T_5-CONFIGURATIONS AND NON-RIGID SETS OF MATRICES

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ABSTRACT. In 2003 B. Kirchheim-D. Preiss constructed a Lipschitz map in the plane with 5 incompatible gradients, where incompatibility refers to the condition that no two of the five matrices are rank-one connected. The construction is via the method of convex integration and relies on a detailed understanding of the rank-one geometry resulting from a specific set of five matrices. The full computation of the rank-one convex hull for this specific set was later carried out in 2010 by W. Pompe [Pom10] by delicate geometric arguments.

For more general sets of matrices a full computation of the rank-one convex hull is clearly out of reach. Therefore, in this short note we revisit the construction and propose a new, in some sense generic method for deciding whether convex integration for a given set of matrices can be carried out, which does not require the full computation of the rank-one convex hull.

1. INTRODUCTION

In this paper we consider differential inclusions of the type

$$Du(x) \in K \quad x \in \Omega, \tag{1}$$

where $K \subset \mathbb{R}^{n\times m}$ is a given compact set of matrices, $\Omega \subset \mathbb{R}^n$ is a bounded domain with Lipschitz boundary, and $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ is a Lipschitz mapping. Being Lipschitz, by Rademacher's theorem u is differentiable almost everywhere and hence (8) makes sense almost everywhere.

Following [Kir01, Kir03] we call a compact set $K \subset \mathbb{R}^{m\times n}$ *non-rigid*, if the differential inclusion (8) admits non-affine Lipschitz solutions. It is clear that this definition is independent of the choice of Ω. It is moreover well known that if $A, B \in K$ with $\mathrm{rank}(A-B) = 1$, then there exists non-affine solutions of (8); these have locally the form $u(x) = Cx + ah(x \cdot \xi)$, where $A - B = a \otimes \xi$, $C \in \mathbb{R}^{m\times n}$ and $h : \mathbb{R} \to \mathbb{R}$. Such pairs of matrices are called rank-one connections. The more interesting question is to characterize non-rigid sets K which do not contain rank-one connections.

Such problems have received considerable attention in the last couple of decades, in part due to the relevance to problems in non-linear elasticity, but also due to applications of the method of construction to various systems of partial differential equations [KŠM03, MŠ03, SJ04b, AFSJ08, PD05, Zha06, DLSJ09, CFG11, Shv11, SJ12]. In analogy with the well-understood one-dimensional case [Cel05, BF94], a general method for constructing solutions is to consider the relaxation of the problem (8), and then to conclude that *typical* solutions of the relaxed problem (in a suitable topology) are in fact solutions of the original problem. For the higher dimensional case $m, n \geq 2$ there are two difficulties with this strategy, which need to be overcome:

(a) First, at variance with the one-dimensional case the relaxation is in general not given by the convex hull K^{co}, but could be potentially much smaller.

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(b) Second, the iteration for obtaining solutions from relaxed solutions requires suitable modifications.

Concerning (b) there are by now several ways in which the iteration can be carried out; either by a Baire category argument [Kir01, DM97], or by an explicit construction, known as convex integration [MŠ03]; we refer to the lecture notes [SJ14] for a general discussion and comparison of these techniques. The common denominator in these methods is that one needs to find a suitable open (or in case of constraints relatively open) subset $U \subset \mathbb{R}^{m\times n}$ and define approximate solutions of (8) as solutions the corresponding inclusion

$$Du(x) \in U \quad \text{a.e. } x \in \Omega. \tag{2}$$

In general the properties required on U will imply that U is a subset of the rank-one convex hull K^{rc} (for definitions see Section 2.1 below), but the specific requirements vary from approach to approach. Then, in each particular example of a differential inclusion, one has to construct such a set U.

In this paper we are interested in the stability properties of such a construction. Recall that the map $K \mapsto K^{rc}$ is upper semicontinuous, but in general not lower semicontinuous [Kir03, p.80]. In [Kir01] Kirchheim gave a generic construction of a finite set K without rank-one connections for which the corresponding inclusion (8) admits non-affine solutions and moreover K is stable in the sense that small perturbations of K still have the same property. These sets are finite, but the number of matrices is quite large as the set K is obtained via a compactness argument. On the other hand it is known that the number of matrices in a non-rigid set without rank-one connections can be quite small: an example of Kirchheim and Preiss [Kir03, p.100] shows that 5 matrices suffice (moreover, in [CK02] it was shown that 4 matrices do not suffice, so that 5 is the minimal number). The example of Kirchheim-Preiss is the following: Let $K = \{X_1, \dots, X_5\}$ with

$$\begin{aligned} X_1 &= \begin{pmatrix} \sqrt{3} & -2 \\ -2 & \sqrt{3} \end{pmatrix}, X_2 = \begin{pmatrix} \sqrt{3} & 2 \\ 2 & \sqrt{3} \end{pmatrix}, X_3 = \begin{pmatrix} -\sqrt{3}+2 & 0 \\ 0 & -\sqrt{3}-2 \end{pmatrix}, \\ X_4 &= \begin{pmatrix} -\sqrt{3}-2 & 0 \\ 0 & -\sqrt{3}+2 \end{pmatrix}, X_5 = \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{3}{4} \end{pmatrix}. \end{aligned} \tag{3}$$

