Geometrical properties of polyconvex polytopes

Marcus Wagner

1. Introduction.

Motivated by existence and semicontinuity problems in multidimensional calculus of variations, a number of generalized convexity notions for functions has been introduced since the 50s of the last century. In its hierarchical order, the most important of these notions are polyconvexity, quasiconvexity and rank-one convexity. [01] However, the necessity to employ these semiconvexity notions as well for *point sets* in the space $\mathbb{R}^{nm}$ was recognized much later. [02] As a consequence, the properties of semiconvex sets are still less thoroughly investigated than those of semiconvex functions. The present paper, which originated in the context of the study of multidimensional control problems of Dieudonné-Rashevsky type with nonconvex data, [03] makes a contribution to the geometry of polyconvex sets and, particularly, of polyconvex polytopes. For the description of these sets, we introduce the following notations.

Definition 1.1. (The operator T) *Let n, $m \geqslant 1$ and denote* $\mathrm{Min}(n,m) = n \wedge m$. *Considering elements $v \in \mathbb{R}^{nm}$ as (n,m)-matrices, we define $T(v) = \big(v, T_2(v), T_3(v), \dots, T_{(n\wedge m)}(v)\big) \in \mathbb{R}^{\tau(n,m)} = \mathbb{R}^{\sigma(1)} \times \mathbb{R}^{\sigma(2)} \times \mathbb{R}^{\sigma(3)} \times \dots \times \mathbb{R}^{\sigma(n\wedge m)}$ as the row vector consisting of all minors of v: $T_2(v) = \mathrm{adj}_2(v)$, $T_3(v) = \mathrm{adj}_3(v)$, $\dots$, $T_{(n\wedge m)}(v) = \mathrm{adj}_{(n\wedge m)}(v)$. Consequently, we have $\sigma(k) = \binom{n}{k} \cdot \binom{m}{k}$, $1 \leqslant k \leqslant n \wedge m$. The sum of the dimensions is denoted by $\tau(n,m) = \sigma(1) + \dots + \sigma(n \wedge m)$.*

We call a set $\mathrm{P} \subseteq \mathbb{R}^{nm}$ *polyconvex* iff it can be described with the aid of a convex set $\mathrm{Q} \in \mathbb{R}^{\tau(n,m)}$ ("convex representative of P") through the relation $v \in \mathrm{P} \iff T(v) \in \mathrm{Q}$ (see Definition 2.1. below). Obviously, this notion allows for the generation of a polyconvex hull $\mathrm{Pco}\,(\mathrm{E})$ of a given set $\mathrm{E} \subseteq \mathbb{R}^{nm}$ (Definition 2.5.) without reference to the notion of a polyconvex function. [03] In analogy to convex analysis, sets arising as the polyconvex hull of finitely many points will be called *polyconvex polytopes* (Definition 2.7.).

The main results of our investigation may be summarized as follows. First, we obtain a characterization theorem for polyconvex polytopes (Proposition 2.8.), which states that a polyconvex set in $\mathbb{R}^{nm}$ is a polyconvex polytope iff the intersection of its convex representatives forms a convex polytope in $\mathbb{R}^{\tau(n,m)}$. The polyconvexity of a nonempty, compact set $\mathrm{P} \subset \mathbb{R}^{nm}$ is characterized by the possibility to approximate P in Hausdorff distance from inwards as well as from outwards with arbitrary precision by polyconvex polytopes (Theorem 4.3.). Moreover, the selection theorem of Blaschke can be generalized within the polyconvex framework: every uniformly bounded sequence of nonempty, compact, polyconvex sets admits a subsequence, which converges to a nonempty, compact, polyconvex set in Hausdorff distance (Theorem 4.8.).

On the other hand, there are essential results from convex analysis, which cannot be extended to polyconvex sets. Polyconvexity is invariant neither under rotations nor under reflections (Counterexamples 2.16. and 2.17.). Although the separation concept for convex sets can be naturally extended (separation of polyconvex sets by quasiaffine hypersurfaces instead of hyperplanes), most separation properties from convex analysis

---

[01] These notions go back to [Morrey 52] , cf. [Dacorogna 08] , p. 156 f., Definition 5.1. A key publication enforcing the study of polyconvexity was [Ball 77] .

[02] Cf. [Dacorogna 08] , p. 316 f., Definition 7.2. and Remark 7.3.

[03] For more details, cf. [Wagner 10] , [Wagner 11] , pp. 218 – 220, [Wagner 12] and [Wagner 14].

