Seminar Partial Differential Equations: Spacelike initial data in general relativity

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1 Introduction

In this seminar, we are going to look into the spacelike initial data in general relativity, which is a fundamental aspect of understanding the dynamics of the gravitational field. The talk essentially involves specifying initial data on a spacelike hypersurface and then determining the subsequent evolution of this data according to Einstein's equation.

The book *General Relativity*, by Robert Wald [Wal84] provides a rigorous treatment of this topic as it emphasizes the mathematical formalism of differential geometry and the theory of partial differential equations. Hence, we consider Wald's book as reference throughout the seminar talk.

1.1 Motivation

What we know so far is that in general relativity we describe spacetime structures and gravitation by a spacetime (M, g_{ab}) where M is a 4-dimensional manifold and g_{ab} is a Lorentzian metric satisfying Einstein's equation

$$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R, \tag{1}$$

where G_{ab} is the Einstein tensor, R_{ab} the Ricci curvature tensor and R is the scalar curvature. Equation (1) forms a set of coupled, nonlinear partial differential equations that govern the dynamics of spacetime. We are able to deduce some exact solutions out of Einstein's equation which give us physical predictions concerning cosmology but solving them is not enough to prove that general relativity is a physically viable theory. There exists a wide class of solutions of Einstein's equation, thus if a large class of solutions failed to exist, then we are being forced to reject general relativity as a correct theory of nature.

The issue starts when we look at classical physics. In classical physics we have a great deal of physical control over the initial conditions of the systems, like for ordinary particle mechanics we have control over the initial positions and velocities. Therefore, we are able to completely determine their behavior as they evolve freely without any outside interference.

Hence, it is natural to believe that the same applies for gravitational problems, but practical ability to control initial conditions for a gravitational problem is very limited. But we might still try for much smaller regions than cosmological scales to be able to control initial conditions so that for any given initial data, Einstein's equation would determine the subsequent evolution of a system.

We propose then that a theory possesses an initial value formulation, if appropriate initial data is specified such that the subsequent dynamical evolution of the system is uniquely determined and the initial value formulation is well posed, if it satisfies the two following properties.

(1) Small changes in the initial data should produce small changes in the solutions over any fixed, compact region of spacetime.

(2) Changes in the initial data in a region S of the initial data surface should not produce any changes in the solutions outside the causal future $J^+(S)$ of this region.

2 Initial Value Formulation of General Relativity

The analysis of Einstein's equation (1) starts with an analogy to the situation in electrodynamics, since it is conceptually similar by the fact that we need to make a gauge choice, i.e., a choice of coordinates, so that Einstein's equation will take a desired form in the sense as we will see for Maxwell's equations.

2.1 Maxwell's equations

Let's consider the vacuum Maxwell's equations for some vector potential A_a in Minkowski spacetime, taking the form

$$\partial^a \left(\partial_a A_b - \partial_b A_a\right) = 0. \tag{2}$$

We choose a surface Σ at constant inertial time $t = 0$ as our initial hypersurface. We see that Maxwell's equations (2) contain no second time derivatives at all in the time component

$$\partial^a \left(\partial_a A_0 - \partial_t A_a\right) = 0, \tag{3}$$

or equivalently

$$\nabla^2 A_0 - \vec{\nabla} \cdot \left(\frac{\partial \vec{A}}{\partial t}\right) = 0, \tag{4}$$

which gives an initial value constraint on the initial data $(A_\mu, \partial A_\mu/\partial t)$. Therefore, any initial data which fails to satisfy the equation (3), cannot possibly yield a solution to Maxwell's equations. The remaining three spatial components do give rise to second order time derivatives of the spatial components of A_a, hence we can solve for $\partial^2 A_\mu/\partial t^2$ in the manner required by the following Cauchy-Kowalewski theorem.