Observe that $K \subset \mathbb{R}^{2\times 2}_{sym}$, the space of 2×2 symmetric matrices. Furthermore, it is easy to check that K contains no rank-one connections. The statement in [Kir03, p.100] is the following:

Theorem 1.1. There exists a relatively open subset $U \subset \mathbb{R}^{2\times 2}_{sym}$ such that for any $F \in U$ there exists a Lipschitz map $u : \Omega \to \mathbb{R}^2$ satisfying

$$\begin{aligned} Du &\in K \quad \text{a.e. } x \in \Omega \\ u(x) &= Fx \quad x \in \partial\Omega. \end{aligned} \tag{4}$$

Moreover, there exists $\varepsilon > 0$ such that for any $\tilde X_i \in \mathbb{R}^{2\times 2}_{sym}$ with $|X_i - \tilde X_i| < \varepsilon$, $i = 1, \dots, 5$, the set $\tilde K = \{\tilde X_1, \dots, \tilde X_5\}$ has the same property (with some perturbed subset $\tilde U$).

From this statement it follows immediately that K (and any small perturbation $\tilde K$ in symmetric 2×2 matrices) is non-rigid. The proof of existence of the set U in Theorem 1.1 is based on an explicit geometric construction. Subsequently, W. Pompe calculated in [Pom10] the full rank-one convex hull K^{rc} (and even showed that this agrees with the quasiconvex hull K^{qc}), and that one can take $U = \text{rel int } K^{rc}$, the topological interior of K^{rc} relative in $\mathbb{R}^{2\times 2}_{sym}$.

The aim of this paper is to give a new and in some sense more systematic proof of Theorem 1.1 for five-point sets K as in (3), which moreover shows the stability in the full space $\mathbb{R}^{2\times 2}$. Noting that generic 5-point configurations in $\mathbb{R}^{2\times 2}$ do not lie in any 3-dimensional subspace, this shows that non-rigid sets with minimal number of elements are stable with respect to generic perturbations. A further advantage of our characterization of non-rigid 5-element sets is that it allows for an algebraic criterion (see Theorem 2.3 below) which can be easily implemented numerically without having to compute the rank-one convex hull.

Our main theorem can be stated as follows:

Theorem 1.2. Let $K = \{X_1, \ldots, X_5\} \subset \mathbb{R}^{2\times 2}$ be a *large* T_5 *set*. Then K is non-rigid.

The definition of *large* T_5 *set* will be given below in Definition 2.6. It follows from Lemma 2.4 below that the property to be a *large* T_5 *set* is stable with respect to generic perturbations.

As explained above, the property of a set K to be non-rigid depends on certain properties of the rank-one convex hull of K^{rc}. In this paper we will adopt the approach of [MŠ99, MŠ03] and use the notion of *in-approximation* of K. Since 5-point sets in the space $\mathbb{R}^{2\times 2}$ lie generically in a constrained set given by the determinant (see Lemma 2.5 for the precise statement), we recall the version of convex integration applicable for constraints from [MŠ99]. In what follows, $\Omega \subset \mathbb{R}^2$ is a bounded domain and $\Sigma \subset \mathbb{R}^{2\times 2}$ denotes either the set of matrices

$$\Sigma = \{X \in \mathbb{R}^{2\times 2} : \det X = 1\} \text{ or } \Sigma = \{X \in \mathbb{R}^{2\times 2} : X \text{ is symmetric}\}.$$

The relevant definition and corresponding theorem, specialized to our situation, is as follows:

Definition 1.3. Let $K \subset \Sigma$ compact. We call a sequence of relatively open sets $\{U_k\}_{k=1}^{\infty}$ in Σ an in-approximation of K if

- $U_k \subset U_{k+1}^{rc}$ for all i;
- $\sup_{X \in U_k} \text{dist}(X, K) \to 0$ as $k \to \infty$.

Theorem 1.4 ([MŠ99])**.** Let $K \subset \Sigma$ be a compact set and suppose $\{U_k\}_{k=1}^{\infty}$ is an in-approximation of K. Then for each piecewise affine Lipschitz map $v : \Omega \to \mathbb{R}^2$ with $Dv(x) \in U_1$ in Ω there exists a Lipschitz map $u : \Omega \to \mathbb{R}^2$ satisfying

$$\begin{aligned} Du(x) &\in K \quad \text{a.e. in} \quad \Omega, \\ u(x) &= v(x) \quad \text{on} \quad \partial\Omega. \end{aligned}$$

In the statement of the theorem above we have included the case when Σ is the set of 2×2 symmetric matrices. Whilst this case[1] is not included in [MŠ99], it was treated in [Kir03] Proposition 3.4 and Theorem 3.5. With this result at hand, the proof of Theorem 1.2 reduces to showing that any large T_5 set admits an in-approximation. This is the content of Theorem 2.8 below.

[1] In some sense this case can be seen as a limiting case from $\Sigma_t = \{X : \det X = t\}$ with $t \to \infty$, see the proof of Lemma 2.5 below.

