Derivation of the Schwarzschild solution

PDE II seminar talk

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1 Constructing the form of the metric

The Schwarzschild metric is the solution to the Einstein equations

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi \mathbb{T}_{\mu\nu} \tag{1}$$

for a static and spherically symmetric spacetime in vacuum. It describes the gravitational field of the exterior of a round body, that does not change in time. The goal is to find all 10 independent metric components $g_{\mu\nu}$ of the general metric

$$g = \sum_{\mu,\nu=0}^{3} g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu. \tag{2}$$

Definition 1 (Static spacetime)**.** *A spacetime is called static if there exists a one-parameter group of isometries, ϕ_t, whose orbits are timelike curves. And there exists a spacelike hypersurface Σ, which is orthogonal to the orbits of the isometries.*

This corresponds to the condition, that there exists a timelike Killing vector field ξ (along the orbits of the isometries ϕ_t), which is hypersurface orthogonal. A Killing vector field satisfies

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0. \tag{3}$$

And by Frobenius's theorem, which can be found in Wald's General Relativity, hypersurface orthogonality is equivalent to the Killing vector field satisfying

$$\xi_{[a} \nabla_b \xi_{c]} = 0, \tag{4}$$

where the brackets around the indices [a b c] denote the total antisymmetrization of the Tensor $\xi_a \nabla_b \xi_c$.
These isometries describe the time translation and time reversal symmetry of a static spacetime.

The next step is to choose coordinates for the static spacetime. Every point p in Σ lies on a unique orbit of ξ. Therefore, they can be labelled by the parameter t and the other (spatial) coordinates in Σ, which we call x^1, x^2 and x^3. The points p can then be translated along the orbits to another hypersurface Σ_t. Note that Σ_t is still orthogonal to ξ and the coordinates of p from Σ are conserved along the orbits, because the parameter t is independent of the chosen coordinates on Σ. Thus, there are no cross terms $\mathrm{d}t\mathrm{d}x^i$ and the metric takes the form

$$g = -f(x^1, x^2, x^3)\mathrm{d}t^2 + \sum_{i,j=1}^{3} h_{ij}(x^1, x^2, x^3)\mathrm{d}x^i\mathrm{d}x^j. \tag{5}$$

We also know that the function f takes the form $f = -\xi_a \xi^a$.

Definition 2 (Spherically symmetric)**.** *A spacetime is called spherically symmetric if its isometry group contains a subgroup that is isomorphic to $SO(3)$. The orbits of that isometry are 2-spheres.*

The elements of $SO(3)$ can be interpreted as rotations, hence the spacetime is invariant under rotation. Each orbit 2-sphere is a multiple of the unit 2-sphere. Thus, the orbit 2-sphere can be completely characterized by its total area A. By introducing spherical coordinates (r, θ, ϕ), where the radial coordinate r is defined as

$$r = \left(\frac{A}{4\pi}\right)^{\frac{1}{2}}, \tag{6}$$

we get the induced metric on each orbit 2-sphere

$$g = r^2(\mathrm{d}\theta^2 + \sin\theta\mathrm{d}\phi^2) =: r^2\mathrm{d}\Omega^2. \tag{7}$$

It is important to note that r can not necessarily be interpreted as the distance from the surface of the sphere to its center, because the manifold could look like $\mathbb{R} \times \mathbb{S}^2$ rather than $\mathbb{R}^3 = \mathbb{R}_{\geq 0} \times \mathbb{S}^2$. In this case there is no center of the sphere.

If a spacetime is both static and spherically symmetric, then the Killing vector field ξ (if unique) must be orthogonal to the orbit 2-spheres. Thus, ξ is invariant under all rotational symmetries and the orbit 2-spheres lie wholly within the hypersurface Σ_t.
The full coordinates on the spacetime can now be constructed by selecting a sphere in Σ_t and choosing spherical coordinates θ and ϕ on this sphere. These coordinates can be extended to other spheres in Σ_t by means of geodesics. So if $\nabla_a r \neq 0$, we have coordinates (r, θ, ϕ) in Σ_t and by translating along the Killing vector field ξ, we can finally choose the coordinates (t, r, θ, ϕ). In these coordinates a metric of a static spherically symmetric spacetime takes the form

$$g = -f(r)\mathrm{d}t^2 + h(r)\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin\theta\mathrm{d}\phi^2). \tag{8}$$

