Special Relativity: Global and Causal Structure of Minkowski Spacetime

Aaron Thede

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In Special Relativity we investigate isolated systems and ignore the influence of matter at far distances. The Minkowski spacetime represents the geometry of a *static and highly symmetric universe*. Last week, Oskar introduced the Minkowski spacetime and some of its basic properties.

Recap.

- $g(X,Y) = \eta_{\mu\nu} x^\mu y^\nu$, where $\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
- In other words: $g = -dt^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2$
- A vector $X \in \mathbb{R}^{3+1}$ is called $\begin{cases} \text{spacelike if} & g(X,X) > 0 \\ \text{timelike if} & g(X,X) < 0 \\ \text{null if} & g(X,X) = 0 \end{cases}$

1 Double Null Foliation

Today, we want to describe the *global and asymptotic structure of Minkowski spacetime*. To do so, one could start by considering the following *foliation of Minkowski*

$$\mathbb{R}^{3+1} = \bigcup_{\tau \in \mathbb{R}} \mathcal{H}_\tau,$$

where $\mathcal{H}_\tau = \{t = \tau\}$ is a spacelike hypersurface.

Remark. 1. A hypersurface $\mathcal{H}$ is called *spacelike*, if the Normal N_x at each point $x \in \mathcal{H}$ is timelike. In this case, $g \mid_{T_x\mathcal{H}}$ is positive-definite (i.e. $\mathcal{H}$ is a Riemannian manifold).

2. A hypersurface $\mathcal{H}$ is called *timelike*, if the Normal N_x at each point $x \in \mathcal{H}$ is spacelike. In this case, $g\,|_{T_x\mathcal{H}}$ has signature $(-,+,+)$.

3. A hypersurface $\mathcal{H}$ is called *null*, if the Normal N_x at each point $x \in \mathcal{H}$ is null. In this case, $g\,|_{T_x\mathcal{H}}$ is degenerate.

Example (timelike hypersurfaces)**.** 1. The hypersurfaces $\mathcal{T}_\tau := \{x^1 = \tau\}$ are timelike for each τ since their normal is the spacelike vector field ∂_{x_1}. Then $(\mathcal{T}_\tau, g\,|_{T_x\mathcal{T}_\tau})$ is isometric to the Minkowski space $\mathbb{R}^{2+1}$.

2. The hypersurface

$$H^3_+ = \{X : g(X,X) = 1 \text{ and } X \text{ future-directed}\}$$

is a timelike hypersurface.

Example (null hypersurfaces)**.** 1. Let $n = (n_0, n_1, n_2, n_3)$ be a null vector. The planes given by the equation

$$P_n = \{(t, x^1, x^2, x^3) : n^0 t = n_1 x^1 + n_2 x^2 + n_3 x^3\}$$

are null hypersurfaces, since their normal is the null vector n.

2. The (future null) cone

$$C = \left\{(t, x^1, x^2, x^3) : t = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}\right\}$$

is a null hypersurface. Its tangent plane at the endpoint of n is the plane P_n, hence its normal is the null vector n.

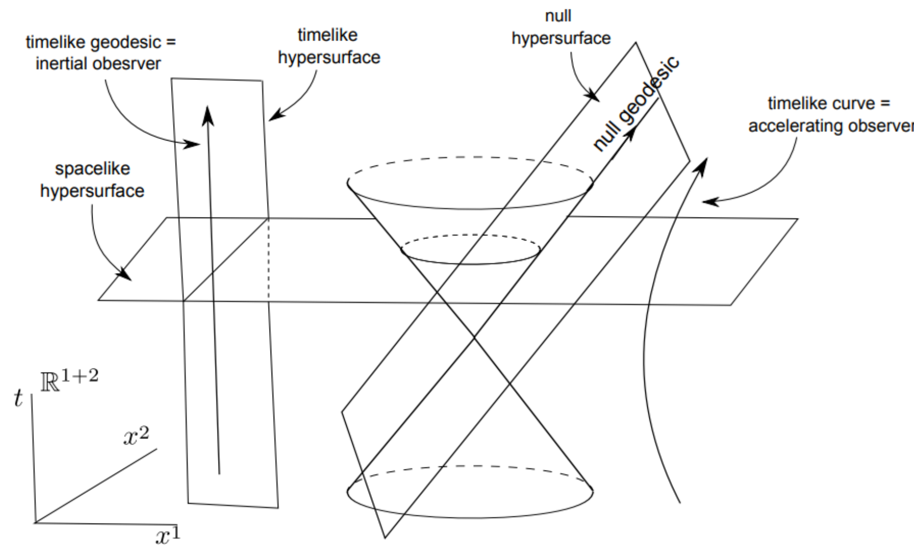


Figure 1: Summary hypersurfaces

Note however, this foliation does not capture the properties of null geodesics whose importance is manifest from the fact that signals (i.e. light) travel along such curves.