[03] In contrast to the alternative concept proposed e. g. by [Müller 99] , p. 135, or [Zhang 98] , p. 145, where the polyconvex hull of a set is obtained via level sets of certain polyconvex functions; cf. [Dacorogna 08] , pp. 331 ff.

disappear (Counterexamples 3.6. and 3.7.). The only case, which allows for separation, involves a rank-one segment or a rank-one ray, respectively (Propositions 3.3. – 3.5.).
The structure of the paper is as follows: In Section 2, we introduce the notions of polyconvex sets, polyconvex hulls, polyconvex polytopes and their precise convex representatives, studying their main properties and providing a number of examples and counterexamples. Section 3 is devoted to the polyconvex support-and-separation concept. We prove the separation theorem for rank-one objects and provide counterexamples for the possible extension of the weak and strong separation theorem of convex analysis. In Section 4, we study the Hausdorff approximation properties of compact polyconvex sets and prove the theorems announced above.

Notations.

The rank of a (n,m)-matrix $v \in \mathbb{R}^{nm}$ is denoted by $\mathrm{Rg}\,(v)$. For a set A, $\mathrm{int}\,(\mathrm{A})$ and $\mathrm{co}\,(\mathrm{A})$ denote the interior and the convex hull, respectively. For a convex set C, $\mathrm{relint}\,(\mathrm{C})$ denotes its relative interior and $\mathrm{ext}\,(\mathrm{C})$ the set of the extremal points of C. The symbol $\mathfrak{o}$ denotes, depending on the context, the null vector or the null matrix of appropriate dimension.

2. Polyconvex polytopes.

a) Basic notions of polyconvex analysis.

Definition 2.1. (Polyconvex set) [04] *Consider elements $v \in \mathbb{R}^{nm}$ as (n,m)-matrices. A set $\mathrm{P} \subseteq \mathbb{R}^{nm}$ is called polyconvex iff there exists a convex set $\mathrm{Q} \subseteq \mathbb{R}^{\tau(n,m)}$ such that $\mathrm{P} = \big\{\, v \in \mathbb{R}^{nm} \;\big|\; T(v) \in \mathrm{Q} \,\big\}$. The set Q is called a convex representative for the polyconvex set P.*

Note that the convex representative of a polyconvex set is not necessarily uniquely determined. By the following lemma, the smallest possible convex representative is singled out, which will be called the precise representative $\widetilde{\mathrm{Q}}$ of P.

Lemma 2.2. (Precise representative of a polyconvex set) *1)* [05] *If $\mathrm{P} \subseteq \mathbb{R}^{nm}$ is a polyconvex set then $\widetilde{\mathrm{Q}} = \mathrm{co}\,\big\{\, T(v) \in \mathbb{R}^{\tau(n,m)} \;\big|\; v \in \mathrm{P} \,\big\}$ forms a convex representative of P.*

2) For any convex representative $\mathrm{Q} \subseteq \mathbb{R}^{\tau(n,m)}$ of P, it holds that $\widetilde{\mathrm{Q}} \subseteq \mathrm{Q}$. Consequently, $\widetilde{\mathrm{Q}}$ is obtained as the intersection of all convex representatives of P.

Proof. The proof of Part 2) is obvious. ∎

Lemma 2.3. (Rank-one segments in polyconvex sets) *Every polyconvex set $\mathrm{P} \subseteq \mathbb{R}^{nm}$ is rank-one convex, i. e. $v_1,\, v_2 \in \mathrm{P}$ and $\mathrm{Rg}\,(v_1 - v_2) = 1$ imply $\lambda\, v_1 + (1-\lambda)\, v_2 \in \mathrm{P}$ for all $0 < \lambda < 1$.*

Proof. This is an immediate consequence of [Dacorogna 08], p. 254 f., Proposition 5.65., (vi). ∎.

Examples 2.4. *1) Every convex set $\mathrm{C} \subseteq \mathbb{R}^{nm}$ is polyconvex* since the set $\mathrm{Q} = \mathrm{C} \times \mathbb{R}^{\sigma(2)} \times \ldots \times \mathbb{R}^{\sigma(N \wedge m)} \subseteq \mathbb{R}^{\tau(n,m)}$ forms a convex representative of C. The space $\mathbb{R}^{nm}$ admits even $\mathbb{R}^{\tau(n,m)}$ as precise representative. [06]

2) Every closed quasiaffine half-space $\mathrm{S} = \{\, v \in \mathbb{R}^{nm} \mid \langle\, W,\, T(v)\,\rangle \geqslant \alpha \,\}$ with $W \in \mathbb{R}^{\tau(n,m)}$, $\alpha \in \mathbb{R}$ is polyconvex since the closed convex half-space $\mathrm{Q} = \{\, V \in \mathbb{R}^{\tau(n,m)} \mid \langle\, W,\, V\,\rangle \geqslant \alpha \,\}$ forms a convex representative of S.