2. T_N-Configurations

2.1. **Definitions.** A function $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ is said to be *rank-one convex* if for any $A, B \in \mathbb{R}^{m\times n}$ with rank $B = 1$ the restriction $t \mapsto f(A + tB)$ is convex. For a compact set $K \subset \mathbb{R}^{m\times n}$ the rank-one convex hull is defined as

$$K^{rc} = \left\{ A \in \mathbb{R}^{m\times n} : f(A) \leq \sup_{X\in K} f(X) \text{ for all rank-one convex } f : \mathbb{R}^{m\times n} \to \mathbb{R} \right\}.$$

It is easy to see that rank-one convexity is invariant under linear transformations of the form

$$X \mapsto PXQ + B, \tag{5}$$

where P, Q are invertible $m \times m$ and $n \times n$ matrices respecively, and $B \in \mathbb{R}^{m\times n}$. In particular, if $PKQ + B = \{PXQ + B : X \in K\}$ then $(PKQ + B)^{rc} = PK^{rc}Q + B$.

For a square matrix X we denote by cof X the cofactor matrix, and by $\langle X, Y\rangle :=$ tr $(X^T Y)$ the natural scalar product of matrices. In particular, for 2×2 matrices we have $\det X = \frac{1}{2}\langle \text{cof } X, X\rangle$.

We denote by $\{X_1, \ldots, X_N\}$ the unordered set of matrices $X_i, i = 1, \ldots, N$ and by $(X_1, \ldots, X_N)$ the ordered N-tuple.

Definition 2.1 (T_N-configuration)**.** Let $X_1, \ldots, X_N \in \mathbb{R}^{m\times n}$ be N matrices such that rank $(X_i - X_j) > 1$ for all $i \neq j$. The ordered set $(X_1, \ldots, X_N)$ is said to be a T_N *configuration* if there exist $P, C_i \in \mathbb{R}^{m\times n}$ and $\kappa_i > 1$ such that

$$\begin{aligned} X_1 &= P + \kappa_1 C_1 \\ X_2 &= P + C_1 + \kappa_2 C_2 \\ &\vdots \\ X_N &= P + C_1 + \ldots + C_{N-1} + \kappa_N C_N, \end{aligned} \tag{6}$$

and furthermore rank $(C_i) = 1$ and $\sum_{i=1}^{N} C_i = 0$.

Note that it is certainly possible for a fixed set of N matrices $\{X_1, \ldots, X_N\}$ to lead to several T_N-configurations corresponding to different orderings. The significance of T_N-configurations is given by the following well-known lemma (see for instance [MŠ03, Tar93]):

Lemma 2.2. Suppose $(X_i)_{i=1}^N$ is a T_N-configuration. Then

$$\{P_1, \ldots, P_N\} \subset \{X_1, \ldots, X_N\}^{rc},$$

where $P_1 = P$ and $P_i = P + \sum_{j=1}^{i-1} C_j$ for $i = 2, \ldots, N$.

A direct consequence is that the rank-one segments

$$\{P_i + tC_i | 0 \leq t \leq \kappa_i\}$$

are also contained in $\{X_1, \ldots, X_N\}^{rc}$.

Although Definition 2.1 gives no easy way to decide whether a given ordered N-tuple is a T_N-configuration, we recall the following characterization from [SJ05]:

Theorem 2.3 (Algebraic criterion)**.** Suppose $(X_1, \dots, X_N) \in (\mathbb{R}^{2\times 2})^N$ and let $A \in \mathbb{R}^{N\times N}$ with $A_{ij} = \det(X_i - X_j)$. Then $(X_1, \dots, X_N)$ is a T_N-configuration if and only if there exist $\lambda_1, \dots, \lambda_N > 0$ and $\mu > 1$ such that $A^\mu \lambda = 0$.

Here, for $\mu \in \mathbb{R}$ and $A \in \mathbb{R}^{N\times N}_{sym}$ with $A_{ii} = 0 \quad \forall \quad i = 1, \dots, N$, we define

$$A^\mu = \begin{pmatrix} 0 & A_{12} & A_{13} & \dots & A_{1N} \\ \mu A_{12} & 0 & A_{23} & \dots & A_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu A_{1N} & \mu A_{2N} & \mu A_{3N} & \dots & 0 \end{pmatrix}. \tag{7}$$

In fact, from μ and $\lambda = (\lambda_1, \dots, \lambda_N)$ we can easily compute the parametrization (P, C_i, κ_i) of the T_N-configuration $(X_1, \dots, X_N)$. In particular, recalling the definition of P_i from Lemma 2.2, we have (see [SJ05]):

$$\begin{aligned} P_1 &= \frac{1}{\lambda_1 + \dots + \lambda_N} (\lambda_1 X_1 + \dots + \lambda_N X_N) \\ P_2 &= \frac{1}{\mu\lambda_1 + \lambda_2 + \dots + \lambda_N} (\mu\lambda_1 X_1 + \lambda_2 X_2 + \dots + \lambda_N X_N) \\ &\vdots \\ P_N &= \frac{1}{\mu\lambda_1 + \dots + \mu\lambda_{N-1} + \lambda_N} (\mu\lambda_1 X_1 + \dots + \mu\lambda_{N-1} X_{N-1} + \lambda_N X_N) \end{aligned} \tag{8}$$

2.2. **Stability.** Now we consider the question how T_5 configurations in the $\mathbb{R}^{2\times 2}$ behave with respect to small perturbations. Similar problems have been considered in [MŠ03] (T_4-configurations in $\mathbb{R}^{4\times 2}$), [Kir03] (T_4-configurations in $\mathbb{R}^{2\times 2}$) and [SJ04a] (T_5-configurations in $\mathbb{R}^{4\times 2}$). Whilst a simple dimension-count (as in [MŠ03, Kir03, SJ04a]) shows that *generic* T_5-configurations (in the sense of generic choices of P, C_i, κ_i in the parametrization (6)) are stable with respect to small perturbations in $\mathbb{R}^{2\times 2}$, the argument below shows that they are *always* stable.

