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Introduction to Curvature in General Relativity

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Summary

This is the script of Seminar talk in 'Introduction to Curvature in General Relativity' I gave with Dr. Stefan Czimek. The talk addresses derivative operators, the concept of parallel transport, and finally Riemann tensor. The talk is referred to the Book 'General Relativity' (Wald 1984)

1 Introduction

There are two important notions of curvature in General Relativity. Intrinsic and Extrinsic. For extrinsic notion, we need embedding in a higher dimension (For example: A 2 Dimensional surface embedded in $\mathbb{R}^n$). Here we consider intrinsic notion, which needs no higher dimensional embedding. These kinds of curvatures are defined by means of parallel transport of a vector through a curve and it's failure to do so in a closed loop on a curved surface. For defining parallel transport we need the notion of a derivative operator and for defining curvature we need a tensor that governs the failure of a vector to undergo parallel transform along a closed loop on a curved surface.

2 Derivative Operator

To define Parallel transport, we first need a derivative operator. We can obviously choose a normal partial differential operator ∂_a. But since we are going to deal with different tangent spaces referring to different points on a curve, we need a more general operator known as Derivative Operator
A Derivative Operator ∇_a takes a smooth Tensor field (k,l) to a smooth Tensor field (k,l+1). The action of a derivative operator on a tensor field is denoted by

$$\nabla_a T^{a_1 \cdots a_k}{}_{b_1 \cdots b_l}$$

It satisfies 5 conditions:

1. Linearity: For all A,B $\in \mathfrak{T}(k, l)$ and $\alpha, \beta \in \mathbb{R}^n$

$$\nabla_c(\alpha A^{a_1 \cdots a_k}{}_{b_1 \cdots b_l} + \beta B^{a_1 \cdots a_k}{}_{b_1 \cdots b_l}) = \alpha \nabla_c A^{a_1 \cdots a_k}{}_{b_1 \cdots b_l} + \beta \nabla_c B^{a_1 \cdots a_k}{}_{b_1 \cdots b_l}$$

2. Leibniz rule: For all A $\in \mathfrak{T}(k,l)$ and B $\in \mathfrak{T}(k',l')$

$$\nabla_e[A^{a_1\cdots a_k}{}_{b_1\cdots b_l}B^{a_1\cdots a'_k}{}_{b_1\cdots b'_l}] = [\nabla_e A^{a_1\cdots a_k}{}_{b_1\cdots b_l}]B^{a_1\cdots a'_k}{}_{b_1\cdots b'_l} + A^{a_1\cdots a_k}{}_{b_1\cdots b_l}[\nabla_e B^{a_1\cdots a'_k}{}_{b_1\cdots b'_l}]$$

3. Commutativity with contraction

$$\nabla_d(A^{a_1\cdots c\cdots a_k}{}_{b_1\cdots c\cdots b_l}) = \nabla_d A^{a_1\cdots c\cdots a_k}{}_{b_1\cdots c\cdots b_l}$$

4. Consistency with tangent vectors as derivatives on scalar fields. For all f$\in \mathfrak{F}$ and $t^a \in V_p$

$$t(f) = t^a \nabla_a f$$

5. Torsion free: For all f$\in \mathfrak{F}$

$$\nabla_a \nabla_b f = \nabla_b \nabla_a f$$

Remark: We can write commutator of two vectors in terms of ∇_a
Using fourth property for all vector v,w $\in$ V

$$[v,w]^b = v^a \nabla_a w^b - w^a \nabla_a v^b$$

Example: Let Ψ be a coordinate system and {dx^μ} and {$\partial/\partial x^\mu$} be a coordinate basis. Let ∂_a, the partial derivative operator be the derivative operator namely Ordinary derivative.
Partial Derivatives won't follow commutativity with contraction, which fails it to make the Covariant derivative.
∂_a is not naturally associated with a manifold because ∂_a changes with the choice of coordinates.

3 Uniqueness of ∇_a and disagreement on action on tensor fields

By condition 4 any two derivative operators ∇_a and $\tilde{\nabla}_a$ must agree on their action on scalar fields. To find their possible disagreements on tensors let us consider the tensor of the next highest rank, namely the dual vector field
Let us consider the difference $\nabla_a(f\omega_b)$ - $\tilde{\nabla}_a(f\omega_b)$. By Leibniz rule it becomes

$$f(\tilde{\nabla}_a\omega_b - \nabla_a\omega_b)$$

We take this difference for two different dual vectors ω_b, ω'_b acting on the same point. We can prove using advanced calculus that

$$\tilde{\nabla}_a\omega'_b - \nabla_a\omega'_b = \tilde{\nabla}_a\omega_b - \nabla_a\omega_b$$