Theorem 1 (Cauchy-Kowalewski theorem)**.** *Let $t, x^1, \ldots, x^{m-1}$ be coordinates of $\mathbb{R}^n$. Consider a system of n partial differential equations for n unknown functions $\phi_1, \ldots, \phi_n$ in $\mathbb{R}^m$, having the form*

$$\frac{\partial^2 \phi_i}{\partial t^2} = F_i\left(t, x^\alpha; \phi_j; \frac{\partial \phi_j}{\partial t}; \frac{\partial \phi_j}{\partial x^\alpha}; \frac{\partial^2 \phi_j}{\partial t \partial x^\alpha}; \frac{\partial^2 \phi_j}{\partial x^\alpha x^\beta}\right) \tag{5}$$

where each F_i is an analytic function of its variables. Let $f_i(x^\alpha)$ and $g_i(x^\alpha)$ be analytic functions. Then there is an open neighborhood $\mathcal{O}$ of the hypersurface $t = t_0$ s.t. within $\mathcal{O}$ there exists a unique solutions of the equation s.t.

$$\phi_i(t_0, x^\alpha) = f_i(x^\alpha) \quad \textit{and} \quad \frac{\partial \phi_i(t_0, x^\alpha)}{\partial t} = g_i(x^\alpha). \tag{6}$$

One could argue that if we differentiate Maxwell's equations (2), we might end up with a second order time derivative which lets us formulate an initial value formulation in the sense of the Cauchy-Kowalewski theorem but by the identity

$$\partial^b \partial^a (\partial_a A_b - \partial_b A_a) = 0. \tag{7}$$

we see that the time derivative vanishes identically if the spatial components of Maxwell's equations are satisfied. Hence, we have an underdetermined system for the vector potential A_a, since there are three spatial equations and an initial constraint for four unknown functions. Therefore, we need to make a gauge choice, i.e., we specify A_0 arbitrarily throughout the spacetime to obtain a solution. Thus, on account of this gauge arbitrariness, Maxwell's equations cannot possibly be expected to determine A_a from initial conditions.

2.2 Spacelike initial data

As we discussed the analogy to electrodynamics, we turn our attention now to Einstein's equation in a vacuum, meaning that the Einstein tensor vanishes, i.e.,

$$G_{ab} = 0. \tag{8}$$

The first issue is the nature of the initial value formulation in general relativity, as we question ourselves what are the quantities to prescribe general relativity initially. We know from classical physics, that if we are given a spacetime background, our task is then to determine the time evolution of the quantities in the background from the initial values and time derivatives. But in general relativity, we are solving for the spacetime itself which we see in Einstein's equation (1) which is a non-linear partial differential equation for components of the metric g_{ab} where second order derivatives occur in the Einstein tensor G_{ab}. However, it is possible to formulate Einstein's equation as a Cauchy problem.

So let (M, g_{ab}) be a globally hyperbolic spacetime. This means, that if the spacetime has no grossly pathological causal features, i.e., there is a spacelike embedded hypersurface $\Sigma \subset M$ with the property that every inextendible causal curve $\gamma : (a, b) \to M$ (meaning that $\dot{\gamma}^a$ is timelike or null everywhere) intersects Σ precisely once, then we say that the spacetime possesses Cauchy surfaces.

Thus, by considering a globally hyperbolic spacetime (M, g_{ab}) we are able to foliate the spacetime itself by these 3-dimensional Cauchy surfaces Σ_t which we parametrize by a global time function t. We introduce the unit normal vector field n^a to the Cauchy surface Σ_t which then allows the spacetime metric to induce the spatial metric by

$$h_{ab} = g_{ab} + n_a n_b, \tag{9}$$

which is a 3-dimensional Riemannian metric on each of the Cauchy surfaces. This consideration indeed holds as we define a vector field t^a on M representing the flow of time throughout spacetime. The vector field t^a then satisfies the condition

$$t^a \nabla_a t = 1, \tag{10}$$

which gives us a clue about how we move in spacetime between two Cauchy surfaces. To understand this movement, we decompose the vector field t^a into its normal and tangential part to the Cauchy surface Σ_t by defining a so-called lapse function N and a shift vector N^a with respect to t^a (see Figure 1):

$$\begin{aligned} N &= -t^a n_a = (n^a \nabla_a t)^{-1}, \\ N_a &= h_{ab} t^b. \end{aligned} \tag{11}$$

Hence, we view the effect of moving forward in time as that of changing the spatial metric on an abstract 3-dimensional manifold Σ from $h_{ab}(0)$ to $h_{ab}(t)$. We then conclude that a globally hyperbolic spacetime (M, g_{ab}) represents a time development of a Riemannian metric on a fixed 3-dimensional manifold Σ where we like to specify our initial data on.