Q: What is a null geodesic? A null geodesic is a geodesic starting in a null vector. i.e. a vector $X \in \mathbb{R}^{3+1}$ s.t. $g(X, X) = 0$. Note that in the Minkowski case, the curvatures (Riemann, Ricci, scalar) are all zero and therefore the geodesics are just lines with respect to the coordinate system (t, x^1, x^2, x^3). In particular photons travel along null geodesics (light speed).
(Remark: Since g is not positive definite (signature $(-,+,+,+)$) X being a null vector does not necessarily mean that $X = 0$. For example $X = (1, 1, 0, 0)$ has $g(X, X) = 0$ but is not equal to zero.)

Indeed, an observer located far away from an isolated system under investigation must understand the asymptotic behavior of null geodesics in order to be able to measure radiation and other information sent from this system. For this reason, we will consider a foliation of Minkowski spacetime which captures the geometry of null geodesics emanating from points of a timelike geodesic. This is the so-called *double null foliation.*

Consider the timelike geodesic $\alpha(t) = (t, 0, 0, 0)$. Recall the *future null cone* C_τ with vertex at $\alpha(\tau)$

$$C_\tau = \left\{(t, x^1, x^2, x^3) : t - \tau = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}\right\}$$

and *past null cone* $\underline{C}_\tau$

$$\underline{C}_\tau = \left\{(t, x^1, x^2, x^3) : t - \tau = -\sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}\right\}.$$

In order to simplify the above expressions and capture the symmetry of the null cones, we introduce spherical coordinates (r, θ, ϕ) for the Euclidean hypersurface $\mathcal{H}_\tau$, s.t. $r = 0$ corresponds to the curve α. Then, in (t, r, θ, ϕ) coordinates, the Minkowski metric takes the form

$$g = -dt^2 + dr^2 + r^2 \cdot g_{\mathbb{S}^2(\theta,\phi)},$$

Where $g_{\mathbb{S}^2(\theta,\phi)} = d\theta^2 + (\sin\theta)^2 d\phi^2$ is the standard metric on the unit sphere. Then we have

$$C_\tau = \{(t, r, \theta, \phi) : t - r = \tau\}$$
$$\underline{C}_\tau = \{(t, r, \theta, \phi) : t + r = \tau\}$$

Now it is very convenient to convert to *null coordinates* (u, v, θ, ϕ) defined s.t.

$$u = t - r$$
$$v = t + r$$

Note also that $v \geq u$ and $u = v$ iff $r = 0$. The metric with respect to null coordinates takes the form