So ∇_a - $\tilde{\nabla}_a$ maps dual vector fields to (0,2) tensor fields. That implies it results in a (1,2) tensor field denoted as C^c_{ab}

$$\nabla_a\omega_b = \tilde{\nabla}_a\omega_b - C^c_{ab}\omega_c$$

Using property 4 we can say that for every vector field t^a and dual vector field ω_a

$$(\tilde{\nabla}_a - \nabla_a)(\omega_b t^b) = 0$$

By Leibniz rule and previous equations of tensor field arise from difference of two derivative operators

$$(\tilde{\nabla}_a - \nabla_a)(\omega_b t^b) = (C^c_{ab}\omega_c)t^c + \omega_b(\tilde{\nabla}_a - \nabla_a)t^b$$

By contracting and index substitution we can obtain for all ω_b that

$$\nabla_a t^b = \tilde{\nabla}_a t^b - C^b_{ac}t^c$$

This can be generalised to a higher rank tensor

$$\begin{aligned}\nabla_a T^{b_1\cdots b_k}{}_{c_1\cdots c_l} =& \tilde{\nabla}_a T^{b_1\cdots b_k}{}_{c_1\cdots c_l} + \sum_i C^{b_i}_{ad} T^{b_1\cdots d\cdots b_k}{}_{c_1\cdots c_l} \\ &- \sum_j C^d_{ac_j} T^{b_1\cdots b_1}{}_{c_1\cdots d\cdots c_l}.\end{aligned}$$

When $\tilde{\nabla}_a$ is ordinary derivative ∂_a, C^c_{ab} is denoted as Γ^c_{ab} and it is known as Christoffel Symbol. Thus we can write

$$\nabla_a t^b = \partial_a t^b + \Gamma^b_{ac}t^c$$

4 Parallel Transport

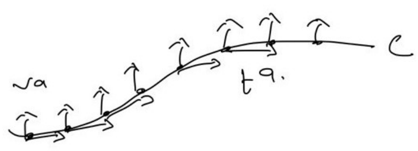

Figure 1: A visual representation of Parallel Transport

Given a curve C with tangent t^a we can define the notion of parallel transport for a vector v^a given at each point if we move along the curve, if the following equation satisfies

$$t^a \nabla_a v^b = 0$$

For an arbitrary tensor

$$t^a \nabla_a T^{b_1\cdots b_k}{}_{c_1\cdots c_l} = 0$$

Ina coordinate system,

$$t^a \partial_a v^b + t^a \Gamma^b_{ac} v^c = 0$$

In terms of components and parameter t

$$\frac{dv^\nu}{dt} + \sum_{\mu,\lambda} t^\mu \Gamma^\nu_{\mu\lambda} v^\lambda = 0$$

This is a differential equation with solution as the vector that got transported
Two tangent spaces can be easily mapped if we know the curve and the ∇_a and by means of parallel transport. the mathematical structure arsing from that is called connections
Many distinct derivative operators can be chosen and no derivative operaor is special. However if we choose a metric g_{ab} a natural choice of derivative operator can be made. This is because metric can arise a condition which we can associate naturally to parallel transport.
Given two vectors v^a and w^b which individually obey parallel transport condition, we can see that their inner product is always parallel transported irrespective of the curve. That is

$$t^a \nabla_a (g_{bc} v^b w^c) = 0$$

By Leibniz rule we can see that

$$\nabla_a g_{bc} = 0$$

This is called Metric Compatibility condition
This equation uniquely determines ∇_a which is shown by following theorem
Theorem: Let Gab be a metric. Then there exists a unique Torsion free operator ∇_a satisfying $\nabla_a g_{bc} = 0$
A sketch of proof proceed like this: We first take a covariant derivative $\tilde{\nabla}_a$ and we solve for C^c_{ab}. We then determine the choice of C^c_{ab} is unique.
Using the representation of action of tensor field on covariant derivative ∇_a we write

$$0 = \nabla_a g_{bc} = \tilde{\nabla}_a g_{bc} - C^d_{ab} g_{dc} - C^d_{ac} g_{bd}$$

Solving this and using index substitution we can arrive

$$C^c_{ab} = \frac{1}{2} g^{cd} \left(\tilde{\nabla}_a g_{bd} + \tilde{\nabla}_b g_{ad} - \tilde{\nabla}_d g_{ab} \right)$$

The choice of C^c_{ab} solves the metric compatibility condition and it is unique.
In terms of an ordinary derivative operator, the Christoffel symbol is

$$\Gamma^c_{ab} = \frac{1}{2} g^{cd} \left(\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab} \right)$$

The coordinate basis components of Christoffel symbol are

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} \sum_\sigma g^{\rho\sigma} \left(\frac{\partial g_{\nu\sigma}}{\partial x^\mu} + \frac{\partial g_{\mu\sigma}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma} \right)$$

5 Riemann Tensor and Curvature

We can use path dependence of parallel transport to define intrinsic notion of curvature.
Let ∇_a be a derivative operator and we calculate the action of two derivative operators applied on $f\omega_a$, where ω_a is dual vector field and f is a smooth function.

$$\nabla_a\nabla_b(f\omega_c) = \nabla_a(\omega_c\nabla_b f + f\nabla_b\omega_c)$$
$$= (\nabla_a\nabla_b f)\omega_c + \nabla_b f\nabla_a\omega_c + \nabla_a f\nabla_b\omega_c + f\nabla_a\nabla_b\omega_c$$