Lemma 2.4. Let $(X_1, \dots, X_5)$ be a T_5-configuration in $\mathbb{R}^{2\times 2}$ with $\det(X_i - X_j) \neq 0$ for all $i \neq j$. Then there exists $\varepsilon > 0$ so that any $(\tilde{X}_1, \dots, \tilde{X}_5)$ with $|\tilde{X}_i - X_i| < \varepsilon$, $i = 1 \dots 5$, is also a T_5-configuration.

Proof. Let $A = (\det(X_i - X_j))_{i,j=1\dots 5}$ and A^μ be defined as in (7). Since the first column of A^μ contains μ as a factor, it is clear that $\det A^\mu|_{\mu=0} = 0$. Moreover, since $(A^\mu)^T = \mu A^{\mu^{-1}}$, we have that $\det A^\mu = \mu^5 \det(A^{\mu^{-1}})$. This shows that $\det A^\mu|_{\mu=-1} = 0$. Since $\mu \mapsto \det A^\mu$ is a polynomial of degree 4, we deduce

$$\begin{aligned} \det A^\mu &= \mu(\mu+1)(a + b\mu + a\mu^2) \\ &= a\mu(\mu+1)(\mu - \mu^*)(\mu - \frac{1}{\mu^*}) \end{aligned}$$

for some $a, b \in \mathbb{R}$ and $\mu^* \in \mathbb{C}$. Furthermore, using Theorem 2.3, since we assume that $(X_1, \dots, X_5)$ is a T_5-configuration, we have that $\mu^* > 1$ and there exists $\lambda^* \in \mathbb{R}^5$ with $\lambda_i^* > 0$ for all $i = 1 \dots 5$ such that $A^{\mu^*} \lambda^* = 0$.

Next, observe that μ^* is a root of $\mu \mapsto \det A^\mu$ with multiplicity 1, hence

$$0 \neq \frac{d}{d\mu}\bigg|_{\mu=\mu^*} \det A^\mu = \left\langle \text{cof } (A^{\mu_*}), \frac{d}{d\mu}\bigg|_{\mu=\mu_*} A^\mu \right\rangle$$

whereas clearly

$$\left(\frac{d}{d\mu}A^{\mu}\right)_{ij}=\begin{cases}\det(X_i-X_j) & i<j\,,\\ 0 & i\geq j\,.\end{cases}$$

In particular this implies that adj $(A^{\mu_*})\neq 0$, so that rank $(A^{\mu^*})=4$. Consequently the map

$$A\mapsto(\mu,\lambda)$$

defined by the equations $\det A^{\mu}=0$ and $A^{\mu}\lambda=0$ is continuous (hence smooth, being a polynomial) in a neighbourhood of (μ^*,λ^*). But then it easily follows that for all $(\tilde X_1,\dots,\tilde X_5)$ with $|\tilde X_i-X_i|$ sufficiently small the corresponding matrix $\tilde A$ admits a solution $\tilde\mu>1$ and $\tilde\lambda$ with $\tilde\lambda_i>0$, $i=1\dots 5$. □

We summarize: T_5 configurations are stable with respect to small perturbations, and in particular there exists a smooth map

$$(X_1,\dots,X_5)\mapsto(P_1,\dots,P_5)$$

in a neighbourhood of any fixed T_5-configuration, which maps nearby (ordered) 5-tuples to the associated points in Lemma 2.2 and (8).

It was noted in [SJ04a] (see Figure 2.2) that the set $K=\{X_1,\dots,X_5\}$ in (3) corresponds to 12 different T_5 configurations, associated to the orderings

$$[1,2,3,5,4],\ [1,2,4,5,3],\ [1,2,5,3,4],\ [1,2,5,4,3]$$
$$[1,3,2,5,4],\ [1,3,5,4,2],\ [1,4,2,5,3],\ [1,4,5,3,2]$$
$$[1,5,3,2,4],\ [1,5,3,4,2],\ [1,5,4,2,3],\ [1,5,4,3,2].$$

Then, according to Lemma 2.4 each of these orderings leads to a T_5-configuration for small perturbations $\{\tilde X_1,\dots,\tilde X_5\}$ in the *full space* $\mathbb{R}^{2\times 2}$. Now, generic 5-point sets in $\mathbb{R}^{2\times 2}$ need not satisfy any affine constraint, but they nevertheless satisfy a *polyaffine* constraint; this is the content of the following lemma:

Lemma 2.5. Let $(X_1,\dots,X_5)$ be a T_5-configuration in $\mathbb{R}^{2\times 2}$. Then there exist invertible matrices $P,Q\in\mathbb{R}^{2\times 2}$ and a matrix $B\in\mathbb{R}^{2\times 2}$ such that one of the following holds for the transformed 5-tuple $(Y_1,\dots Y_5)$, where $Y_i=PX_iQ+B$:

(i) Y_i is symmetric for all i; or
(ii) $\det(Y_i)=1$ for all i.

