Manifolds and Vectors

Seminar course

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Introduction

This is the first talk in the seminar series on General Relativity held in the summer term 2024 at the univeristy Leipzig under supervision of Prof. Dr. Stefan Czimek. The first talk is meant to give the fundamental mathematical structure for the coming weeks. I will present the concepts of Manifolds and Vectors which are the very basis on which the whole General Relativity concepts lies.
The Goal of a Manifold is to give a Vector space like structure on which one can investigate physical phenomenon. Still in General Relativity the main difference is to not be bounded by the idea of space time behaving like a four dimensional vector space. So a Manifold behaves locally like a vector space but not globally, which differentiates special and general Relativity.

1 Manifolds

Manifolds are the basic mathematical structure for us to search for the properties of the space-time structure.

1.1 Definition (Manifolds)**.** Let $n \in \mathbb{N}$, $0 \leq k \leq \infty$ and $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. A n-dimensional C^k, $\mathbb{K}$ *Manifold* M consist of a Set M, a collection of subsets $\{\Omega_\alpha\}_\alpha$ and a collection of (homeomorphic) functions $\{\psi_\alpha\}_\alpha$ ($\alpha \in \mathcal{I}$). Now $\{\Omega_\alpha\}_\alpha$ and $\{\psi_\alpha\}_\alpha$ be the maximum possible sets with the following properties.

(i) $\bigcup_\alpha \Omega_\alpha = M$

(ii) for all α there exists a map $\psi_\alpha : \Omega_\alpha \to \mathbb{R}^n$ and $\psi_\alpha(\Omega_\alpha) = U_\alpha$ be open such that ψ_α is a bijection on U_α.

(iii) For $\alpha, \beta \in \mathcal{I}$ with $\Omega_\alpha \cap \Omega_\beta \neq \emptyset$ we can consider the map $\psi_\beta \circ \psi_\alpha^{-1}$: $\psi_\alpha(\Omega_\alpha \cap \Omega_\beta) \to \psi_\beta(\Omega_\alpha \cap \Omega_\beta)$. We demand $\psi_\alpha(\Omega_\alpha \cap \Omega_\beta), \psi_\beta(\Omega_\alpha \cap \Omega_\beta)$ to be open and $\psi_\beta \circ \psi_\alpha^{-1}$ to be C^k

The Maps ψ_α are called *charts* or *coordinate systems*. The two sets $\{\Omega_\alpha\}_\alpha$ and $\{\psi_\alpha\}_\alpha$ only need to be maximal so a Manifold cannot be created by adding or removing one subset or a function. We will focus on C^∞, real Manifolds. If we define M to be a topological space and the charts to be homeomorphisms we would be able to deduce more properties. Although I think for this seminar we don't need that.

1.2 Example. General example of the set M with subsets and two charts:
Another easy and important example is the 2-Sphere:

$$M = S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1)^2 + (x_2)^2 + (x_3)^2 = 1\}$$

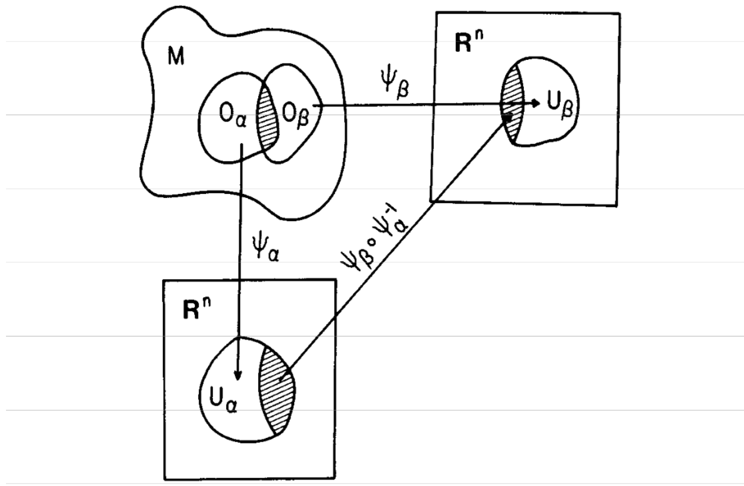

Fig. 2.1. An illustration of the map $\psi_\beta \circ \psi_\alpha^{-1}$ arising when two coordinate systems overlap.

