box_g H^\mu = G(H^\mu, \partial H^\mu)$$

with

$$(H^\mu, \partial_t H^\mu) = (0,0) \text{ at } t = 0$$

this proves that $H^\mu = 0$ throughout the region where a solution to (4) exists.

We proved the local existence of a solution of Einstein's equation for initial data sufficiently near that of flat spacetime.