---

[04] [Dacorogna 08], p. 316, Definition 7.2. (ii), going back to [Dacorogna/Ribeiro 06], p. 108, Definition 3.1. (ii).

[05] [Dacorogna 08], p. 317, Theorem 7.4. (iii).

[06] [Bevan 06], p. 24, (2.2).

3) Every quasiaffine hypersurface $\mathrm{H} = \{ v \in \mathbb{R}^{nm} \mid \langle W, T(v) \rangle = \alpha \}$ *with* $W \in \mathbb{R}^{\tau(n,m)}$, $\alpha \in \mathbb{R}$ *is polyconvex* since the hyperplane $\mathrm{Q} = \{ V \in \mathbb{R}^{\tau(n,m)} \mid \langle W, V \rangle = \alpha \}$ forms a convex representative of H.

4) The intersection $\mathrm{P} = \bigcap_{\alpha \in \mathrm{A}} \mathrm{P}_\alpha$ *of an arbitrary family* $\{ \mathrm{P}_\alpha \}_{\alpha \in \mathrm{A}}$ *of polyconvex sets* $\mathrm{P}_\alpha \subseteq \mathbb{R}^{nm}$ *is polyconvex:* If $\mathrm{Q}_\alpha \subseteq \mathbb{R}^{\tau(n,m)}$ is a convex representative of P_α, $\alpha \in \mathrm{A}$, then $\mathrm{Q} = \bigcap_{\alpha \in \mathrm{A}} \mathrm{Q}_\alpha$ is a convex set, which forms a convex representative of $\bigcap_{\alpha \in \mathrm{A}} \mathrm{P}_\alpha$.

Definition 2.5. (Polyconvex hull) [07] *For a given set* $\mathrm{E} \subseteq \mathbb{R}^{nm}$, *we define the polyconvex hull* $\mathrm{Pco}\,(\mathrm{E})$ *as the intersection of all polyconvex sets* $\mathrm{P} \subseteq \mathbb{R}^{nm}$ *with* $\mathrm{E} \subseteq \mathrm{P}$.

Proposition 2.6. (Carathéodory representation of the polyconvex hull) [08] *For a given set* $\mathrm{E} \subseteq \mathbb{R}^{nm}$, *the polyconvex hull is described by the formula*

$$\mathrm{Pco}\,(\mathrm{E}) \;=\; \Big\{ v \in \mathbb{R}^{nm} \;\Big|\; T(v) = \sum_{s=1}^{\tau(n,m)+1} \lambda_s \, T(v_s)\,,\; v_s \in \mathrm{E}\,,\; \lambda_s \geqslant 0\,,\; \sum_{s=1}^{\tau(n,m)+1} \lambda_s = 1 \Big\}\,. \quad \blacksquare \tag{2.1}$$

Formula (2.1) implies that the polyconvex hull of an open or compact set is again open or compact, respectively.

b) Polyconvex polytopes.

In analogy to convex analysis, we single out an elementary class of polyconvex sets.

Definition 2.7. (Polyconvex polytope) *A set* $\mathrm{P} \subset \mathbb{R}^{nm}$, *which arises as the polyconvex hull* $\mathrm{P} = \mathrm{Pco}\,\{ v_1, \dots, v_N \}$ *of a finite subset of* $\mathbb{R}^{nm}$, *is called a polyconvex polytope.*

By Proposition 2.6., every polyconvex polytope is compact.