Taken form `Wald`

We define 6 subsets:

$$\Omega_i^{\pm} = \{(x_1, x_2, x_3) \in S^2 : \pm x_i > 0\}$$

Let $D = \{(x_1, x_2) \in \mathbb{R} : x_1^2 + x_2^2 < 1\}$ be the unit disk.Now define the charts: $f_i^{\pm} : \Omega_i^{\pm} \to D$ with $f_1^{+}(x_1, x_2, x_3) = (x_2, x_3)$ and so on. The overlapping functions $f_i^{\pm} \circ (f_j^{\pm})^{-}1$ are indeed C^{∞} functions in there domain:

Proof. We will proof it for the case $i = 1$, $j = 2$ and $+$. So we have the function $f = f_1^{+} : \Omega_1^{+} \to D$ with $(x_1, x_2, x_3) \mapsto (x_2, x_3)$ and $g = (f_2^{+})^{-1} : D \to \Omega_2^{+}$ with $(x_1, x_2) \mapsto (x_1, \sqrt{1 - x_1^2 - x_2^2}, x_2)$ this is well defined. If we now concatenate those functions we get: $f \circ g : g^{-1}(\Omega_1^{+} \cap \Omega_2^{+}) \to f(\Omega_1^{+} \cap \Omega_2^{+})$ with $(x_1, x_2) \mapsto (\sqrt{1 - x_1^2 - x_2^2}, x_2)$ which also is well defined and infinitely continuous differentiable because the square and the power of two is. For all the other cases the proof is analogue. There for S^2 is a 2-dimensional, C^{∞}, real Manifold. $\square$

1.3 Definition. Let M and M' be Manifolds of dimension n and n' respectively. Then we can define on the product of the two sets ($M \times M' = \{(p, p') : p \in M, p' \in M'\}$) a $n + n'$ dimensional Manifold. This is called the *product* Manifold. Let $\{\Omega_\alpha \times \Omega_\beta\}_{(\alpha,\beta)}$ be the set of subsets with Ω_α and Ω_β a subset of the Manifolds M and M'. We also have the charts $\psi_\alpha : \Omega_\alpha \to \mathbb{R}^n$ and $\psi_\beta : \Omega_\beta \to \mathbb{R}^{n'}$. We define the chart on our new Manifold to be $\psi_{(\alpha,\beta)} : \Omega_\alpha \times \Omega_\beta \to \mathbb{R}^n \times \mathbb{R}^{n'}$ with $\psi_{(\alpha,\beta)}(p, p') = (\psi_\alpha(p), \psi_\beta(p'))$. It can be checked that this is indeed a Manifold.

The Product Manifolds are especially interesting in the rest of this seminar as most manifolds in Walds book can be expressed as products of $\mathbb{R}^n$ and a sphere S^m.

1.4 Definition (Diffeomorphism)**.** Let M and M' be two Manifolds with $\{\psi_\alpha\}_\alpha$ $(\alpha \in \mathcal{I})$ and $\{\psi'_\beta\}_\beta$ $(\beta \in \mathcal{J})$ denote the charts. Let $f : M \to M'$ be a map. Then f is said to be C^∞ if for each $\alpha \in \mathcal{I}$ and $\beta \in \mathcal{J}$ the map $\psi'_\beta \circ f \circ \psi_\alpha^{-1} : U_\alpha \to U'_\beta$ is in C^∞. In more general a function $f : M \to \mathbb{R}$ or $g : \mathbb{R} \to M$ is considered to be C^∞ if the composition with any chart $\psi : M \to U \subset \mathbb{R}^n$ or its inverse ψ^{-1} respectively, $(f \circ \psi^{-1}) : U \to \mathbb{R}$ or $(\psi \circ g) : \mathbb{R} \to U$ is a C^∞ function in $\mathbb{R}$.
Now $f : M \to M'$ is called a *diffeomorphism* if f is in C^∞, is a bijection and if f^{-1} also is in C^∞. Also then M and M' are called *diffeomirphic*

If one considers the manifolds as topological spaces this definition is equal with the definition of the diffeomorphism of topological spaces.