If we do the same thing, but order of operator action reversed and we subtract the resulting equations we get

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)(f\omega_c) = f(\nabla_a\nabla_b - \nabla_b\nabla_a)\omega_c$$

Using the exact same reasoning we used to derive Christoffel symbol we can say $\nabla_a\nabla_b - \nabla_b\nabla_a$ maps dual vector fields to (0,3) tensor fields. That implies it results in a (1,3) tensor field denoted as R_{abc}^d. This tensor is called Riemann curvature Tensor. For all dual vectors ω_c

$$\nabla_a\nabla_b\omega_c - \nabla_b\nabla_a\omega_c = R_{abc}^d\omega_d$$

To show Riemann tensor governs the failure of vector to parallely transport we consider the following diagram

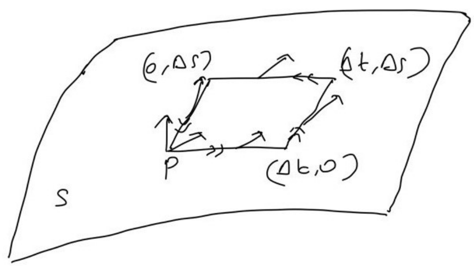

Figure 2: Parallel Transport of a vector over a closed loop on a curved surface

Here we are analysing the change in the scalar $v^a\omega_a$ as we traverse the loop. On the first leg of loop, for small Δt the change is

$$\delta_1 = \Delta t\frac{\partial}{\partial t}\left(v^a\omega_a\right)\Big|_{(\Delta t/2,0)}$$

by evaluating the derivative at the midpoint, this expression is accurate to second order in the displacement Δt. We may rewrite δ_1 as

$$\delta_1 = \Delta t v^a T^b\nabla_b\omega_a\big|_{(\Delta t/2,0)},$$

where T^b is the tangent to the curves of constant s and T^b obeys parallel transport equation. Let us combine the variations δ_1 and δ_3

$$\delta_1 + \delta_3 = \Delta t \left\{ v^a T^b \nabla_b \omega_a \big|_{(\Delta t/2,0)} - v^a T^b \nabla_b \omega_a \big|_{(\Delta t/2,\Delta s)} \right\},$$

δ_2 and δ_4 combine similarly. The term in brackets vanishes as $\Delta s \to 0$, and that implies that to first order in Δt and Δs, the total change in $v^a \omega_a$ (and thus the total change in v^a) vanishes. That is parallel transport is path independent to the first order. Now we evaluate the second order change in $v^a \omega_a$ by evaluating the terms in brackets up to first order

$$\delta_1 + \delta_3 = -\Delta t \Delta s v^a S^c \nabla_c \left(T^b \nabla_b \omega_a \right),$$

Where S^c is tangent to the curves of constant t.
Adding similar contributions from δ_2 and δ_4, we find the total change in $v^a \omega_a$ is

$$\delta \left(v^a \omega_a \right) = \Delta t \Delta s v^a T^c S^b R_{cba}{}^d \omega_d;$$

This equation can hold for all ω_c if and only if the total change in v^a (accurate to second order in Δt)

$$\delta v^a = \Delta t \Delta s v^d T^c S^b R_{cbd}{}^a$$

This equation shows that Riemann tensor measures the path dependence of parallel transport or Riemann tensor governs the failure of a vector to parallel transport around a closed loop on a curved surface.
We can use an analogous procedure to derive the expression of tensor acted upon by a covariant derivative, to derive the expression of action of commutator of derivative operators on an arbitrary tensor field in terms of Riemann Tensor. Let t^a be a vector field and ω_c its dual

$$0 = (\nabla_a \nabla_b - \nabla_b \nabla_a)(t^c \omega_c)$$

This gives us

$$\omega_c (\nabla_a \nabla_b - \nabla_b \nabla_a) t^c + t^c \omega_d R_{abc}{}^d$$

From this we can obtain

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) t^c = -R_{abd}{}^c t^d$$

By induction we can obtain, the expression of the action on an arbitrary tensor field

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) T^{c_1 \cdots c_k}{}_{d_1 \cdots d_l} = -\sum_{i=1}^{k} R_{abe}{}^{c_i} T^{c_1 \cdots e \cdots c_k}{}_{d_1 \cdots d_l} + \sum_{j=1}^{l} R_{abd_j}{}^e T^{c_1 \cdots c_k}{}_{d_1 \cdots e \cdots d_l}$$

Riemann Tensor have following properties

1. R^d_{abc} = -R^d_{bac}

2. $R^d_{[abc]}$ = 0

3. For a derivative operator ∇_a naturally associated with the metric g_{bc} and the metric compatibility condition, we have

$$R_{abcd} = -R_{abdc}$$

4.The Bianchi identity

$$\nabla_{[a}R^e_{bc]d} = 0$$

References

Wald, Robert M. 1984. *General relativity*. The University of Chicago Press: Chicago and London, pp 30-41.