$$g = -dudv + \frac{1}{4}(u - v)^2 \cdot g_{\mathbb{S}^2(\theta,\phi)}$$

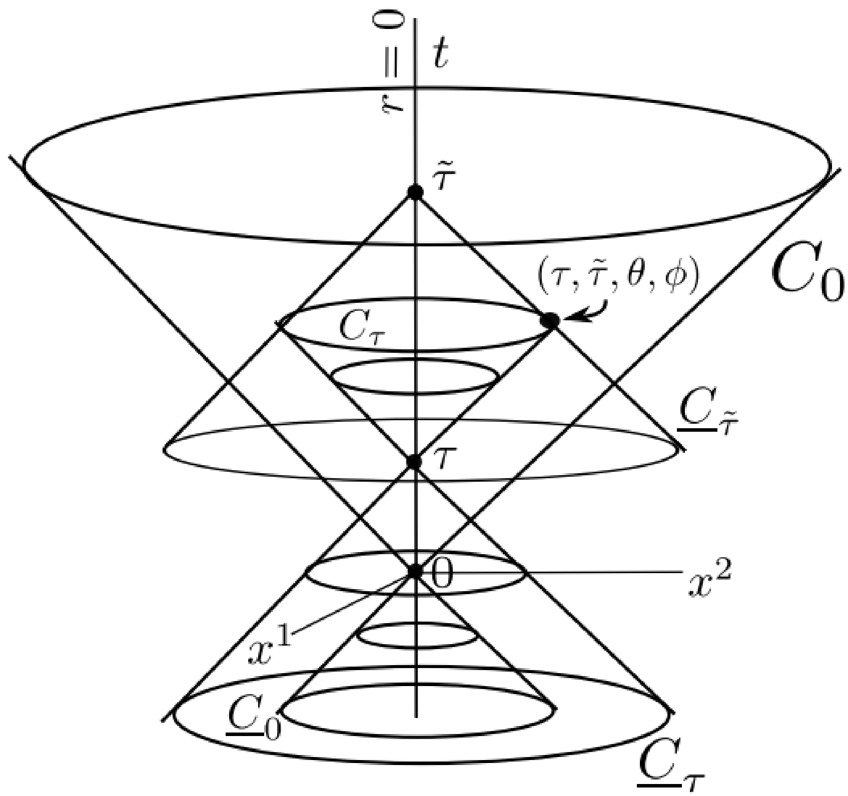
and then we end up with

$$C_\tau = \{(u, v, \theta, \phi) : u = \tau\},$$

$$\underline{C}_\tau = \{(u, v, \theta, \phi) : v = \tau\}.$$

and the *double null foliation* is given by

$$\mathbb{R}^{3+1} = \bigcup_{\tau \in \mathbb{R}} C_\tau \cup \underline{C}_\tau.$$



2 The Penrose Diagram

The aim is to describe the asymptotic structure of Minkowski space. In Particular, we want to draw a "bounded" diagram whose boundary represents infinity and somehow respects the causal structure of Minkowski.
Clearly, $v \to \infty$ along the null cones C_τ and $u \to \infty$ along $\underline{C}_\tau$. In Order to bring the endpoint of null geodesics in finite distance, we consider the following change of coordinates:

$$\tan p = v,$$

$$\tan q = u,$$

with $p, q \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $p \geq q$. Then in (p, q, θ, ϕ) coordinates the metric takes the form

$$g = \frac{1}{\cos^2 p \cdot \cos^2 q} \left(-dpdq + \frac{1}{4} \sin^2(p-q) \cdot g_{\mathbb{S}^2(\theta,\phi)} \right)$$

At first glance, this metric does not seem very pretty, because a consequence of the boundedness of the range of p, q is that the left factor blows up as $p, q \to \pm\frac{\pi}{2}$. In order to overcome this degeneracy, we consider the metric $\tilde{g}$ which now takes the form

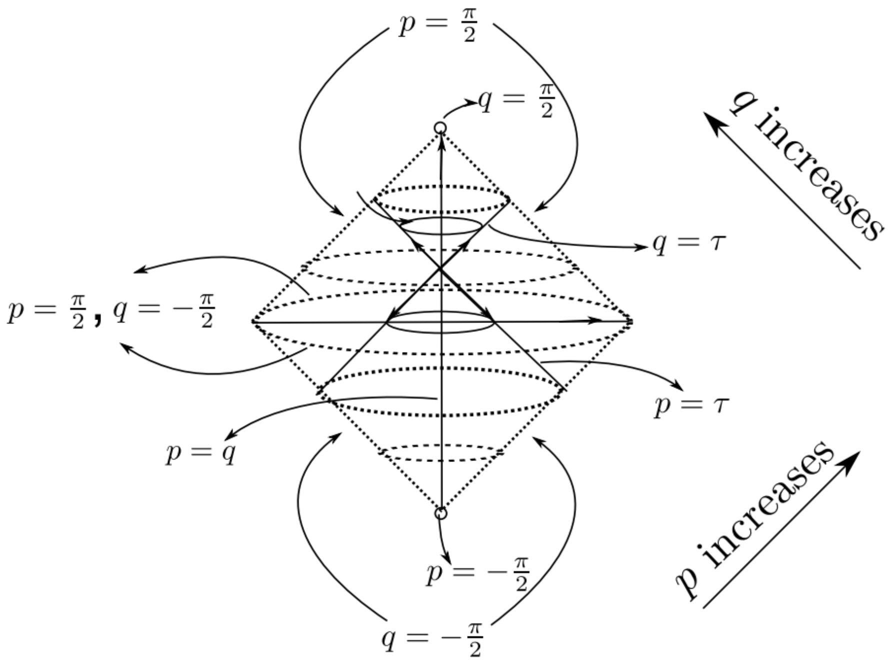
$$\tilde{g} = \left(-dpdq + \frac{1}{4} \sin^2(p-q) \cdot g_{\mathbb{S}^2(\theta,\phi)} \right).$$