2 Vectors

As mentioned above our manifold should have locally some vector space properties. We will achieve that by defining a vector space at each point in our Manifold. As a build up one can remember the 2-sphere again. For very short distances on this sphere it ”looks like” a plane, so it behaves locally like a $\mathbb{R}^2$ vector space. More precisely we take the tangent plane at a point as our vector space.
So in the next steps we will introduce directional derivatives or tangent vectors on the manifold.

2.1 Definition. Let M be a Manifold and let $\mathcal{F}$ denote all C^∞ functions from M to $\mathbb{R}$. Let $v : \mathcal{F} \to \mathbb{R}$ be a map. Then v is a *tangent vector* at point $p \in M$, if it obeys

(i) for all $f, g \in \mathcal{F}$ and all $a, b \in \mathbb{R}$ it holds $v(af + bg) = av(f) + bv(g)$ (linearity)

(ii) for all $f, g \in \mathcal{F}$ it holds $v(fg) = f(p)v(g) + v(f)g(p)$ (Leibnitz rule)

2.2 Lemma. Let $F : \mathbb{R}^n \to \mathbb{R}$ be a C^∞ function then for every $a = (a_1, \dots, a_n) \in \mathbb{R}^n$ we have C^∞ functions H_μ such that for all $x \in \mathbb{R}^n$

$$F(x) = F(a) + \sum_{\mu=1}^{n} (x_\mu - a_\mu) H_\mu(x)$$

And we have

$$H_\mu(a) = \frac{\partial F}{\partial x_\mu}|_{x=a}$$

Proof. First for the $n = 1$ case we choose $H(x) := \frac{F(x)-F(a)}{x-a}$ if $x \neq a$. And $H(a) := \lim_{x\to a} H(x)$. As this is the difference quotient the limes exists and $H(a) = \frac{\partial F}{\partial x}|_{x=a}$.
For $n > 1$ we will prove it by induction. Assume the statement holds for $n-1$, show it for n. Let $a \in \mathbb{R}^n$ and $F : \mathbb{R}^n \to \mathbb{R}$ in C^∞. Now the function $g(x_1, \ldots, x_{n-1}) = F(x_1, \ldots, x_{n-1}, a_n)$ is a function from $\mathbb{R}^{n-1}$ to $\mathbb{R}$ and is also C^∞ so by our induction assumption we get functions $H_1, \ldots, H_{n-1}$ such that

$$g(x_1, \ldots, x_{n-1}) = g(a_1, \ldots, a_{n-1}) + \sum_{i=1}^{n-1} (x_i - a_i) H_i(x_1, \ldots, x_{n-1})$$

Now we consider the function $f : \mathbb{R} \to \mathbb{R}$ which is defined as $f(x_n) = F(x_1, \ldots, x_n)$ for a solid $x_1, \ldots, x_{n-1} \in \mathbb{R}^{n-1}$. We get a function $H_n(x_n)$ with

$$f(x_n) = f(a_n) + (x_n - a_n) H_n(x_n)$$

But the function H_n depends on the $x_1, \ldots, x_{n-1}$ so we consider it to be the function $H(x_1, \ldots, x_n)$. Now we use that $f(a_n) = f(x_1, \ldots, x_{n-1}, a_n) = g(x_1, \ldots, x_{n-1})$ and $g(a_1, \ldots, a_{n-1}) = F(a_1, \ldots, a_n)$, we can deduce that

$$\begin{aligned} &F(x_1, \ldots, x_n) = f(x_n) \\ &= g(x_1, \ldots, x_{n-1}) + (x_n - a_n) H_n(x_1, \ldots, x_n) \\ &= F(a_1, \ldots, a_n) + \sum_{i=1}^{n} (x_i - a_i) H_i(x_1, \ldots, x_n) \end{aligned}$$

