What is martingale?

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A Digression
A digression - conditional expectation

In the following, let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space.

Let \(A\) and \(B\) be two events. Then the conditional probability is given by

\[
\mathbb{P}(A \mid B) = ?.
\]
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\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.
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A digression - conditional expectation

In the following, let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space.

Let \(A = \{X = x\}\) and \(B = \{Y = y\}\) be two events. Then the conditional probability is given by

\[
\mathbb{P}(X = x \mid Y = y) = \frac{\mathbb{P}(\{X = x\} \cap \{Y = y\})}{\mathbb{P}(Y = y)}.
\]

The expectation of \(X\) is

\[
\mathbb{E}(X) = \sum_x x \mathbb{P}(X = x).
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The conditional expectation of $X$ given $Y = y$ is

$$\mathbb{E}(X \mid Y = y) = \sum_x x \mathbb{P}(X = x \mid Y = y).$$
Example

We flip a coin twice. \( X \deq \# \{ \text{heads in 1 flip} \} \)
\( Y \deq \# \{ \text{heads in 2 flips} \} \)

Get for all possible values of \( Y \):

\[
\mathbb{E}(X \mid Y = 0) = 0, \quad \mathbb{E}(X \mid Y = 1) = \frac{1}{2}, \quad \mathbb{E}(X \mid Y = 2) = 1.
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Upshot \( \mathbb{E}(X \mid Y) = \frac{Y}{2} = g(Y) \) expectation of \( X \) given \( Y \) is function of \( Y \)
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Nice properties

(i) \( \mathcal{H} \subset \mathcal{F} \implies \mathbb{E}(\mathbb{E}(X \mid \mathcal{F}) \mid \mathcal{H}) = \mathbb{E}(X \mid \mathcal{H}) \) (tower property)
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(ii) \( X \perp \mathcal{F} \implies \mathbb{E}(X \mid \mathcal{F}) = \mathbb{E}X \) (rôle of independance)
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Upshot $\mathbb{E}(X \mid Y) = \frac{Y}{2} = g(Y)$ expectation of $X$ given $Y$ is function of $Y$

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Martingales
Setting

\((\Omega, \mathcal{F}, (\mathcal{F}_n)_{n \geq 0}, \mathbb{P})\) filtered probability space, i.e. probability space and

\[ \mathcal{F}_n \subset \mathcal{F} \text{ sub } \sigma\text{-algebras, } \mathcal{F}_n \uparrow \text{ events observable up to time } n \]

We gamble with fixed bet and winning 1.

\[ \xi_j \in L^1 \text{ iid winning in the } n^{th} \text{ draw} \]

\[ S_n = \xi_1 + \ldots + \xi_n \text{ overall winning after } n \text{ draws} \]

\[ S_0 = 0 \]

\[ \mathcal{F}_n = \sigma(\xi_1, \ldots, \xi_n) \text{ information about all draws } 1, 2, \ldots, n \]

\[ = \sigma(S_1, \ldots, S_n) \]

\[ \mathbb{E}(S_{n+k} \mid \mathcal{F}_n) \text{ prediction of future winning, based on current information up to } n \]
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\( = \sigma(S_1, \ldots, S_n) \)

\[ \mathbb{E}(S_{n+k} \mid \mathcal{F}_n) \]

prediction of future winning, based on current information up to \(n\)

Question: What can we say about \(\mathbb{E}(S_{n+k} \mid \mathcal{F}_n)\)?
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Overall winning after \( n \) draws

Prediction of future winning, based on current information up to \( n \)

\[
\mathbb{E}(S_{n+k} \mid \mathcal{F}_n) = \mathbb{E}(S_n + \xi_{n+1} + \ldots + \xi_{n+k} \mid \mathcal{F}_n)
\]
\begin{align*}
\xi_j \in L^1 \text{ iid} \\
S_n &= \xi_1 + ... + \xi_n \\
S_0 &= 0 \\
\mathcal{F}_n &= \sigma(\xi_1, ..., \xi_n) \\
&= \sigma(S_1, ..., S_n) \\
\mathbb{E}(S_{n+k} \mid \mathcal{F}_n) &= \text{prediction of future winning, based on current information up to } n
\end{align*}

