PDE 2 SEMINAR

General Relativity

submitted by,

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submitted to,

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1) <u>Introduction</u>

In the last seminars we introduced the Mathematical tools and notions that we needed to Introduce the Einstein Equations.
So in this seminar I am going to introduce the Einstein's Equation which I believe is the heart of General Relativity.
Einstein's equation is equivalent to a coupled system of non linear second order partial differential equations for the metric components $g_{\mu\nu}$.
Before that I am just going to summarise some important tools that were introduce last time.

2) Review of Mathematical tools

2.1) Levi-Civita Connection

Levi-Civita Connection is defined as a bilinear(over $\mathbb{R}$) map ∇:$\mathcal{TM} \times \mathcal{TM} \longrightarrow \mathcal{TM}$, with (X,Y) $\mapsto \nabla_X Y$ where X,Y $\in \mathcal{TM}$ and $f \in C^\infty(\mathcal{M})$
That obeys

1. leibniz rule: $\nabla_X(fY) = \nabla_X(f)Y + f\nabla_X(Y)$
2. linearity over $\mathcal{C}^\infty(\mathcal{M})$: $\nabla_{fX}Y = f\nabla_X Y$
3. metric compatibility:$\nabla g = 0$
4. torsion free: $\nabla_X(Y) - \nabla_Y(X) = [X, Y]$

In coordinates $(U, \mathbf{x})$ we can expand Levi Civita connection as :-

for vector fields X,Y $\in \mathcal{TM}$

$$(\nabla_X Y)^\sigma = X^\nu \partial_\nu Y^\sigma + \Gamma^\sigma_{\mu\nu} X^\mu Y^\nu \tag{1}$$

for covector fields(smooth section of a cotangent bundle) $\omega \in \mathcal{T}^*\mathcal{M}$ and X $\in \mathcal{TM}$

$$(\nabla_X \omega)_\sigma = X^\nu \partial_\nu \omega_\sigma - \Gamma^\mu_{\nu\sigma} \omega_\mu X^\nu \tag{2}$$

For a (1,1) tensor field it looks like :-

$$\nabla_X T^\mu_\nu = X^\sigma \partial_\sigma (T^\mu_\nu) + \Gamma^\mu_{\sigma\alpha} X^\sigma T^\alpha_\nu - \Gamma^\alpha_{\sigma\nu} X^\sigma T^\mu_\alpha \tag{3}$$

Similarly we can construct for (k,l) tensor fields (refer any GR book) Γ's are called the Christoffel symbols

some facts about Γ's :-

1. Γ's are the freedom that is left to choose a ∇,that is we can tell you exactly which ∇ we are using by telling you the Γ's
2. Γ's are coordinate dependent functions
3. Γ's *doesn't transform like a tensor*, because under coordinate transformation using tensor transformation law we get the following equation

$$\Gamma^i_{(y)jk} = \frac{\partial y^i}{\partial x^q} \frac{\partial^2 x^q}{\partial y^i \partial y^k} + \frac{\partial y^i}{\partial x^q} \frac{\partial x^p}{\partial y^j} \frac{\partial x^s}{\partial y^k} \Gamma^q_{(x)sp} \tag{4}$$

as we can see in (4) the first term on the right hand side doesn't allow Γ to transform like a tensor, but that is fine as Γ's are constructed in such a way that the combination (1) transforms like a tensor(due to the torsion free condition).

4. In special relativity as the metric is flat(Minkowski) and we can cover it with a single chart and define Γ's to be 0(Normal coordinates). But under the change of charts the appear again.

Relationship between Γ's and the metric tensor

There is a relationship between the Γ's and the metric tensor g for a Levi-civita connection,it is given (In coordinate charts) as the following:-

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2}(g^{-1})^{\sigma\rho}\left(\partial_{\mu}g_{\nu\rho} + \partial_{\nu}g_{\mu\rho} - \partial_{\rho}g_{\mu\nu}\right) \tag{5}$$

It is worth to note that (5) comes from the metric compatibility of the Levi Civita connection and is a nice and very important equation as we can calculate Γ's from a given metric.