2 Calculating the Ricci tensor and solving the Einstein vacuum equations

For the exterior of the body, the Energy-Momentum tensor $\mathbb{T}_{\mu\nu}$ vanishes, because there is no matter field terms. Therefore, the Einstein equations in vacuum simplify to

$$R_{\mu\nu} = 0. \tag{9}$$

A straightforward way to get the Ricci curvature tensor is to first calculate all Christoffel symbols $\Gamma^{\rho}_{\mu\nu}$ for the metric from equation (8) with

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}\sum_{\sigma} \left(g^{-1}\right)^{\rho\sigma} \left(\frac{\partial g_{\nu\sigma}}{\partial x^{\mu}} + \frac{\partial g_{\mu\sigma}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}}\right). \tag{10}$$

This leads to the following nine independent, non-zero components.

$$\begin{aligned}
\Gamma^{t}_{tr} &= \frac{1}{2}f^{-1}f', & \Gamma^{r}_{tt} &= \frac{1}{2}h^{-1}f', \\
\Gamma^{r}_{rr} &= \frac{1}{2}h^{-1}h', & \Gamma^{r}_{\theta\theta} &= -rh^{-1}, \\
\Gamma^{r}_{\phi\phi} &= -rh^{-1}\sin^2\theta, & \Gamma^{\theta}_{\phi\phi} &= -\sin\theta\cos\theta, \\
\Gamma^{\theta}_{\theta r} &= r^{-1}, & \Gamma^{\phi}_{\phi r} &= r^{-1}, \\
\Gamma^{\phi}_{\phi\theta} &= \frac{\cos\theta}{\sin\theta}.
\end{aligned} \tag{11}$$

All other non-zero components can be constructed through the symmetric property of the Christoffel symbols, $\Gamma^{\rho}_{\mu\nu} = \Gamma^{\rho}_{\nu\mu}$. The Ricci curvature can now be derived from the contraction of the Riemann curvature

$$R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu} = \partial_{\alpha}\Gamma^{\alpha}_{\mu\nu} - \partial_{\nu}\Gamma^{\alpha}_{\mu\alpha} + \Gamma^{\beta}_{\mu\nu}\Gamma^{\alpha}_{\beta\alpha} - \Gamma^{\beta}_{\mu\alpha}\Gamma^{\alpha}_{\beta\nu}. \tag{12}$$

Then we finally get the Ricci curvature tensor components as

$$R_{tt} = \frac{1}{2}h^{-1}f'' - \frac{1}{4}h^{-2}h'f' - \frac{1}{4}\left(hf\right)^{-1}\left(f'\right)^2 + \left(hr\right)^{-1}f', \tag{13}$$

$$R_{rr} = -\frac{1}{2}f^{-1}f'' + \frac{1}{4}\left(hf\right)^{-1}h'f' + \frac{1}{4}f^{-2}\left(f'\right)^2 + \left(hr\right)^{-1}h', \tag{14}$$

$$R_{\theta\theta} = 1 - h^{-1} + \frac{1}{2}h^{-2}rh' - \frac{1}{2}\left(hf\right)^{-1}rf'. \tag{15}$$

Additionally, we have $R_{\phi\phi} = \sin^2\theta R_{\theta\theta}$ and all other components of the Ricci tensor are vanishing. By setting all components to zero and adding f^{-1} times equation (13) to h^{-1} times equation (14), we get

$$\frac{f'}{f} + \frac{h'}{h} = 0, \tag{16}$$

which has the solution

$$f = Kh^{-1}, \tag{17}$$

where K is a constant. To check if the constant has any physical relevance or if they can be chosen freely as nice numbers, we re-scale the time coordinate to $t \to K^{1/2}t$. The metric then gives

$$\begin{aligned} g &= -Kh^{-1}(r)\mathrm{d}t^2 + h(r)\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \\ &= -h^{-1}(r)\mathrm{d}(K^{1/2}t')^2 + h(r)\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \\ &= -h^{-1}(r)\mathrm{d}t'^2 + h(r)\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2. \end{aligned} \tag{18}$$