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\]

\[
\begin{cases}
= S_n & \mathbb{E}\xi_1 = 0 \quad \text{– fair game} \\
\geq S_n & \mathbb{E}\xi_1 \geq 0 \quad \text{– profitable game} \\
\leq S_n & \mathbb{E}\xi_1 \leq 0 \quad \text{– adverse game}
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\( \mathbb{E}(S_{n+k} \mid \mathcal{F}_n) \)

- winning in the \( n^{th} \) draw
- overall winning after \( n \) draws
- information about all draws 1, 2, \ldots, \( n \)
- prediction of future winning, based on current information up to \( n \)

\[
\mathbb{E}(S_{n+k} \mid \mathcal{F}_n) = \mathbb{E}( \underbrace{S_n + \xi_{n+1}}_{\in \mathcal{F}_n} + \ldots + \xi_{n+k} \mid \mathcal{F}_n ) \\
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Let’s vary the game and allow variable bets, i.e.

\[ e_{n+1} = e_{n+1}(\xi_1, ..., \xi_n) \geq 0 \]

bet in the \( n + 1 \text{th} \) draw.

Note \( e_{n+1} \) does not depend on \( \mathcal{F}_n \), but on \( \mathcal{F}_{n+1} \).
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**Note**  $e_{n+1}$ does not depend on $\mathcal{F}_n$, but on $\mathcal{F}_{n+1}$

$$\Rightarrow S_{n+1} = S_n + e_{n+1}\xi_{n+1}$$

$$\Rightarrow \mathbb{E}(S_{n+1} \mid \mathcal{F}_n) = S_n + \mathbb{E}(e_{n+1}\xi_{n+1} \mid \mathcal{F}_n)$$

$$= S_n + e_{n+1} \mathbb{E}(\xi_{n+1} \mid \mathcal{F}_n)$$

$$\geq 0$$

$$= \mathbb{E}\xi_{n+1}$$

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Interpretation even a good strategy does not change the fundamental character of our game
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**Interpretation**  even a good strategy does not change the fundamental character of our game
Question: What is a martingale?

(Heuristic) Answer: A martingale is the generalisation of the concept of a (fair) game.

**Definition**

Let $I$ be an index set (ordered) and $(\mathcal{F}_t)_{t \in I}$ a filtration. A stochastic process $(X_t)_{t \geq 0}$ satisfying, $\forall t \geq 0$,

(i) $X_t$ is $\mathcal{F}_t$-adapted, i.e. $X_t \in \mathcal{F}_t$ measurable;

(ii) $X_t \in L^1$, i.e. $\mathbb{E}|X| < \infty$;

(iii) $\mathbb{E}(X_t \mid \mathcal{F}_s) = X_s \forall s \leq t$.

is called a martingale.
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Aide-memoire  $(X_t)_t$ martingale $\implies \mathbb{E}(X_t \mid \mathcal{F}_s) = X_s$

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Example (Doubling strategy)

Let $X_1, ..., X_n$ be iid random variables such that

$$\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = \frac{1}{2}.\]

We think of $X_i$ describing the result of a coin flipping game:

- win $1\€$ if coin comes up heads
- lose $1\€$ if coin comes up tails

Consider **doubling strategy**, i.e. keep doubling the bet until we eventually win.

once we win, we stop and our initial bet is $1\€$. 
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Note that the size of bet on $n^{th}$ play is $2^{n-1}$ (assuming we still play at time $n$).

Denote $W_n$ total winnings after $n$ coin tosses assuming $W_0 = 0$. 

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Denote $W_n$ total winnings after $n$ coin tosses assuming $W_0 = 0$

Then $W_n$ is a martingale!
$W_n$ only takes two possible values, $W_n \in \{1, -2^n + 1\}$ ($n \in \mathbb{N}$). Why?