2.2) The curvature Tensor

The Riemann curvature of a Levi Civita Connection is a (1,3) tensor field denoted as *Riem* i.e

Riem: $\mathcal{TM} \times \mathcal{TM} \times \mathcal{TM} \times \mathcal{T}^*\mathcal{M} \longrightarrow C^{\infty}(\mathcal{M})$ is defined for vector fields X,Y,Z $\in \mathcal{TM}$ and covector field $\omega \in \mathcal{T}^*\mathcal{M}$ as:-

$$Reim(Z, X, Y, \omega) := \omega(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z) \tag{6}$$

In Charts, (6) reads as :-

$$R_{\mu\nu\rho}{}^{\sigma} := Riem(\partial_{\mu}, \partial_{\nu}, \partial_{\rho}, d\mathbf{x}^{\sigma}) = \partial_{\nu}\Gamma^{\sigma}_{\rho\mu} - \partial_{\rho}\Gamma^{\sigma}_{\nu\mu} + \Gamma^{\alpha}_{\rho\mu}\Gamma^{\sigma}_{\nu\alpha} - \Gamma^{\alpha}_{\nu\mu}\Gamma^{\sigma}_{\rho\alpha} \tag{7}$$

By contracting the Riemann Tensor we get Ricci tensor and on further contraction we get Ricci scalar.

2.2.1) Ricci tensor

The Ricci tensor (field) is a (0, 2)-tensor field that we denote by Ric with components $R_{\mu\nu}$ with respect to an arbitrary coordinate chart satisfying:

$$Ric := R_{\mu\sigma\nu}{}^{\sigma} = (g^{-1})^{\sigma\rho}R_{\mu\sigma\nu\rho} \tag{8}$$

2.2.2) Ricci scalar

The Ricci scalar R$\in C^{\infty}(\mathcal{M})$ is defined as follows:

$$R := (g^{-1})^{\mu\nu}R_{\mu\nu} \tag{9}$$

2.2.3) Symmetric Properties of the Riemann Curvature tensor

Riemann curvature tensor has the following properties:-

1. ${R_{(ab)c}}^d = 0$ (Anti-symmetric in first two indices)
2. ${R_{[abc]}}^d = 0$
3. $R_{abcd} = R_{cdab}$
4. $\nabla_{[a} {R_{bc]d}}^e = 0$ (Bianchi Identity)

In the above properties the Bianchi identity(property 4.) is very important as by contracting the contracted Bianchi identity(which is a (0,4) tensor,we get it from the metric compatibity condition) twice and in few steps we get the Einstein tensor defined as:-

$$\boxed{G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R} \tag{10}$$

which will be the left hand side of the Einstein Equations representing the geometry of spacetime.

Note: G_{ab} is symmetric and obeys contracted Bianchi identity i.e $\boxed{\nabla^a G_{ab} = 0}$ during the derivation actually we first

arrive at $\nabla^a G_{ab} = 0$ and from that we extract G_{ab}

2.2.3) Relationship between Metric and Curvature

From (5) and (7) we can derive a relationship between Curvature tensor and the metric of the manifold in a arbitrary coordinate chart, which is given by:-

$$R_{\rho\sigma\mu\nu} = \frac{1}{2}(\partial_\mu \partial_\sigma g_{\rho\nu} - \partial_\mu \partial_\rho g_{\nu\sigma} - \partial_\nu \partial_\sigma g_{\rho\nu} + \partial_\nu \partial_\rho g_{\nu\sigma}) \tag{11}$$

2.3) The Stress-Energy-Momentum Tensor

In the last section we saw what will be the left hand side of the Einstein's Equation and in this section we are going to Introduce the right hand side of the Einstein's Equation that represents matter distribution in the universe, this tensor is called the stress energy momentum tensor.

Stress Energy Momentum tensor(field) $T_{\mu\nu}$ is a (0,2) tensor field which is *symmetric, non-negative and divergence free*, it should represent matter in the Special and General relativity.

