PDE seminar talk - script

Max Dalitz

Tensors; the Metric Tensor

In the last talk we talked about and concretised our notion of an "infinitesimal displacement". This gives rise to the notion of tensors when considering other quantities of interest. It turns out that a lot of quantities in physics linearly depend on displacements. Wald gives two examples of that:

Firstly, consider a magnetic probe in three-dimensional space. Around it you can measure the magnetic field strength. But, if you change the orientation of the probe with respect to the observer, the field strength is going to change so theoretically we might assume to need an infinite number of measurements to determine the magnetic field. However, since the field strength is obviously linearly dependent on the orientation of the probe, only three readings with the probe oriented in three linearly independent directions are needed. Similarly, take an object in three-dimensional space in equilibrium. At a point in the object we can run a plane with normal vector $\vec{n}$ through it, splitting the object into two sides. One can then measure the force applied by one side onto the other in the direction of a chosen vector $\vec{l}$. Again, the force, F, is linearly dependent on our choices $(\vec{n}, \vec{l})$ and instead of an infinite number of readings, we only need $3 * 3 = 9$, namely the values F takes when $\vec{n}$ and $\vec{l}$ point in basis directions. Thus, we have a bilinear map $(\vec{n}, \vec{l}) \mapsto F$, called the *stress tensor*.

In this talk we will introduce tensors as a way of understanding linear dependence on displacements and apply them to our case of interest in order to gather another tool to work with going forward, and introduce a metric on the tangent space at a point defined earlier.

First, let's recall some concepts from linear algebra and functional analysis:
Let V be an arbitrary finite-dimensional vector space over $\mathbb{R}$. Its *dual space* is defined by $V^* := \{f : V \to \mathbb{R} \mid \text{linear}\}$; elements in it are called dual vectors. Defining addition and scalar multiplication in the obvious way, we turn V^* into a vector space of its own. We have, that $\dim V^* = \dim V$. This is proven by the following definition: Given a basis $v_1, ..., v_n$ of V, we define elements $v^{1*}, ..., v^{n*} \in V^*$ by

$$v^{\mu *}(v_\nu) = \delta_{\mu,\nu}$$

This obviously defines a basis on V^*, called the *dual basis* to the basis $\{v_\mu\}$.

Since V^* is a finite dimensional vector space, we can define its dual, called the double dual of V, V^{**}. Since the dual basis is dependent on the basis we chose for V, there is actually no natural way of identifying V^* with V even though they are apparently isomorphic. There is however such a way for V^{**} with the canonic embedding τ defined by

$$\tau : V \to V^{**}, v \mapsto \tau_v$$

where $\tau_v(w^*) := w^*(v)$.
This map is obviously linear and injective and, since $\dim V = \dim V^{**}$, bijective.

This is enough of a recap to now introduce tensors. We define a *tensor* of type (k, l) to be a multilinear map

$$T : V^* \times ... \times V^* \times V \times ... \times V \to \mathbb{R}$$

where there are k slots for dual vectors and l slots for ordinary vectors.

Thus, a tensor of type $(0, 1)$ is just an element of V^*. Similarly, a tensor of type $(1, 0)$ is an element of V^{**} which we can identify with an element of V.
Because of this identification, we can view tensors of higher type in different ways. For instance, take a tensor T of type $(1, 1)$. If we fix any $v \in V$, the remaining map is $T(\cdot, v) : V^* \to \mathbb{R} \in V^{**}$ which we can identify with an element of V. We have thus input a vector in V and recieved another vector in V in a linear fashion. So, we can view T to be a map $V \to V$, respectively as a map $V^* \to V^*$.

We denote the space of tensors of type (k, l) as $\mathscr{T}(k, l)$. Defining addition and scalar multiplication of multilinear maps in the obvious way, $\mathscr{T}(k, l)$ becomes a vector space. Since such a multilinear map is uniquely defined by the values it takes on basis vectors of V and V^*, and since there are $k + l$ slots with n possible basis vectors each, we have $\dim \mathscr{T}(k, l) = n^{k+l}$.
Let's take a look at two important operations on $\mathscr{T}(k, l)$. For that let's fix a basis $\{v_\mu\}$ and its dual basis $\{v^{\mu *}\}$.
The first operation we call the *contraction* of a tensor with respect to the i-th and j-th coordinates and define it as

$$C : \mathscr{T}(k, l) \to \mathscr{T}(k - 1, l - 1)$$

$$CT := \sum_{\sigma=1}^{n} T(\cdot .. \cdot v^{\sigma *} \cdot .. \cdot, \cdot .. \cdot v_\sigma \cdot .. \cdot)$$

where the vectors of the basis and dual basis are put into the $i - th$ and $j - th$ slot respectively. As we can see all this operation does is move the tensor it is applied to to a lower type by determining two slots by filling them with basis vectors.
Note that the contraction of a vector T of type $(1, 1)$ is just its trace if viewed as a map from V to V.
The other important operation is called the *outer product*. Given two tensors $T \in$