Clearly the metric $\tilde{g}$ is conformal to g (since $\exists \varphi$ smooth $: \tilde{g} = \varphi g$). Note that $\nabla p = -\partial_q, \nabla q = -\partial_p$, where ∇ considered with respect to $\tilde{g}$, and therefore, the hypersurfaces

$$\tilde{C}_\tau = \{(p, q, \theta, \phi) : q = \tau, \, \tau \in \mathbb{R}\},$$

$$\underline{\tilde{C}}_\tau = \{(p, q, \theta, \phi) : p = \tau, \, \tau \in \mathbb{R}\}$$

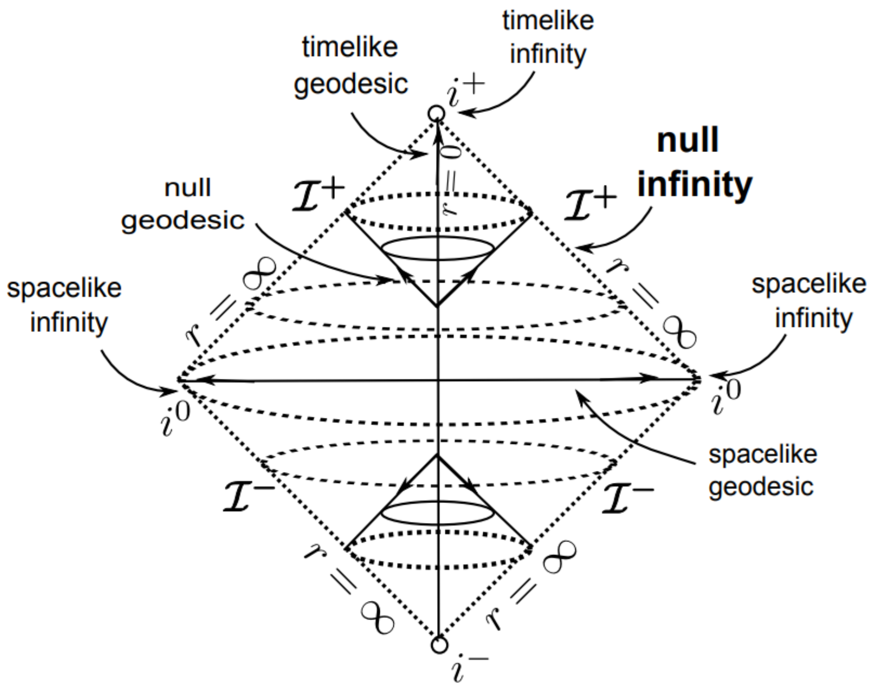
are null (with respect to $\tilde{g}$).
Hence, if we suppress one angular direction, we can globally depict the manifold $(\tilde{M}, \tilde{g})$ covered by the coordinates (p, q, θ, ϕ) as follows:



The Manifold $(\tilde{M}, \tilde{g})$, which is conformal to Minkowski $\mathbb{R}^{3+1}$.

We define:

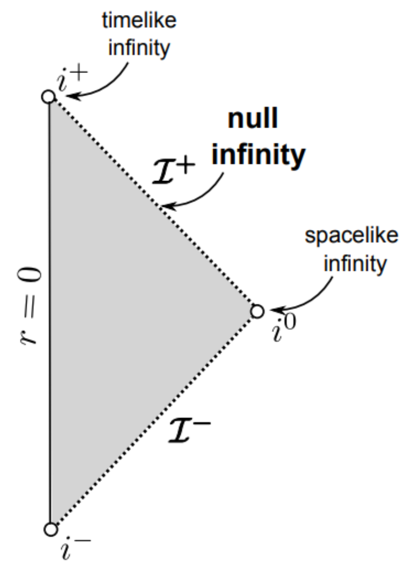
- *Future null infinity* $\mathcal{I}^+$ to be the endpoints of all future-directed null geodesics along, which $r \to +\infty$.
- *Future timelike infinity* i^+ to be the endpoints of all future-directed timelike geodesics.
- *Spacelike Infinity* i^0 to be the endpoint of all space geodesics. This is in fact a point, and not a sphere, which can be thought of as the point at infinity of the one-point compactification of, say, the spacelike hypersurface $t = 0$.