As it represents Energy and Momentum it should satisfy the Energy Momentum conservation law, which implies:-

1. flat spacetime(Special Relativity) is given by:-

$$\partial^\mu T_{\mu\nu} = 0 \tag{12}$$

2. In General relativity is given by:-

$$\nabla^\mu T_{\mu\nu} = 0 \tag{13}$$

One should notice that we can get (13) from (12) by replacing $\partial \longrightarrow \nabla$ and as ∂ is associated with the Minkowski metric(η) and ∇ is associated with a metric with non zero curvature(g) we are also switching the the metric from Minkowski to a metric with non-zero curvature i.e $\eta_{\mu\nu} \longrightarrow g_{\mu\nu}$,
*This rule of switching the derivative operators and associated metric to switch from flat spacetime to curved spacetime is called the **"minimal substitution rule"**.*

More about minimal substitution rule!

Minimal substitution rule: We can switch from flat spacetime to curved spacetime using the two rules given below:-

1. $\partial \longrightarrow \nabla$
2. $\eta_{\mu\nu} \longrightarrow g_{\mu\nu}$

The minimal substitution rule is the consequence of two basic principles that govern laws of Physics in GR :-

1. **Principle of General Covariance** : The metric, g_{ab} and quantities derived from it are only spacetime quantities that can appear in equations of Physics("the laws of general relativity take the same form when expressed with respect to a different coordinate chart")
2. **Compatibility with Special Relativity** : Equations must reduce to equations of Special Relativity when metric g_{ab} is flat (Minkowski).

Example :

1. The most natural generalisation of the equation satisfied the Klein-Gordon field scalar field $\phi \in C^\infty(\mathcal{M})$ from flat spacetime to curved spacetime is given by :-

$$\partial^a \partial_a \phi \longrightarrow \nabla^a \nabla_a \phi \tag{14}$$

The generalisation of Stress Energy tensor T_{ab} for this field is :-

$$T[\eta]_{ab} := \partial_a\phi\partial_b\phi - \frac{1}{2}\eta_{ab}(\partial_c\phi\partial^c\phi + m^2\phi^2) \longrightarrow T[g]_{ab} := \nabla_a\phi\nabla_b\phi + \frac{1}{2}g_{ab}(\nabla_c\phi\nabla^c\phi + m^2\phi^2) \tag{15}$$

Note that $\nabla^a T[g]_{ab} = 0$

2. Equations of motion for a perfect fluid in GR is given as follows:-

$$(\rho + P)u^a\nabla_a u_b + (g_{ab} + u_a u_b)\nabla^a P = 0 \tag{16}$$

The stress energy tensor for the perfect fluid is given by :-

$$T_{ab} = \rho u_a u_b + P(g_{ab} + u_a u_b) \tag{17}$$

Note that we derive (16) from the divergence free condition ($\nabla^a T_{ab} = 0$) of the stress energy tensor for the perfect fluid.

3)Spacetime Geometry and Matter Distribution

It's time to talk some Physics!

3.1) Why Einstein Created New Theory Of Gravity Instead Of Modifying Newtons Theory ?

Two key ideas that motivated Einstein to not modify Newtons theory of gravity instead seek an entirely new theory of spacetime are the **1)***Equivalence Principle* and **2)***Mach's Principle.*

- **Equivalence Principle** : All bodies are influenced by gravity and indeed , all bodies fall precisely the same way in a gravitational field i.e

  inertial mass $\times$ acceleration = gravitational mass $\times$ intensity of the gravitational field

  by careful experiments it is found that:-

  inertial mass = gravitational mass $\Longrightarrow$ acceleration = gravitational field

  $\Rightarrow$ Motion is independent of the nature of bodies $\therefore$ path of freely falling bodies define a preferred set of curves in spacetime (geodesics) , independent of nature of bodies and this opens the possibility of describing the properties of gravitational field into structure of spacetime itself.

- **Mach's Principle** : All matter in the universe should contribute to the local definition of "non-accelerating" and "non-rotating" (In a universe devoid of matter there should be no meaning to these concepts).
  But in Special Relativity , structure of spacetime is given once and for all and is unaffected by material bodies that may be present.

The new theory for spacetime and gravity

The new theory for spacetime and gravity proposed by Einstein states that :-

The intrinsic, observer-independent, properties of spacetime is given by a spacetime metric as in Special Relativity but this spacetime metric need not be flat. Indeed curvature, which is the deviation from flatness, accounts for the physical effects usually ascribed to a gravitational field.

Furthermore the curvature of spacetime related to the stress-energy tensor of matter in spacetime via equations formulated by Einstein.