$\mathcal{T}(k,l)$ and $T' \in \mathcal{T}(k',l')$, we say that the outer product of T and T', denoted $T \otimes T'$, is a tensor of type $(k+k', l+l')$ defined by

$$T{\otimes}T'(v^{1^*},..,v^{k+k'^*},v_1,..,v_{l+l'}) := T(v^{1^*},..,v^{k^*},v_1,..,v_l){\cdot}T(v^{k+1^*},..,v^{k+k'^*},v_{l+1},..,v_{l+l'})$$

Therefore the outer product presents one way of constructing tensors of higher type. Recalling "$\mathcal{T}(1,0) = V$" and $\mathcal{T}(0,1) = V^*$, we find that we can construct tensors by taking the outer product of vectors and dual vectors. We can therefore define a basis of $\mathcal{T}(k,l)$ with respect to a basis $\{v_\mu\}$ of V and its dual basis $\{v^{\mu *}\}$, by considering the n^{k+l} elements

$$\{v_{\mu_1} \otimes .. \otimes v_{\mu_k} \otimes v^{\nu^{1^*}} \otimes .. \otimes v^{\nu^{l^*}}\}$$

where into each of the $k+l$ slots we put one of the n (dual) basis vectors. Because these elements are oviously linearly independent and span a subspace of dimension n^{k+l}, we have indeed found a basis. We can therefore write T with respect to that basis, yielding

$$T = T^{\mu_1,..,\mu_k}_{\nu_1,..,\nu_l} \; v_{\mu_1} \otimes ... \otimes v^{\nu^{l^*}}$$

employing Einstein index notation. We call the $T^{\mu_1,..,\mu_k}_{\nu_1,..,\nu_l}$ the components of T with respect to the basis $\{v_\mu\}$.
The operations introduced earlier can obviously also be written in this way. Their components are given by

$$(CT)^{\mu_1,..,\mu_{k-1}}_{\nu_1,..,\nu_{l-1}} = \sum_{\sigma=1}^{n} T^{\mu_1,..,\sigma,..,\mu_{k-1}}_{\nu_1,..,\sigma,..,\nu_{l-1}}$$

where the σ are put into the i-th and j-th slots, and

$$(T \otimes T')^{\mu_1,..,\mu_{k+k'}}_{\nu_1,..,\nu_{l+l'}} = T^{\mu_1,..,\mu_k}_{\nu_1,..,\nu_l} \cdot T'^{\mu_{k+1},..,\mu_{k+k'}}_{\nu_{l+1},..,\nu_{l+l'}}$$

respectively.

Now that we have a foundation of understanding of tensors, let us take a look at the case we are most interested in in this seminar, namely when V is the tangent space at a point p on a manifold M, V_p. Its dual space, V_p^*, is called the cotangent space, vectors in it cotangent vectors. If we define a coordinate basis $\{\partial/\partial x^\mu\}$, the associated dual basis is denoted as $\{d/dx^\mu\}$. It follows from the vector transformation law

$$v'_{\mu'} = \sum_{\mu=1}^{n} v_\mu \frac{\partial x'^{\mu'}}{\partial x^\mu}$$

and the definition of the associated dual basis vectors that the components w_μ of a dual vector transform by

$$w'_{\mu'} = \sum_{\mu=1}^{n} w_\mu \frac{\partial x^\mu}{\partial x'^{\mu'}}$$

when changing coordinate systems. Similarly, the components of tensors transform by

$$T'^{\mu'_1,..,\mu'_k}_{\nu'_1,..,\nu'_l} = \sum_{\mu_1,..,\nu_l=1}^{n} T^{\mu_1,..,\mu_k}_{\nu_1,..,\nu_l} \frac{\partial x'^{\mu'_1}}{\partial x^{\mu_1}} ... \frac{\partial x^{\nu_l}}{\partial x'^{\nu'_l}}.$$

This is known as the tensor transformation law.

We are now ready to introduce the metric tensor. A metric tells us the "infinitesimal squared distance" associated with an "infinitesimal displacement". In the last talk we found that our notion of an "infinitesimal displacement" is exactly captured in the definition of a tangent vector. Therefore, we define a *metric* to be a linear map $V_p \times V_p \to \mathbb{R}$, so a tensor g of type $(0, 2)$. In addition to that we require g to be symmetric and nondegenerate. We can write g in terms of its components by

$$g = \sum_{\mu,\nu} g_{\mu\nu} dx^{\mu} \otimes dx^{\nu}.$$

Given a metric we can always find an orthonormal basis $\{v_i\}$ of the tangent space at each point such that $g(v_\mu, v_\nu) = \pm\,\delta_{\mu,\nu}$.
There are of course other orthonormal bases of V_p but it turns out that the number of positive/negative signs of $g(v_\mu, v_\nu)$ are always the same. We call that number the *signature* of the metric. One mostly deals with positive definite metrics, so metrics of the signature $(+ + ...+)$, also called *Riemannian* metrics. However, the metric of spacetime has the signature $(- + ++)$ and is called *Lorentzian*.