Proof. **Step 1.** Let $z_i=(X_i,\det X_i)\in\mathbb{R}^{2\times 2}\times\mathbb{R}$, $i=1\dots 5$. If the vectors $z_1,\dots,z_5$ are linearly independent, there exists $F\in\mathbb{R}^{2\times 2}$ and $f\in\mathbb{R}$ such that

$$\langle F,X_i\rangle+f\det X_i=1\qquad\text{for all } i=1\dots 5.$$

On the other hand if the vectors $z_1,\dots,z_5$ are linearly dependent, then there exists $F\in\mathbb{R}^{2\times 2}$ and $f\in\mathbb{R}$ such that $(F,f)\neq(0,0)$ and

$$\langle F,X_i\rangle+f\det X_i=0\qquad\text{for all } i=1\dots 5.$$

In either case there exist a nontrivial pair $(F,f)\in\mathbb{R}^{2\times 2}\times\mathbb{R}$ such that

$$\langle F,X_i\rangle+f\det X_i=\alpha\qquad\text{for all } i=1\dots 5\tag{9}$$

for some $\alpha\in\mathbb{R}$.

Step 2. Suppose $f=0$. Then $\tilde X_i:=X_i-\alpha\frac{F}{|F|^2}$ satisfies $\langle F,\tilde X_i\rangle=0$ for all i. Assume for a contradiction that $\det F=0$, so that $F=\eta\otimes\xi$ for some nonzero $\eta,\xi\in\mathbb{R}^2$. By choosing suitable invertible matrices P,Q we deduce that $Y_i=P\tilde X_iQ$ satisfies

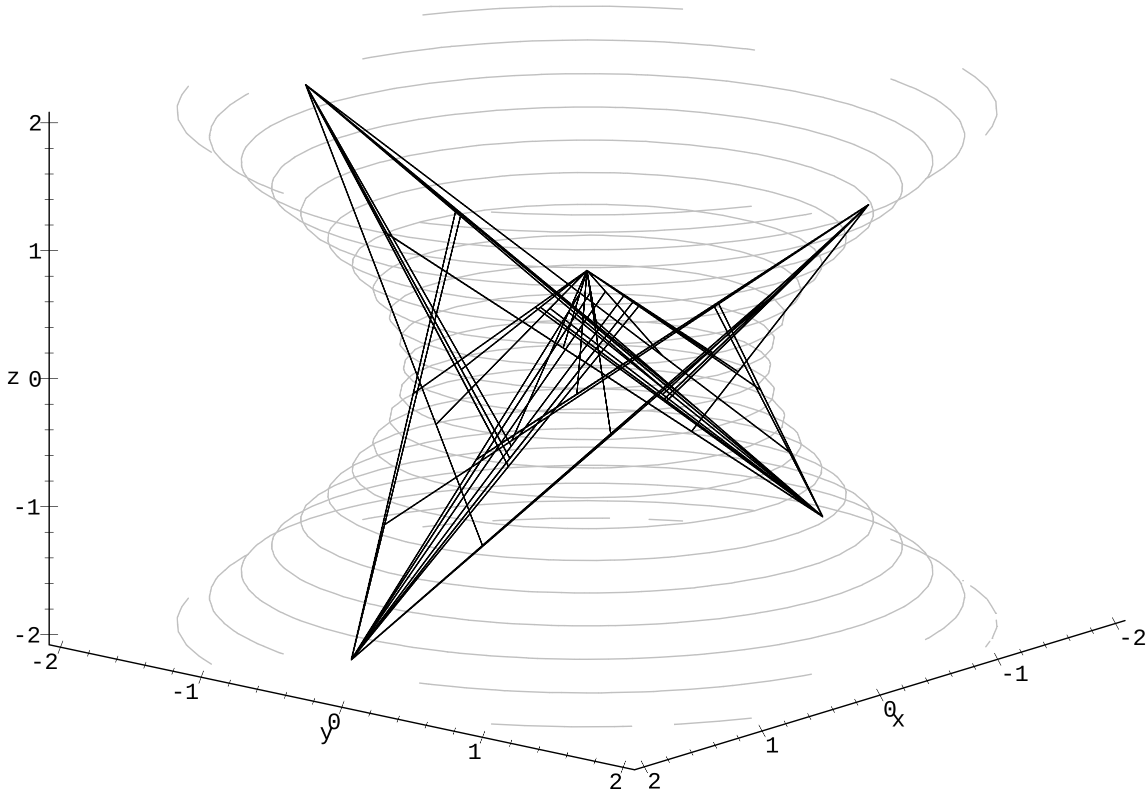

FIGURE 1. The plot from [SJ04a] showing the 12 different T_5 configurations associated to the set $\{X_1, \ldots, X_5\}$ in (3). The one-sheeted hyperboloid corresponding to $\{\det = -1\}$ is shown in grey.

$\langle Y_i, e_1 \otimes e_2 \rangle = 0$ for all i, in other words Y_i is lower-triangular. Let $\tilde{Y}_i$ be the projection of Y_i onto the diagonal. Then $\det(\tilde{Y}_i - \tilde{Y}_j) = \det(Y_i - Y_j) = c \det(X_i - X_j)$ with $c = \det(PQ) \neq 0$, so that, since $(X_1, \ldots, X_5)$ is a T_5-configuration, so is $(\tilde{Y}_1, \ldots, \tilde{Y}_5)$. However, in the diagonal plane there exist no T_5 configurations; Indeed, if $\tilde{C}_i$ are the corresponding rank-one vectors, the condition $\det(\tilde{Y}_i - \tilde{Y}_j) \neq 0$ require that $\tilde{C}_i$ is not parallel to $\tilde{C}_{i+1}$ (with $\tilde{C}_6 = \tilde{C}_1$). However, in the diagonal plane there are only two rank-one directions, making this requirement an impossibility.