$\Rightarrow$ *Structure of spacetime is related to the matter content of spacetime in accordance with some (not all!) of Mach's ideas.*

3.2) The Einstein's Equation

In this section we are going to derive the Einstein's equation using the minimal substitution rule that we Introduced in earlier part. For this let's compare Newtonian Gravity and General Relativity:-

In Newtonian theory

1. **Tidal Acceleration**(acceleration experienced by a body falling under non-uniform gravitational field)
   In Newtonian gravity tidal acceleration(a_N) is given by the following equation:-

$$a_N := -(\vec{x} \cdot \vec{\nabla})\vec{\nabla}\phi = -\partial_b \partial^a \phi \tag{18}$$

   where ϕ is the gravitational field (potential) and $\vec{x}$ is the Separation vector of particles.

2. **Poisson's Equation**(Gravitational field equation in Newtonian Gravity) is give as follows,

$$\nabla^2 \phi = 4\pi\rho \tag{19}$$

   Where ρ is the mass density

   Note:
   1. Poisson's Equation determines the gravitational force law depending on the masses around.
   2. $\nabla^2 = \partial_a \partial^a$ (In Newtonian gravity)

In General Relativity

1. **Tidal Acceleration** is given by,

$$a^a = -R_{cbd}{}^a v^c x^b v^d \tag{20}$$

   where v^a is the 4-velocity of a particle and x^a is the deviation vector.

2. **Matter distribution** In Special and General Relativity the Matter distribution is given by the stress tensor, T_{ab}

Comparing Newtonian gravity and General Relativity

From the the above discussion we can can conclude the following correspondences,

1. from (18) and (20) we get,:-

$$R_{cbd}{}^a v^c x^b v^d \longleftrightarrow \partial_b \partial^a \phi \tag{21}$$

2. From mass density(ρ) and stress tensor (T_{ab}) we get :-

$$\rho \longleftrightarrow T_{ab} v^a v^b \tag{22}$$

3. from (19), (21) and (22) we can write,

$$R_{cad}{}^{a} v^c v^d = 4\pi T_{cd} v^c v^d \tag{23}$$

this implies,

$$\boxed{R_{cd} = 4\pi T_{cd}} \tag{24}$$

Note that The equation (24) was first postulated by Einstein, but it had a serious defect :-

The stress energy tensor satisfies $\nabla^c T_{cd} = 0$, on the other hand from the contracted Bianchi identity we know ,

$$\nabla^c (R_{cd} - \frac{1}{2} g_{cd} R) = 0 \tag{25}$$

then from (24) we get,

$$\begin{aligned} \nabla^c R_{cd} &= 0 \\ from(25) \implies \nabla^c R_{cd} &= \frac{1}{2}\nabla^c (g_{cd} R) = \frac{1}{2}\nabla_d R = 0 \\ then(24) \implies g^{cd} R_{cd} &= g^{cd} T_{cd} \iff R^c{}_c = T^c{}_c \\ \implies \frac{1}{2}\nabla_d R &= 4\pi \nabla_d (T^c{}_c) = 0 \\ \implies \nabla_d T &= 0 \end{aligned}$$

This means that matter distribution is constant throughout the universe, which is a highly unphysical restriction on matter distribution, this forces us to reject (24)

Solution to this defect

If instead of (24) we consider the following equation ,

$$\boxed{G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R = 8\pi T_{ab}} \tag{26}$$

Then we can observe that there is no longer a conflict between Bianchi identity and conservation of Energy as from contracted Bianchi identity we know $\nabla^a G_{ab} = 0$ and from conservation of Energy we know $\nabla^a T_{ab} = 0$.
Moreover the correspondence that motivated (24) is not destroyed.

proof (Correspondence of (26) to (24))

Taking the trace of (26) we get

$$\begin{aligned} g^{ab} R_{ab} - \frac{1}{2} g^{ab} g_{ab} R &= 8\pi g^{ab} T_{ab} \\ R - \frac{4}{2} R &= 8\pi T \\ R &= -8\pi T \end{aligned} \tag{27}$$

where in second line we used $g^{ab} g_{ab} = \delta^a_a = 4$.