We can see that the constant K can be absorbed into the time coordinate and we continue with the re-scaled time. This is equivalent to choosing $K = 1$. Inserting $f = h^{-1}$ into equation (15) now yields

$$-f'r + 1 - f = 0 \quad \Leftrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}r}(rf) = 1. \tag{19}$$

A solution to this differential equation is given by

$$f = 1 + \frac{C}{r}, \tag{20}$$

where C is a constant. Thus, the metric that solves the vacuum Einstein equations for a static and spherically symmetric spacetime is

$$g = -\left(1 + \frac{C}{r}\right)\mathrm{d}t^2 + \left(1 + \frac{C}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2. \tag{21}$$

3 The Newtonian limit

To determine the constant C, we can check the solution of the Einstein equations in the Newtonian limit, because the new theory should agree with the well tested Newton mechanics. For that, it makes sense to look at radial free fall in the Schwarzschild spacetime for $r \gg |C|$. The radial free fall is a geodesic curve of the form

$$\gamma = \begin{pmatrix} \gamma^t \\ \gamma^r \\ \gamma^\theta \\ \gamma^\phi \end{pmatrix} = \begin{pmatrix} t \\ r \\ \theta_0 \\ \phi_0 \end{pmatrix}, \tag{22}$$

where θ_0 and ϕ_0 are constant. The geodesic equation in coordinates is

$$\frac{\mathrm{d}^2\gamma^\mu}{\mathrm{d}\tau^2} - \Gamma^\mu_{\alpha\beta}\frac{\mathrm{d}\gamma^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}\gamma^\beta}{\mathrm{d}\tau} = 0. \tag{23}$$

In the case of radial free fall only $\frac{\mathrm{d}\gamma^r}{\mathrm{d}\tau}$ and $\frac{\mathrm{d}\gamma^t}{\mathrm{d}\tau}$ is $\neq 0$. So the geodesic equation for radial free fall is

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} - \Gamma^r_{tt} \left(\frac{\mathrm{d}t}{\mathrm{d}\tau} \right)^2 = 0, \tag{24}$$

if we also assume that $\frac{\mathrm{d}t}{\mathrm{d}\tau} \gg \frac{\mathrm{d}r}{\mathrm{d}\tau}$. Calculating the Christoffel symbol gives

$$\begin{aligned} \Gamma^r_{tt} &= \frac{1}{2} \left(g^{-1}\right)^{r\alpha} \left(\frac{\partial g_{t\alpha}}{\partial t} + \frac{\partial g_{\alpha t}}{\partial t} - \frac{\partial g_{tt}}{\partial x^\alpha} \right) \\ &= -\frac{1}{2} \left(g^{-1}\right)^{rr} \frac{\partial g_{tt}}{\partial r} = -\frac{1}{2} f (-f)' \\ &= -\frac{C}{2r^2} \left(1 + \frac{C}{r} \right) \approx -\frac{C}{2r^2}. \end{aligned} \tag{25}$$

For r big enough the radial geodesic equation gives

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} \approx \frac{C}{2r^2}, \tag{26}$$

while Newtonian mechanics gives

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = -\frac{GM}{r^2}, \tag{27}$$

with M being the mass of the body and G being the gravitational constant. Thus, for both theories to agree, it must hold

$$C = -2MG. \tag{28}$$

In natural units one sets $G = 1$. And finally the Schwarzschild metric is

$$g = -\left(1 - \frac{2M}{r} \right) \mathrm{d}t^2 + \left(1 - \frac{2M}{r} \right)^{-1} \mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2. \tag{29}$$

In addition to the singularity at $r = 0$, the Schwarzschild metric has a breakdown in coordinates at $r_S = 2M$, which is called the Schwarzschild radius. Restoring the units leads to the value

$$r_S = \frac{2GM}{c^2} \approx 3 \left(\frac{M}{M_\odot} \right) \mathrm{km} \tag{30}$$

for the Schwarzschild radius, where $M_\odot$ is the mass of the sun. So for normal bodies (i.e. close to the size and mass of the sun), the Schwarzschild radius lies inside the body, where the coordinates of the exterior solution are not valid. It therefore has no relevance. But if the mass of the body is compacted enough , r_S lies outside of the body and we have a black hole. The Schwarzschild radius is then the event horizon of the black hole.

Reference

Wald, R. M. (1984), General Relativity, University Of Chicago Press.