1. Suppose we win for the first time on the $n^{th}$ bet. Then

\[
W_n = -(1 + 2 + \ldots + 2^{n-2}) + 2^{n-1} \\
= -(2^{n-1} - 1) + 2^{n-1} \\
= 1
\]

2. If we have not yet won after $n$ bets, then

\[
W_n = -(1 + 2 + \ldots + 2^{n-1}) \\
= -2^n + 1.
\]

Show $\mathbb{E}(W_{n+1} \mid W_n) = W_n \implies \mathbb{E}(W_{n+m} \mid W_n) = W_n$
\( W_n \) only takes two possible values, \( W_n \in \{1, -2^n + 1\} (n \in \mathbb{N}) \). Why?

(1) Suppose we win for the first time on the \( n^{th} \) bet. Then

\[
W_n = -(1 + 2 + ... + 2^{n-2}) + 2^{n-1}
\]

= losses until \((n - 1)^{th}\) bet

\[
= -(2^{n-1} - 1) + 2^{n-1}
\]

= 1

(2) If we have not yet won after \( n \) bets, then

\[
W_n = -(1 + 2 + ... + 2^{n-1})
\]

= \(-2^n + 1\).

Show \( \mathbb{E}(W_{n+1} \mid W_n) = W_n \) \( \Rightarrow \mathbb{E}(W_{n+m} \mid W_n) = W_n \)
(1) \( W_n = 1 \): then \( \Pr(W_{n+1} = 1 \mid W_n = 1) = 1 \), so

\[
\mathbb{E}(W_{n+1} \mid W_n = 1) = 1 = W_n.
\]

(2) \( W_n = -2^n + 1 \): bet \( 2^n \) on \((n + 1)\)th toss, so \( W_{n+1} \in \{1, -2^{n+1} + 1\} \). Clearly,

\[
\Pr \left( W_{n+1} = 1 \mid W_n = -2^n + 1 \right) = \frac{1}{2},
\]

\[
\Pr \left( W_{n+1} = -2^{n+1} + 1 \mid W_n = -2^n + 1 \right) = \frac{1}{2}
\]

so that

\[
\mathbb{E}(W_{n+1} \mid W_n = -2^n + 1) = 1 \cdot \frac{1}{2} + (-2^{n+1} + 1) \frac{1}{2} = -2^n + 1 = W_n.
\]
Applications
• Doob’s optional stopping theorem → on average, nothing can be winninged by stopping play based on the information obtainable w/o looking into the future

• most essential continuous time process, Brownian Motion, is a martingale (zero drift stochastic process)

• martingales are essential to stochastic integration: $\mathbb{E}B_t^2 = t$, Doob-Meyer decomposition.

• solve PDEs by stochastic/martingale methods

• stochastic differential geometry, stochastic calculus on manifolds (and beyond)

• finance: martingality of an asset is equivalent to not being able to conduct arbitrage through trades in that asset

• ...
Most prominent example

**Example (Brownian motion)**

Let \((M, g)\) be a smooth Riemannian manifold, \(B = (B_t)\) Brownian motion on \(M\) and \(A = \frac{1}{2} \Delta_M\) with \(\Delta_M\) the Laplace-Beltrami operator on \(M\), i.e. \(\Delta = \sum_{i=1}^n \partial_i^2\) on \(\mathbb{R}^n\). Then,

\[
\forall f \in \mathcal{C}_c^\infty(\mathbb{R}^n), \ t \geq 0 : \ f(B_t) - f(B_0) - \int_0^t \frac{1}{2} \Delta f(B_s) ds
\]

is a martingale.
Most prominent example

Example (Brownian motion)

Let \((M, g)\) be a smooth Riemannian manifold, \(B = (B_t)\) Brownian motion on \(M\) and \(\mathbf{A} = \frac{1}{2} \Delta_M\) with \(\Delta_M\) the Laplace-Beltrami operator on \(M\), i.e. \(\Delta = \sum_{i=1}^{n} \partial_i^2\) on \(\mathbb{R}^n\). Then,

\[
\forall f \in C_0^\infty(\mathbb{R}^n), \ t \geq 0 : \quad f(B_t) - f(B_0) - \int_0^t \frac{1}{2} \Delta f(B_s) \, ds
\]

is a martingale.
If the weather forecast was a martingale, what would be the best prediction for tomorrow?
Merci pour votre attention !