We assume that the appropriate initial data of a globally hyperbolic spacetime then consists of a Riemannian metric h_{ab} and its time derivative on a 3-dimensional manifold Σ.

The time-derivative of the spatial metric on a Cauchy surface Σ can be represented by the notion of the extrinsic curvature, which describes how Σ is embedded in the spacetime manifold M. This interpretation is equivalent to the second fundamental

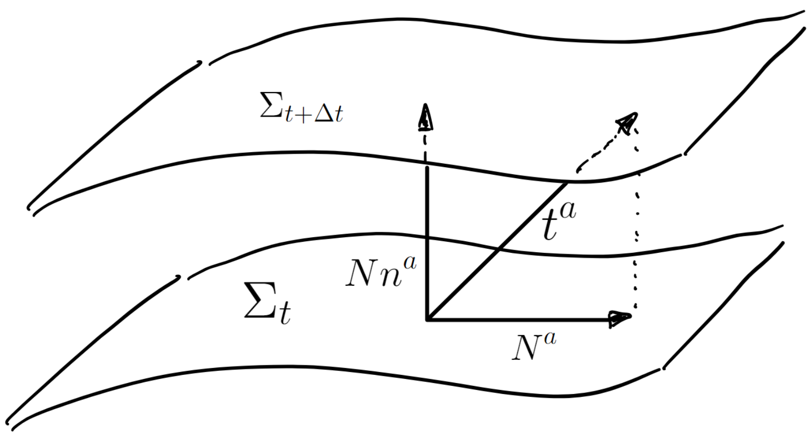

Figure 1: The definition of the lapse function N and shift vector N^a.

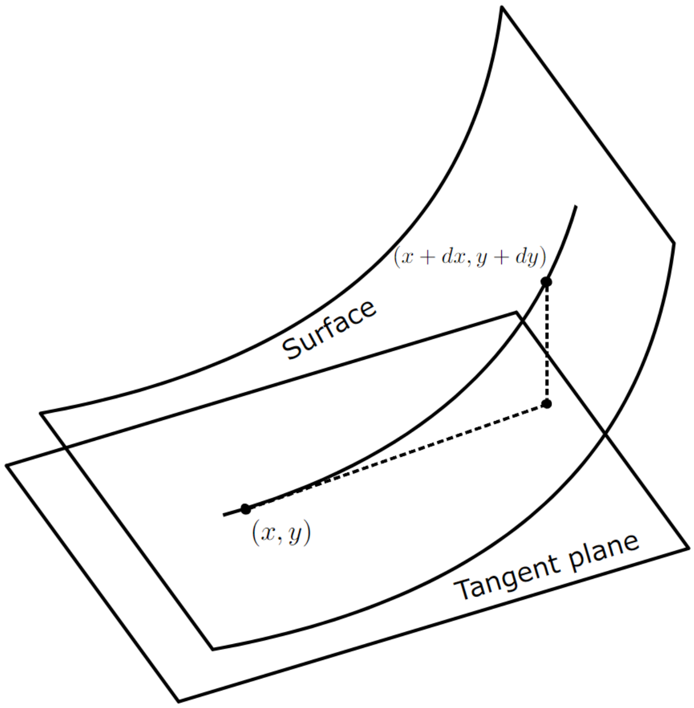

Figure 2: Second fundamental form

form (see Figure 2) on the tangent plane of the Cauchy surface Σ which is given by the expression

$$K_{ab} = h_a{}^c \nabla_c n_b, \tag{12}$$

where n^a is any unit timelike vector field normal to Σ. The formula (12) shows that K_{ab} directly measures the failure of a vector to coincide while it is parallel transported along the Cauchy surface in spacetime (see Figure 3).

We might conclude our considerations that appropriate initial data consists of triples (Σ, h_{ab}, K_{ab}), where (Σ, h_{ab}) is a 3-dimensional Riemannian manifold and K_{ab} is a symmetric covariant 2-tensor on Σ.

3 Well posed initial value formulation in general relativity

We know what our initial data consists of, thus I like to discuss now the well posed initial value formulation of general relativity.