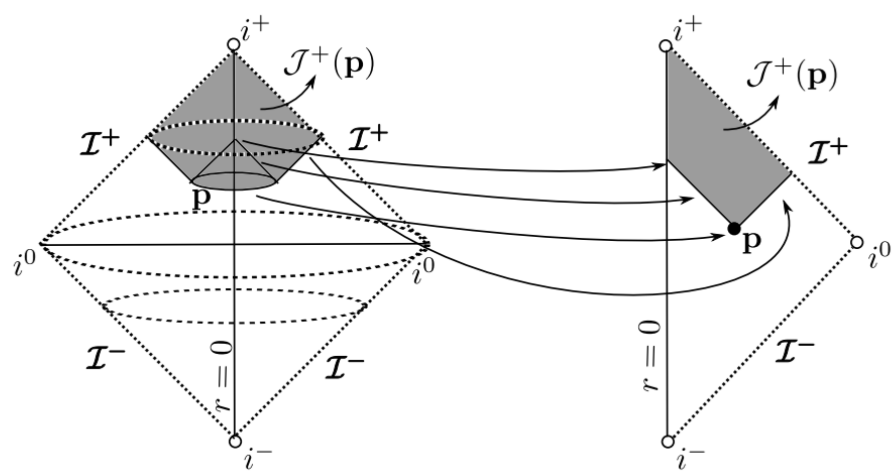
Note that if a curve $\alpha(\tau) = (t(\tau), r(\tau), \theta(\tau), \phi(\tau))$ is such that as $\tau \to +\infty$

- $t \to \infty$ and $r < \infty$, then α approaches i^+.
- $t \sim r \to \infty$, then α approaches $\mathcal{I}^+$.
- $|t| < \infty$ and $r \to -\infty$, then α approaches i^0.
- $t \sim -r \to -\infty$, then α approaches $\mathcal{I}^-$
- $t \to -\infty$ and $r < \infty$, then α approaches i^-.

We can proceed further by suppressing all angular directions. Formally speaking, we consider the quotient $\tilde{M}/SO(3)$. Then the metric $\tilde{g}$ reduces to $\hat{g} = -dpdq$, which coincides with the Minkowski spacetime $\mathbb{R}^{1+1}$. Hence, if we consider a planar section of the cone, then the resulting bounded 2-dimensional domain is embedded in the Minkowski spacetime $\mathbb{R}^{1+1}$:

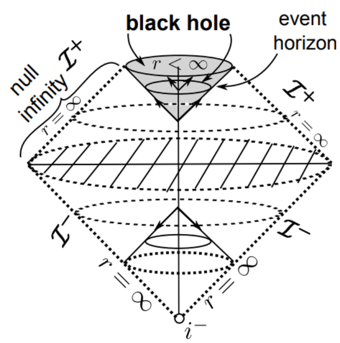


This diagram is called the Penrose diagram of Minkowski Spacetime $\mathbb{R}^{3+1}$. All cones collapsed to lines. Using the above diagram one can read off the causal structure of spacetime as follows:



More generally, one defines the Penrose diagram of a spherically symmetric spacetime to be the image of a bounded conformal transformation of the quotient spacetime in Minkowski spacetime $\mathbb{R}^{1+1}$. The Importance of such diagramms is that they allow one to read off the causal structure and recognize the asymptotic structure of a spacetime.

For example, we see that in Minkowski, the past of future null Infinity $\mathcal{I}^+$ is the whole spacetime. However we can construct spacetimes for which this is not the case. In other words there are spacetimes which contain points which cannot communicate with $\mathcal{I}^+$. The conformal diagram of such spacetime would be as follows:



The shaded region cannot send signals to $\mathcal{I}^+$ and for this reason is called *black hole.*

Take Home Message.

1. We learned about spacelike, timelike and null hypersurfaces
2. We can foliate Minkowski spacetime by null cones, the so called *double null foliation*
3. We can transform Minkowski spacetime conformally into a bounded Penrose diagram, where we still can read off the causal structure
4. We got a first glance at the concept of a black hole