We conclude that $\det F \neq 0$. But then setting $P = F^{-T} J$ with

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and $Y_i = P\tilde{X}_i$ leads to the equality $\langle J, Y_i \rangle = 0$, therefore Y_i is symmetric.

Step 3. Now suppose that $f \neq 0$. Then without loss of generality we may assume that (9) is satisfied with $f = 1$. Let $B \in \mathbb{R}^{2\times 2}$ such that $\operatorname{cof} B = -F$ (since for 2×2 matrices $\operatorname{cof} \operatorname{cof} B = B$, we can simply take $B = -\operatorname{cof} F$) and set $\tilde{X}_i = X_i - B$.

Then

$$\begin{aligned}\det \tilde{X}_i &= \det X_i - \langle \text{cof } B, X_i\rangle + \det B\\ &= \alpha - \langle \text{cof } B + F, X_i\rangle + \det B\\ &= \alpha + \det B =: \beta.\end{aligned}$$

Assume for a contradiction that $\beta = 0$. Then $\det(X_i - X_j) = -\langle \text{cof } (\tilde{X}_i), \tilde{X}_j\rangle$. Let $v \in \mathbb{R}^5$ a nonzero vector such that $\sum_{i=1}^5 v_i \tilde{X}_i = 0$ (such a vector exists since $\tilde{X}_i \in \mathbb{R}^{2\times 2}$). Then

$$\sum_{i=1}^{5} v_j \langle \text{cof } (\tilde{X}_i), \tilde{X}_j\rangle = 0 \quad \text{for all } i = 1\dots 5,$$

hence $Av = 0$, where A is as in Theorem 2.3. But as shown in Lemma 2.4, $\mu = 1$ cannot be a zero of the polynomial $\mu \mapsto \det A^\mu$ if A corresponds to a T_5 configuration, a contradiction. We conclude that $\beta \neq 0$. We can then easily choose P so that $Y_i = P\tilde{X}_i$ satisfies $\det Y_i = 1$ for all i. □

We recall that if $K \subset \mathbb{R}^{2\times 2}$ is a compact set such that $K \subset \{\det X = 1\}$, then also K^{rc} (in fact also K^{pc}, the polyconvex hull) is contained in the set $\{\det X = 1\}$. The preceeding lemma therefore implies that in general the rank-one convex hull of T_5-configurations is contained – possibly after performing the transformations $X \mapsto PXQ + B$ – in the subspace of symmetric matrices, or in the 3-dimensional manifold $\{X : \det X = 1\}$.

2.3. **Construction of an in-approximation.** We will use this stability theorem to build an in-approximation for a large T_5-configuration. As shown by the example (3), a 5-point set may give rise to several different T_5-configurations, corresponding to different orderings of the set. In order to analyse such situations, let $\{X_1^0, \dots, X_5^0\}$ be a 5-element set and let S_5 be the permutation group of 5 elements. To any $\sigma \in S_5$ is associated a 5-tuple $(X^0_{\sigma(1)} \dots, X^0_{\sigma(5)})$. If this 5-tuple is a T_5-configuration, then according to Lemma 2.4 there exists a smooth map

$$(X_{\sigma(1)}, \dots, X_{\sigma(5)}) \mapsto (P^\sigma_{\sigma(1)}, \dots, P^\sigma_{\sigma(5)})$$

defined in a neighbourhood of $(X^0_{\sigma(1)}, \dots, X^0_{\sigma(5)})$, where $P^\sigma_{\sigma(i)}$ are the corresponding matrices from Lemma 2.2, so that in particular

$$\text{rank } (P^\sigma_{\sigma(i)} - X_{\sigma(i)}) = 1 \text{ and } P^\sigma_{\sigma(i)} \in \{X_1, \dots, X_5\}^{rc}.$$

Let

$$C_i^\sigma := P_i^\sigma - X_i \tag{10}$$

and define the map $\Phi^\sigma : B_r(X^0) \to (\mathbb{R}^{2\times 2})^5$ by

$$\Phi^\sigma(X) = (C_1^\sigma, \dots, C_5^\sigma), \tag{11}$$

where we write $X^0 = (X_1^0, \dots, X_5^0)$ and $X = (X_1, \dots, X_5)$. By the preceeding discussion we see that, provided σ leads to a T_5-configuration $(X^0_{\sigma(1)} \dots, X^0_{\sigma(5)})$, the map Φ^σ is a well-defined and smooth map in a neighbourhood $B_r(X^0)$ for some $r > 0$.

Definition 2.6. We call a five-point set $\{X_1^0, \ldots, X_5^0\} \subset (\mathbb{R}^{2\times 2})^5$ a *large T_5-set* if there exist at least three permutations $\sigma_1, \sigma_2, \sigma_3$ such that $(X^0_{\sigma_j(1)}, \ldots, X^0_{\sigma_j(5)})$ is a T_5-configuration for each $j = 1, 2, 3$, and moreover the associated rank-one matrices $C_i^{\sigma_1}, C_i^{\sigma_2}, C_i^{\sigma_3}$ are linearly independent for all $i = 1, \ldots, 5$.