The Cauchy-Kowalewski theorem 1 provides a very general statement to a well posed initial value formulation for non-linear second order partial differential equations. For Einstein's equation (1), we need to make some assumptions in order to apply it to the theorem. Therefore, we restrict the solutions of Einstein's equation to a quasilinear form, i.e., they are linear in the highest derivative terms. This has the advantage that many results on linear systems apply locally, and we can call a system of second-order partial differential equations for n unknown functions $\phi_1, \ldots, \phi_n$ on a manifold M a quasilinear, diagonal, second-order hyperbolic system, if it can be

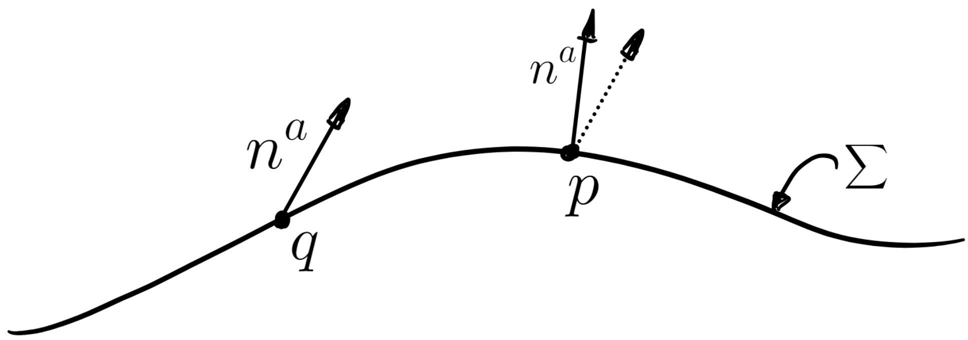

Figure 3: The dashed arrow at p represents the parallel transport of the normal vector n^a at q along a geodesic connecting q and p. The failure of this vector to coincide with n^a at p corresponds intuitively to the bending of Σ in the spacetime in which it is embedded. K_{ab} directly measures the failure of the two vectors at p to coincide for q near p.

put into the form

$$g^{ab}(x;\phi_j;\nabla_c\phi_j)\nabla_a\nabla_b\phi_i = F_i(x;\phi_j;\nabla_c\phi_j), \tag{13}$$

where ∇_a is any derivative operator, g^{ab} is a smooth Lorentzian metric and F_i are smooth functions of each its variables. We note that the metric itself is still depending on the unknown variables and their first derivatives, and the F_i functions have a nonlinear dependence on these variables.

The task is then to show that general relativity has a well posed initial value formulation by casting Einstein's equation exactly into (13). This requires the knowledge about the relations between the spacetime metric, the derivative operator and the curvature on the induced spacelike hypersurface Σ in order to actually deduce initial constraints out of it, which are satisfied by the initial data.

We consider a tensor field on the manifold Σ. We cannot define a covariant derivative on this tensor field, since we do not know how it varies while we move off from Σ. However, if we contract it with the induced metric on Σ, to take no derivatives in directions pointing out of Σ, we are able to obtain the following result.

Lemma 1. *Let (M, g_{ab}) be a spacetime and let Σ be a smooth spacelike hypersurface in M. Let h_{ab} denote the induced metric on Σ and let D_a denote the derivative operator associated with h_{ab}. Then D_a is given by the formula*

$$D_c T^{a_1\cdots a_k}{}_{b_1\cdots b_l} = h^{a_1}{}_{d_1}\cdots h_{b_l}{}^{e_l} h_c{}^f \nabla_f T^{d_1\cdots d_k}{}_{e_1\cdots e_l}, \tag{14}$$

where ∇_a is the derivative operator associated with g_{ab}.