Proposition 2.8. (Characterization of polyconvex polytopes) *A polyconvex set* $\mathrm{P} \subseteq \mathbb{R}^{nm}$ *is a polyconvex polytope iff its precise representative* $\widetilde{\mathrm{Q}} \subseteq \mathbb{R}^{\tau(n,m)}$ *forms a convex polytope.*

Proof. If $\mathrm{Q} \subseteq \mathbb{R}^{\tau(n,m)}$ is some convex representative of $\mathrm{P} = \mathrm{Pco}\,\{ v_1, \dots, v_N \} \subset \mathbb{R}^{nm}$ then Q must contain the points $T(v_1), \dots, T(v_N)$ and its convex hull $\widetilde{\mathrm{Q}}$, cf. Lemma 2.2., 1). On the other hand, Proposition 2.6. implies that

$$\mathrm{P} \;=\; \Big\{ v \in \mathbb{R}^{nm} \;\Big|\; T(v) = \sum_{s=1}^{N} \lambda_s \, T(v_s)\,,\; \lambda_s \geqslant 0\,,\; \sum_{s=1}^{N} \lambda_s = 1 \Big\} \tag{2.2}$$

$$\;=\; \Big\{ v \in \mathbb{R}^{nm} \;\Big|\; T(v) \in \mathrm{co}\,\{ T(v_1), \dots, T(v_N) \} \Big\}\,, \tag{2.3}$$

and $\widetilde{\mathrm{Q}}$ forms itself a convex representative for P. Consequently, the convex polytope $\widetilde{\mathrm{Q}}$ forms even the precise representative of P. Reversely, if $\mathrm{P} \subseteq \mathbb{R}^{nm}$ is a polyconvex set and its precise representative $\widetilde{\mathrm{Q}}$ is a convex polytope with $\mathrm{ext}\,(\widetilde{\mathrm{Q}}) = \{ V_1, \dots, V_N \}$ then from $\widetilde{\mathrm{Q}} = \mathrm{co}\,\{ T(v) \in \mathbb{R}^{\tau(n,m)} \mid v \in \mathrm{P} \}$, we deduce the existence of N points $v_1, \dots, v_N \in \mathrm{P}$ with $T(v_s) = V_s$, $1 \leqslant s \leqslant N$. Consequently,

$$\mathrm{Pco}\,\{ v_1, \dots, v_N \} = \Big\{ v \in \mathbb{R}^{nm} \;\Big|\; T(v) = \sum_{s=1}^{N} \lambda_s \, T(v_s)\,,\; \lambda_s \geqslant 0\,,\; \sum_{s=1}^{N} \lambda_s = 1 \Big\} \tag{2.4}$$

$$= \Big\{ v \in \mathbb{R}^{nm} \;\Big|\; T(v) = \sum_{s=1}^{N} \lambda_s \, V_s\,,\; \lambda_s \geqslant 0\,,\; \sum_{s=1}^{N} \lambda_s = 1 \Big\} \;=\; \Big\{ v \in \mathbb{R}^{nm} \;\Big|\; T(v) \in \widetilde{\mathrm{Q}} \Big\} \;=\; \mathrm{P}\,, \tag{2.5}$$

and $\mathrm{P} = \mathrm{Pco}\,\{ v_1, \dots, v_N \}$ forms a polyconvex polytope. $\blacksquare$

[07] [Dacorogna 08] , p. 323, Definition 3.13.

[08] [Dacorogna 08] , p. 323, Theorem 7.14.

Proposition 2.9. (Transformations of polyconvex polytopes) *Assume that* $\mathrm{P} = \mathrm{Pco}\{\, v_1, \ldots, v_N \,\} \subset \mathbb{R}^{nm}$ *is a polyconvex polytope.*

1) For every $\mu \geqslant 0$ *it holds that* $\mu\,\mathrm{P} = \mathrm{Pco}\{\, \mu\, v_1, \ldots, \mu\, v_N \,\}$*, and* $\mu\,\mathrm{P}$ *forms a polyconvex polytope as well.*

2) For every vector $w \in \mathbb{R}^{nm}$*, it holds that* $\mathrm{P}+w = \mathrm{Pco}\{\, v_1+w, \ldots, v_N+w \,\}$*, and* $\mathrm{P}+w$ *forms a polyconvex polytope as well.*

Proof. 1) If $\mu = 0$ then the singleton $\mu\,\mathrm{P} = \{\, \mathfrak{o} \,\}$ forms a polyconvex polytope. Choosing $\mu > 0$, we observe that $T\big(\mu\, v\big) = \big(\mu\, v,\, \mu^2\, T_2(v), \ldots, \mu^{(n\wedge m)}\, T_{(n\wedge m)}(v)\big)$ and

$$\mu\,\mathrm{P} \;=\; \Big\{\, \mu\, v \in \mathbb{R}^{nm} \;\Big|\; T(v) = \sum_{s=1}^{N} \lambda_s\, T(v_s)\,,\; \lambda_s \geqslant 0\,,\; \sum_{s=1}^{N} \lambda_s = 1 \,\Big\} \tag{2.6}$$