□

2.3 Theorem. Let M be a n-dimensional manifold and $p \in M$. The set of all tangent vectors V_p at p is a vector space with $\dim V = n$. Call this the *tangent vector space* at p

Proof. First V_p is a vector with vector addition $v_1, v_2 \in V_p$ holds $(v_1 + v_2)(f) = v_1(f) + v_2(f)$ and scalar multiplication $c \in \mathbb{R}$ holds $(cv)(f) = cv(f)$ for $f \in \mathcal{F}$. All of the vector space properties can easily be checked and hold because $v(f) \in \mathbb{R}$.
Now we will construct a basis of V_p to show the dimension. First choose a chart $\psi : \Omega \to \mathbb{R}^n$ with $p \in \Omega$ and $\psi(\Omega) = U$. Let $f \in \mathcal{F}$ we have $f \circ \psi^{-1} : U \to \mathbb{R}$ is C^∞. For $\mu = 1, \ldots, n$ we define:

$$X_\mu(f) = \frac{\partial}{\partial x_\mu}(f \circ \psi^{-1})|_{\psi(p)}$$

Where $(x_1, \ldots, x_n)$ are the Cartesian coordinates of $\mathbb{R}^n$. $X_\mu : \mathcal{F} \to \mathbb{R}$ is a function that obeys the two conditions above: Let $f, g \in \mathcal{F}$ and $a, b \in \mathbb{R}$, it holds

$$\begin{aligned}
X_\mu(af + bg) &= \frac{\partial}{\partial x_\mu}((af + bg) \circ \psi^{-1})|_{\psi(p)} \\
&= \frac{\partial}{\partial x_\mu} a(f \circ \psi^{-1}) + b(g \circ \psi^{-1})|_{\psi(p)} \\
&= a\frac{\partial}{\partial x_\mu}(f \circ \psi^{-1})|_{\psi(p)} + b\frac{\partial}{\partial x_\mu}(g \circ \psi^{-1})|_{\psi(p)} \\
&= aX_\mu(f) + bX_\mu(g)
\end{aligned}$$

and

$$\begin{aligned}
X_\mu(f \cdot g) &= \frac{\partial}{\partial x_\mu}(f \cdot g \circ \psi^{-1})|_{\psi(p)} \\
&= \frac{\partial}{\partial x_\mu}(f \circ \psi^{-1}) \cdot (g \circ \psi^{-1})|_{\psi(p)} \\
&= (\frac{\partial}{\partial x_\mu}(f \circ \psi^{-1})) \cdot (g \circ \psi^{-1}) + (f \circ \psi^{-1}) \cdot (\frac{\partial}{\partial x_\mu}(g \circ \psi^{-1}))|_{\psi(p)} \\
&= X_\mu(f) \cdot g(p) + f(p) \cdot X_\mu(g)
\end{aligned}$$

So $X_1, \ldots, X_n$ are tangent vectors and they are linearly independent. Because otherwise there would be a collection $(c_1, \ldots, c_n)$ in $\mathbb{R}^n$ with $\sum_{\mu=1}^n c_\mu X_\mu(f) = 0$ for all $f \in \mathcal{F}$.
Now we show, that they span V_p. For that we use the Lemma. We apply it to $F = f \circ \psi^{-1}$ and $a = \psi(p)$ so for all $x \in \mathbb{R}$ we get

$$(f \circ \psi^{-1})(x) = (f \circ \psi^{-1})(a) + \sum_{\mu=1}^n (x_\mu - a_\mu) H_\mu(x)$$