It can be shown in a straightforward sense that D_a satisfies linearity, Leibniz rule, commutativity with contractions, consistency with notion of tangent vectors as directional derivatives on scalar fields and the property to be torsion free. Furthermore, deduce by using (9) that:

$$D_a h_{bc} = h_a{}^d h_b{}^e h_c{}^f \nabla_d (g_{ef} + n_e n_f) = 0, \tag{15}$$

since $\nabla_d g_{ef} = 0$ and $h_{ab}n^b = 0$. Like with the fact that there exists a unique derivative operator ∇_a for the metric g_{ab} satisfying $\nabla_a g_{bc} = 0$, the induced spatial metric h_{ab} then also uniquely determines a natural derivative operator D_a on Σ. Here, the spatial metric acts as a projection operator from the tangent plane of the manifold M at a point p to the tangent plane of the Cauchy surface Σ at p. Hence, any tensor at $p \in \Sigma$ uniquely gives rise to a spacetime tensor at p. In particular, this leads to the construction of the curvature tensor on Σ, which we denote by ${}^{(3)}R_{abc}{}^d$ and apply a dual field ω_a on Σ to obtain

$$ {}^{(3)}R_{abc}{}^d \omega_d = D_a D_b \omega_c - D_b D_a \omega_c. \tag{16} $$

Plugging the result of lemma 1 into the curvature gives rise to the Gauss-Codacci relations

$$ \begin{aligned} {}^{(3)}R_{abc}{}^d &= h_a{}^f h_b{}^g h_c{}^k h^d{}_j R_{fgk}{}^j - K_{ac}K_b{}^d + K_{bc}K_a{}^d, \\ R_{cd}n^d h^c{}_b &= D_a K^a{}_b - D_b K^a{}_a. \end{aligned} \tag{17} $$

3.1 Initial constraints

Now we have everything we need to formulate initial constraints on a 3-dimensional Riemannian spacetime (Σ, h_{ab}).

We do this by constructing a globally hyperbolic spacetime for which Σ is a Cauchy surface on which initial data is induced. So we write down Einstein's equation (1) for metric components $g_{\mu\nu}$ in a local coordinate system $\{x^\mu\}$ with the time coordinate t chosen s.t. $t = 0$ corresponds to Σ and try to cast them into the quasilinear form (13).

Einstein's equation in vacuum, i.e., $G_{\mu\nu} = 0$ yields a system of ten second-order partial differential equations for the ten unknown metric components. These equations have a quasilinear form, so they are all linear in the second order derivative of the metric. Thus, we express the components of the Einstein tensor in terms of coordinate derivatives of the metric tensor components by using the formula for the Ricci tensor components

$$ R_{\mu\rho} = \frac{\partial}{\partial x^\nu}\Gamma^\nu{}_{\mu\rho} - \frac{\partial}{\partial x^\mu}\Gamma^\nu{}_{\nu\rho} + \Gamma^\alpha{}_{\mu\rho}\Gamma^\sigma{}_{\alpha\nu} - \Gamma^\alpha{}_{\nu\rho}\Gamma^\sigma{}_{\alpha\mu}, \tag{18} $$

and the formula for the coordinate basis components of the Christoffel symbol

$$ \Gamma^\rho{}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left\{ \frac{\partial}{\partial x^\mu} g_{\nu\sigma} + \frac{\partial}{\partial x^\nu} g_{\mu\sigma} - \frac{\partial}{\partial x^\sigma} g_{\mu\nu} \right\}, \tag{19} $$

to cast (1) into

$$ \begin{aligned} G_{\mu\nu} = &-\frac{1}{2} g^{\alpha\beta} \left(-2\partial_\beta \partial_{(\nu} g_{\mu)\alpha} + \partial_\alpha \partial_\beta g_{\mu\nu} + \partial_\mu \partial_\nu g_{\alpha\beta} \right) \\ &+ \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} g^{\rho\sigma} \left(-\partial_\beta \partial_\rho g_{\sigma\alpha} + \partial_\alpha \partial_\beta g_{\rho\sigma} \right) + F_{\mu\nu}(g, \partial g), \end{aligned} \tag{20} $$

where F is a nonlinear function of the metric components $g_{\alpha\beta}$ and their first derivatives. This form does not coincide with the form of (13). But equations

$$ G_{\mu\nu} n^\nu = 0, \tag{21} $$

where n^μ is the (future directed unit) normal to Σ contain no second order time derivatives of the metric components, i.e., these components of $G_{ab} = 0$ at $t = 0$ depend only on the initial data.