In view of the stability result Lemma 2.4 we immediately see that large T_5 sets are stable with respect to small perturbations. Moreover, by Lemma 2.5 each large T_5 set is contained in a 3-dimensional subset Σ, where –modulo a linear transformation of the form (5) – either $\Sigma = \{X : \det X = 1\}$ or $\Sigma = \mathbb{R}^{2\times 2}_{sym}$. Finally, it is not difficult to check directly that the set from (3) is a large T_5 set.

The aim of the following theorem is to construct a stable parametrization of the rank-one convex hull of a large T_5 set.

Proposition 2.7. Let $K = \{X_1^0, \ldots, X_5^0\}$ be a large T_5 set and set $X^0 := (X_1^0, \ldots, X_5^0) \in (\mathbb{R}^{2\times 2})^5$. Then there exists $\delta > 0$ and for each $i = 1, \ldots, 5$ smooth maps

$$p_i : (-\delta, \delta)^3 \times B_\delta(X^0) \to \mathbb{R}^{2\times 2},$$

with the following properties:

(a) the map $\xi \mapsto p_i(\xi, X)$ is an embedding for each X;
(b) $p_i(\xi, X) \in \{X_1, \ldots, X_5\}^{rc}$ for all $\xi \in [0, \delta)^3$;
(c) $p_i(0, X) = X_i$.

Proof. By the discussion preceeding Definition 2.6 there exists $r > 0$ and smooth maps

$$\Phi^{\sigma_j} : B_r(X^0) \to (\mathbb{R}^{2\times 2})^5 \qquad j = 1, 2, 3$$

such that, writing $C_i^{\sigma_j}(X) := \Phi_i^{\sigma_j}(X)$ we have rank $\Phi_i^{\sigma_j}(X) = 1$ and

$$X_i + t\Phi_i^{\sigma_j}(X) \in \{X_1, \ldots, X_5\}^{rc} \quad \text{ for all } t \in [0, 1]$$

for any $X \in B_r(X^0)$ and $i = 1 \ldots 5$.

We fix without loss of generality $i = 1$ and define p_1 as follows. Let $X \in B_{r/8}(X^0)$. For $\xi_1 \in (-r_1, r_1)$, with $r_1 > 0$ to be fixed, define $X^{\sigma_1}(\xi_1)$ to be the 5-tuple

$$X^{\sigma_1}(\xi_1) := (X_1 + \xi_1\Phi_1^{\sigma_1}(X), X_2, \ldots, X_5)\,.$$

Observe that the map

$$(X, \xi_1) \mapsto X^{\sigma_1}(\xi_1)$$

is well-defined and smooth for $(X, \xi_1) \in B_{r/8}(X^0) \times \mathbb{R}$ with $X^{\sigma_1}(0) = X$. Moreover, by the construction of Φ^{σ_1} we have

$$X_1 + \xi_1\Phi_1^{\sigma_1}(X) \in \{X_1, \ldots, X_5\}^{rc} \quad \text{ for all } \xi_1 \in [0, 1]. \tag{12}$$

Fix $r_1 > 0$ so that

$$X^{\sigma_1}(\xi_1) \in B_{r/4}(X^0) \quad \text{ for all } (X, \xi_1) \in B_{r/8}(X^0) \times (-r_1, r_1).$$

Next, for $(\xi_1, \xi_2) \in (-r_1, r_1) \times (-r_2, r_2)$ with $r_2 < r_1$ define

$$X^{\sigma_1\sigma_2}(\xi_1, \xi_2) := (X_1 + \xi_1\Phi_1^{\sigma_1}(X) + \xi_2\Phi_1^{\sigma_2}(X^{\sigma_1}(\xi_1)), X_2, \ldots, X_5)\,.$$

As before, the map

$$(X, \xi_1, \xi_2) \mapsto X^{\sigma_1\sigma_2}(\xi_1, \xi_2)$$

is well-defined and smooth for $(X, \xi_1, \xi_2) \in B_{r/8}(X^0) \times (-r_1, r_1) \times \mathbb{R}$ with $X^{\sigma_1\sigma_2}(\xi_1, 0) = X^{\sigma_1}(\xi_1)$. Consequently we can choose $r_2 > 0$ sufficiently small so that

$$X^{\sigma_1\sigma_2}(\xi_1, \xi_2) \in B_{r/2}(X^0) \quad \text{ for all } (X, \xi_1, \xi_2) \in B_{r/8}(X^0) \times (-r_1, r_1) \times (-r_2, r_2).$$

Furthermore, by the construction of Φ^{σ_2} we have

$$X_1 + \xi_1 \Phi_1^{\sigma_1}(X) + \xi_2 \Phi_1^{\sigma_2}(X^{\sigma_1}(\xi_1)) \in \{X_1 + \xi_1 \Phi_1^{\sigma_1}(X), X_2, \ldots, X_5\}^{rc}$$

for all $\xi_2 \in [0,1]$. In combination with (12) this leads to

$$X_1 + \xi_1 \Phi_1^{\sigma_1}(X) + \xi_2 \Phi_1^{\sigma_2}(X^{\sigma_1}(\xi_1)) \in \{X_1, \ldots, X_5\}^{rc} \quad \text{for all } \xi_1, \xi_2 \in [0, r_2]. \tag{13}$$