Because ψ is a bijection we can replace $x = \psi(q)$. So for all $q \in \Omega$ wqe get

$$f(q) = f(p) + \sum_{\mu=1}^n (\psi(q)_\mu - \psi(p)_\mu) H_\mu(\psi(q))$$

Now let $v \in V_p$ be arbitrary, we want to express v as a linear combination of X_μ. Now we put our representation of f into v and use the Leibnitz rule:

$$\begin{aligned}
v(f) &= v(f(p)) + v(\sum_{\mu=1}^n (\psi_\mu - \psi(p)_\mu) \cdot (H_\mu \circ \psi)) \\
&= \sum_{\mu=1}^n v((\psi_\mu - \psi(p)_\mu)) \cdot (H_\mu \circ \psi)|_p + (\psi_\mu - \psi(p)_\mu)|_p \cdot v(H_\mu \circ \psi) \\
&= \sum_{\mu=1}^n v(\psi_\mu) \cdot H_\mu(\psi(p))
\end{aligned}$$

Because of the lemma $H_\mu(\psi(p)) = H_\mu(a) = \frac{\partial}{\partial x_\mu}(f \circ \psi^{-1})|_{\psi(p)} = X_\mu(f)$, and we have the coefficients $c_\mu = v(\psi_\mu)$ with $\psi_\mu : M \to \mathbb{R}$ by $\psi_\mu(q) = \psi(q)_\mu$. So in total we have

$$v(f) = \sum_{\mu=1}^{n} c_\mu \cdot X_\mu(f)$$

As $f \in \mathcal{F}$ and $v \in V_p$ were arbitrary we showed, that $X_1, \ldots, X_n$ is indeed a basis of the vector space V_p. □

2.4 Definition. For a tangent vector space V_p at $p \in M$ we call the set of $X_\mu(f) = \frac{\partial}{\partial x_\mu}(f \circ \psi^{-1})|_{\psi(p)}$ for $\mu = 1, \ldots, n$ the *coordinate basis.*

2.5 Remark. The coordinates basis X_μ we use in the proof depends on the chart. So with a different chart ψ' instead of ψ we have the coordinate basis X'_μ with transformation formula:

$$\begin{aligned} X_\mu(f) &= \frac{\partial}{\partial x^\mu}(f \circ \psi'^{-1} \circ \psi' \circ \psi^{-1})|_{\psi(p)} \\ &= \sum_{\nu=1}^{n} \frac{\partial(f \circ \psi'^{-1})}{\partial x^\nu}(\psi'(\psi^{-1}(\psi(p)) \cdot \frac{\partial(\psi' \circ \psi^{-1})}{\partial x^\mu}(\psi(p)) \\ &= \sum_{\nu=1}^{n} \frac{\partial x'^\nu}{\partial x^\mu}|_{\psi(p)} X'_\nu(f) \end{aligned}$$

We used mainly the chain rule.
Where x'^ν denotes the ν-th component of the map $\psi' \circ \psi^{-1}$. So we get the coordinate transformation of vectors from one coordinate system to the other:

$$v'^\nu = \sum_{\mu=1}^{n} v^\mu \frac{\partial x'^\nu}{\partial x^\mu}$$

This is also known as the *vector transformation law.*

2.6 Definition. A *smooth curve* C on a Manifold M is a C^∞ map of $\mathbb{R}$ into M, $C : \mathbb{R} \to M$. Each point $p \in M$ on the curve, i.e. exists $t \in \mathbb{R}$ such that $C(t) = p$ can be associated with a tangent vector $T \in V_p$ at p. T is defined as $T(f) = \frac{d(f \circ C)}{dt}|_t$ for $f \in \mathcal{F} = C^\infty(M, \mathbb{R})$. Note that $f \circ C$ is in $C^\infty(\mathbb{R})$.