Like we did for Maxwell's equations, we can then provide initial constraints by using the Gauss-Codacci equations (17) and express them in coordinate invariant form to obtain

$$\begin{aligned} 0 &= D_b K^b{}_a - D_a K^b{}_b \\ 0 &= {}^{(3)}R + (K^a{}_a)^2 - K_{ab}K^{ab}. \end{aligned} \tag{22}$$

The initial constraints (22) are tensor equations on Σ and non-linear elliptic partial differential equations for the initial data h_{ab} and K_{ab}.

Again analogous to the electromagnetic case, we find as a consequence of the Bianchi identity

$$\nabla^a G_{ab} = 0, \tag{23}$$

that if the initial constraints are satisfied initially and the spatial components of Einstein's equation are satisfied everywhere, then the constraints also are satisfied everywhere.

We see this by looking at the identity which relates the time derivative of the components $G_{\mu\nu}n^\nu$ to non-time differentiated components of $G_{\mu\nu}$ and their spatial derivatives. Solving then the purely spatial components of Einstein's equation and set the spatial components of (23) equal to zero and assume the metric components $g_{\mu\nu}$ as known functions, gives us a linear, homogeneous system of four first order equations for the four unknown components $G_{\mu\nu}n^\nu$. It follows by the theory of first order partial differential equations that if these components vanish initially, they must vanish everywhere.

This leaves us with the final statement that Einstein's equation in a vacuum, i.e., $G_{ab} = 0$ is an underdetermined system of equations for the metric components $g_{\mu\nu}$. Hence, we are left with six evolution equations, which happen to be the spatial components of $G_{ab} = 0$, for ten unknown metric components which even confirms the appearance of four arbitrary functions in the tensor transformation law. Thus, we conclude that it is plausible that Einstein's equation contains the correct number of evolution equations and that a well posed initial value formulation exists. The remaining proof is covered by the next seminar.

3.2 Conformal method (Lichnerowicz)

At the end we like to illustrate a special type of solution to the constraint equations which make use of conformal techniques. Hence, we generate a solution by the conformal method by Lichnerowicz where we require that the trace of the second fundamental form vanishes, i.e., $K^a{}_a = 0$.

We give an induced spacetime (Σ, h_{ab}) with an arbitrary Riemannian metric and obtain the first constraint

$$D^a K_{ab} = 0. \tag{24}$$

However, the second constraint is not satisfied in general for initial data (h_{ab}, K_{ab}) but if we do a conformal transformation on the spatial metric $\tilde{h}_{ab} = \phi^4 h_{ab}$ and on the symmetric traceless tensor $\tilde{K}_{ab} = \phi^{-2} K_{ab}$ and define then a derivative operator $\tilde{D}$ associated with $\tilde{h}_{ab}$, it turns out that

$$\tilde{D}^a \tilde{K}_{ab} = 0 \quad \text{and} \quad \tilde{K}^a{}_a = 0. \tag{25}$$

The second constraint for the conformal initial data $(\tilde{h}_{ab}, \tilde{K}_{ab})$ can then be expressed in terms of the original initial data h_{ab} and K_{ab}

$$D^a D_a \phi - \frac{1}{8} R\phi + \frac{1}{8}\phi^{-7} K^{ab} K_{ab} = 0. \tag{26}$$

This equation has local solutions.

4 Conclusion

The spacelike initial value problem in general relativity is a crucial component in understanding the evolution of spacetime and the dynamics of gravitational fields. Throughout this seminar, we have explored how the initial data on a 3-dimensional spacelike hypersurface Σ is used to predict the future development of the spacetime according to Einstein's equation.

The initial data consists of the induced metric h_{ab} and the extrinsic curvature K_{ab}, which must satisfy the constraint equations derived from the Gauss-Codacci equations:

$$\begin{aligned} 0 &= D_b K^b{}_a - D_a K^b{}_b \\ 0 &= {}^{(3)}R + (K^a{}_a)^2 - K_{ab} K^{ab}. \end{aligned} \tag{27}$$

These constraints ensure the consistency of the initial data and lead the way for the application of the evolution equations, which is another set of partial differential equations that dictate how the metric and extrinsic curvature evolve over time.

References

[Wal84] Robert M. Wald. *General Relativity.* Chicago, USA: Chicago Univ. Pr., 1984. DOI: 10.7208/chicago/9780226870373.001.0001.