By Substituting (27) in (26) we get ,

$$R_{ab} = 8\pi(T_{ab} - \frac{1}{2}g_{ab}T) \tag{28}$$

In situations where Newtonian Theory is applicable we have the following ,

Energy of matter measured by an observer who is roughly "at rest" w.r.t masses >> material stresses we get,

$$T \approx -\rho = -T_{00}v^0v^0$$

Then (28) implies that,

$$\begin{aligned} R_{ab} &\approx 8\pi(T_{ab} + \frac{1}{2}g_{ab}T_{00}v^0v^0) \\ R_{00} &\approx 8\pi(T_{00} - \frac{1}{2}T_{00}) \\ &\approx 4\pi T_{00} \\ \implies R_{ab}v^av^b &\approx 4\pi T_{ab}v^av^b \end{aligned} \tag{29}$$

Where in the second line I used for time-like vector fields $g_{ab}v^av^b = -1$.
Hence(29) and (24) are the same.

□

Hence (26) is the desired field equation of GR. It was written by Einstein in 1915 and is called the Einstein's Equation.Entire content of General Relativity can be summarised as follows:-

Spacetime is a four dimensional manifold equipped with a Lorentz metric g_{ab}. The curvature of g_{ab} is related to the matter distribution in spacetime by Einstein's Equation (26).

Note : In four-dimensional spacetime g_{ab} has 10 independent components and so in general relativity we have 10 independent field equations.the Einstein field equations are non-linear in the g_{ab} whereas Newtonian gravity is linear in the field ϕ, Einstein's theory thus involves numerous non-linear differential equations, and so it should come as no surprise that the theory is complicated.

Remarks about Einstein's Equation

1. **Mathematical Character of Einstein's Equation**

   - As said in the introduction, Einstein's equation is equivalent to a coupled system of non linear second order partial differential equations for the metric components $g_{\mu\nu}$.
   - For a metric of Lorentz signature, these equations have a hyperbolic character i.e. we have the correct number of equations and unknown to permit a good initial value formulation.

2. **How one should view Einstein's equation** Einstein's Equation is analogous to Maxwell 's equation with T_{ab} serving as a source of gravitational field in the same way as j_a(Current density) serving as the source of Electromagnetic field. The Important difference between Einstein's equation and Maxwell's equation is , It make sense to solve Maxwell Equation by first specifying j_a then finding A_a.One could try solve Einstein's equation by first specifying T_{ab} then finding g_{ab} but this does not make

sense until g_{ab} is found as we do not know how to physically interpret T_{ab}. This Implies that in GR one should simultaneously solve for spacetime metric and the matter distribution. For these reasons it is difficult to solve Einstein equation when sources are present.
Maxwell 's Equation is given by:-

$$\partial^a(\partial_a A_b - \partial_b A_a) = -4\pi j_a$$

$A, j \in \mathcal{TM}$ is the vector potential and the current density.

3. **Equations of motion of matter**

   Einstein Equation implies the relation $\nabla^a T_{ab} = 0$, which contains great deal of information on the behaviour of matter.

   - For a perfect fluid $\nabla^a T_{ab} = 0$ is the entire content of equation of motion, hence for a Scalar field we may economise our assumptions by just postulating the form of T_{ab} ; equation of motion of fluid is already contained in the Einstein's Equation.
   - Perfect fluid with P=0 (pressure less) is called dust. In this case particles exert no force on each other and equations of motion for dust tells us that individual dust particles move on geodesics.
   - Einstein Equations is self consistent with framework of GR.
   - All bodies that are large enough to feel the tidal forces of gravitational field will deviate from geodesic motion and equation of motion of such bodies also can be found from the condition $\nabla^a T_{ab}$

REFERENCES

(1) Central Lecture Course, WE-Heraeus International Winter School on Gravity and Light by Dr. Frederic P. Schuller :https://youtube.com/playlist?list=PLFeEvEPtX_0S6vxxiiNPrJbLu9aK1UVC_

(2) Lectures on Geometrical Anatomy of Theoretical Physics by Dr. Frederic P. Schuller :https://youtube.com/playlist?list=PLPH7f_7ZlzxTi6kS4vCmv4ZKm9u8g5yic

(3) Robert.M.Wald, 1984, *General Relativity*, University of Chicago press

(4) Sean Carroll, 2003 *Spacetime and Geometry: An Introduction to General Relativity*, Pearson