We can map this curve into $\mathbb{R}^n$ if we choose a coordinate system ψ. We get $z_\mu(t) := (\psi \circ C)_\mu(t)$ for $1 \leq \mu \leq n$. There for we get

$$\begin{aligned} T(f) &= \frac{d(f \circ C)}{dt}|_t = \frac{d(f \circ \psi^{-1} \circ z)}{dt}|_t \\ &= \sum_{\mu=1}^{n} \frac{\partial(f \circ \psi^{-1})}{\partial x_\mu}(z(t)) \cdot \frac{dz_\mu}{dt}|_t \\ &= \sum_{\mu=1}^{n} X_\mu(f)|_{z(t)} \cdot \frac{dz_\mu}{dt}|_t \end{aligned}$$

We used the chain rule, the definition of X_μ and $z(t) = (\psi \circ C)(t) = \psi(p)$. So the individual components of T are $T_\mu = \frac{dz_\mu}{dt}(t)$, therefore

$$T(f) = \sum_{\mu=1}^{n} \frac{dz_\mu}{dt}|_t X_\mu(f)|_{\psi(p)}$$

In the previous we choose a constant point $p \in M$ and investigated the tangent vector space and coordinate basis only for this point. If we now change the point, the tangent vector space and everything else will change. There is no obvious way of identifying those two vector spaces (parallel transport). Still we can look at the change of the tangent space when changing the point of reference.

2.7 Definition. A *tangent field* is a assignment of a tangent vector $v^{(p)} \in V_p$ at each point $p \in M$. This tangent field is set to be *smooth* for each smooth function $f \in \mathcal{F}$ if the function $v(f) : M \to \mathbb{R}$ with $v(f)(p) = v^{(p)}(f)$ is smooth.

Also there exists the concept of a *one parameter group of diffeomorphimsms* which are maps from $\mathbb{R} \times M \to M$ which are in bijective correspondence to the smooth vector fields. A *vector field* on a Manifold is a assignment of a tangent vector at each point $p \in M$. Or a tangent field!

2.8 Definition. Let $\varphi : \mathbb{R} \times M \to M$ with $\varphi_t \in C^\infty$ for fixed $t \in \mathbb{R}$ a diffeomorphism. And for all $t, s \in \mathbb{R}$ we have $\varphi_t \circ \varphi_s = \varphi_{t+s}$. This results in $\varphi_0 = id_M$. Now $\varphi(p) : \mathbb{R} \to M$ is a curve in M for fixed $p \in M$ we call it *orbit*. Define $v^{(p)}$ to be the tangent vector of the curve at point p and $t = 0$. $v^{(p)}(f) = \frac{d(f \circ \varphi(p))}{dt}|_{t=0}$. Which is the association in one direction.
Now for a smooth tangent field $v^{(p)}$ we search for *integral curves* that is a family of curves $\{\varphi(p) : \mathbb{R} \to M\}_{p \in M}$. Such that only one curve passes through p ($\varphi_0(p) = p$) and the tangent at that p is $v^{(p)}$. We select a $p \in M$ and select any coordinate system. The solution is to solve this system of ODEs for the curves in $\mathbb{R}$: $z_\mu(t) = \psi_\mu \circ \varphi_t(p)$

$$\frac{dz_\mu(t)}{dt} = v_\mu \circ \psi^{-1}(z_1(t), \ldots, z_n(t))$$

with v_μ being the μ-th component of v in the coordinate basis $\frac{\partial}{\partial x_\mu}$. It has a unique solution for the starting point $t = 0$. This solution indeed satisfy our conditions od a integral curve:

1. The tangent at point $p \in M$ for any $f \in \mathcal{F}$ is

$$\frac{d(f \circ \varphi_t(p))}{dt}|_{t=0} = \sum_{\mu=1}^{n} X_\mu(f)|_{z(0)} \cdot \frac{dz_\mu}{dt}|_0 = v^{(p)}(f)$$

as seen above and because $v_\mu^{(p)}(f) = \frac{dz_\mu}{dt}(0)$ the starting condition.

2. And there is exactly one curve going through $p \in M$ because two different curves always have a different starting point $\varphi_0(q) = q \neq p = \varphi_0(p)$.

Originator

This script and talk is based on Wald's book General Relativity (University of Chicago press)