Finally, we define $p_1(\xi, X)$ for $X \in B_{r/8}(X^0)$ and $\xi = (\xi_1, \xi_2, \xi_3)$ as

$$p_1(\xi, X) := X_1 + \xi_1 \Phi_1^{\sigma_1}(X) + \xi_2 \Phi_1^{\sigma_2}(X^{\sigma_1}(\xi_1)) + \xi_3 \Phi_1^{\sigma_3}(X^{\sigma_1\sigma_2}(\xi_1, \xi_2)). \tag{14}$$

Then p_1 is well-defined and smooth for $(\xi, X) \in (-r_1, r_1) \times (-r_2, r_2) \times \mathbb{R} \times B_{r/8}(X^0)$ and clearly $p_1(0, X) = X_1$. By the construction of Φ^{σ_3} we have, as before,

$$p_1(\xi, X) \in \{X_1, \ldots, X_5\}^{rc} \quad \text{for all } \xi \in [0, r_2]^3. \tag{15}$$

Next, observe that

$$\frac{\partial}{\partial \xi_j}\Big|_{\xi=0} p_1(\xi, X^0) = \Phi_1^{\sigma_j}(X^0),$$

so that, by the assumption that $\{X_1^0, \ldots, X_5^0\}$ is a large T_5-set, $\partial_\xi p_1(0, X^0)$ has full rank. Consequently, by the implicit function theorem the map

$$\xi \mapsto p_1(\xi, X)$$

is a local embedding near $\xi = 0$ for any X with $|X - X^0|$ sufficiently small.

In summary, we can choose $\delta > 0$ sufficiently small so that the properties (a)-(c) hold for the map p_1. The construction of $p_2, \ldots, p_5$ is entirely analogous. This concludes the proof.

□

Now we are ready to construct an in-approximation of a large T_5 set.

Theorem 2.8. Let $K = \{X_1^0 \ldots, X_5^0\}$ be a large T_5 set. Then there exists an in-approximation $(U_n)_{n\in\mathbb{N}}$ of K.

Proof. Let Σ be the associated constraint set from Lemma 2.5, so that $K \subset \Sigma$ and – without loss of generality – either $\Sigma = \{X : \det X = 1\}$ or $\Sigma = \{X : X^T = X\}$. Define for all $i = 1, \ldots, 5$ and $X \in B_\delta(X^0)$ the sets

$$V_i(X) := \{p_i(\xi, X) | \xi \in (0, \delta)^3\}.$$

Recall from Proposition 2.7 that $V_i(X)$ is relatively open in Σ such that

$$V_i(X) \subset K^{rc}$$

and moreover $V_i(X) \to V_i(X^0)$ if $X \to X^0$.

We construct successively a sequence of 5-tuples

$$X^{(n)} = (X_1^{(n)}, \ldots, X_5^{(n)})$$

and radii $0 < r_n < 1/n$ with the following properties: for all $n = 1, 2, \ldots$

(a) $X_i^{(n)} \in V_i(X^0) \cap B_{1/n}(X_i^0)$;
(b) $V_i(X^{(n+1)}) \supset \overline{B_{r_n}(X_i^{(n)})} \cap \Sigma$.

To start with, fix arbitrary matrices $X_i^{(1)} \in V_i(X^0)$ for $i = 1, \dots 5$. Since $V_i(X^0)$ is relatively open in Σ, there exists $r_1 < 1$ such that

$$\overline{B_{r_1}(X_i^{(1)})} \cap \Sigma \subset V_i\left(X^0\right).$$

Next, having constructed $X^{(k)}, r_k$ for $k = 1, \dots, n$ with the properties (a)-(b) for all $k = 1, \dots, n$, we choose $X_i^{(n+1)} \in V_i(X^0) \cap B_{1/(n+1)}(X_i^0)$ for $i = 1, \dots, 5$ such that

$$\overline{B_{r_n}(X_i^{(n)})} \cap \Sigma \subset V_i(X^{(n+1)}).$$

Such a choice is possible by the continuity of the maps $P \mapsto V_i(P)$ and since $V_i(X^0)$ is relatively open in Σ. Finally, we fix $0 < r_{n+1} < 1/(n+1)$ so that in addition

$$\overline{B_{r_{n+1}}(X_i^{(n+1)})} \cap \Sigma \subset V_i(X^0)$$

for all $i = 1, \dots, 5$.

To conclude with the proof of the theorem, we define

$$U_n := \bigcup_{i=1}^{5} B_{r_n}(X_i^{(n)}) \cap \Sigma.$$

Note that U_n is a relatively open subset of Σ with

$$U_n \subset \bigcup_{i=1}^{5} V_i(X^{(n+1)}) \subset \{X_1^{(n+1)}, \dots, X_5^{(n+1)}\}^{rc} \subset U_{n+1}^{rc}$$

and, since $X_i^{(n)} \to X_i^0$ and $r_n \to 0$ as $n \to \infty$, we also have that

$$\sup_{Y \in U_n} \operatorname{dist}(Y, K) \to 0 \text{ as } n \to \infty.$$

□

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