$$= \Big\{\, \mu\, v \in \mathbb{R}^{nm} \;\Big|\; \mu\, v = \sum_{s=1}^{N} \lambda_s\, \mu\, v_s\,,\; \mu^2\, \mathrm{adj}_2(v) = \sum_{s=1}^{N} \lambda_s\, \mu^2\, \mathrm{adj}_2(v_s)\,, \ldots, \tag{2.7}$$
$$\mu^{(n\wedge m)}\, \mathrm{adj}_{(n\wedge m)}(v) = \sum_{s=1}^{N} \lambda_s\, \mu^{(n\wedge m)}\, \mathrm{adj}_{(n\wedge m)}(v_s)\,,\; \lambda_s \geqslant 0\,,\; \sum_{s=1}^{N} \lambda_s = 1 \,\Big\}$$

$$= \Big\{\, \mu\, v \in \mathbb{R}^{nm} \;\Big|\; T\big(\mu\, v\big) = \sum_{s=1}^{N} \lambda_s\, T\big(\mu\, v_s\big)\,,\; \lambda_s \geqslant 0\,,\; \sum_{s=1}^{N} \lambda_s = 1 \,\Big\} \tag{2.8}$$

$$= \Big\{\, w \in \mathbb{R}^{nm} \;\Big|\; T(w) \in \mathrm{co}\big\{\, T\big(\mu\, v_1\big), \ldots, T\big(\mu\, v_N\big) \,\big\} \,\Big\} \;=\; \mathrm{Pco}\{\, \mu\, v_1, \ldots, \mu\, v_N \,\} \tag{2.9}$$

by Proposition 2.6., and $\mu\,\mathrm{P} = \mathrm{Pco}\{\, \mu\, v_1, \ldots, \mu\, v_N \,\}$ forms a polyconvex polytope.

2) For the investigation of the Minkowski sum $\mathrm{P}+w$, it suffices to consider the case where w is a matrix with a single column different from $\mathfrak{o}$. Without loss of generality, we may assume that $w = \big(w_1\, \mathfrak{o} \ldots \mathfrak{o}\big) \in \mathbb{R}^{nm}$, $w_1 \neq \mathfrak{o}$. In this case, we observe that

$$T\big(v+w\big) = \big(v+w,\, \mathrm{adj}_2(v) + C_2(w)\, v,\, \mathrm{adj}_3(v) + C_3(w)\, \mathrm{adj}_2(v), \ldots\big) \tag{2.10}$$

with certain matrices $C_2(w) \in \mathbb{R}^{\sigma(1)\times\sigma(2)}$, $C_3(w) \in \mathbb{R}^{\sigma(2)\times\sigma(3)}$, ... , whose entries depend on w only. We find

$$\mathrm{P}+w \;=\; \Big\{\, v+w \in \mathbb{R}^{nm} \;\Big|\; v = \sum_{s=1}^{N} \lambda_s\, v_s\,,\; \mathrm{adj}_2(v) = \sum_{s=1}^{N} \lambda_s\, \mathrm{adj}_2(v_s)\,,\; \mathrm{adj}_3(v) = \sum_{s=1}^{N} \lambda_s\, \mathrm{adj}_3(v_s)\,, \ldots, \tag{2.11}$$
$$\mathrm{adj}_{(n\wedge m)}(v) = \sum_{s=1}^{N} \lambda_s\, \mathrm{adj}_{(n\wedge m)}(v_s)\,,\; \lambda_s \geqslant 0\,,\; \sum_{s=1}^{N} \lambda_s = 1 \,\Big\}$$

$$= \Big\{\, v+w \in \mathbb{R}^{nm} \;\Big|\; \lambda_s \geqslant 0\,,\; \sum_{s=1}^{N} \lambda_s = 1\,,\; v+w = \sum_{s=1}^{N} \lambda_s\, \big(\, v_s + w \,\big)\,, \tag{2.12}$$
$$\mathrm{adj}_2(v) + C_2(w)\, v = \sum_{s=1}^{N} \lambda_s\, \mathrm{adj}_2(v_s) + C_2(w) \sum_{s=1}^{N} \lambda_s\, v_s \;=\; \sum_{s=1}^{N} \lambda_s\, \big(\, \mathrm{adj}_2(v_s) + C_2(w)\, v_s \,\big)\,,$$
$$\mathrm{adj}_3(v) + C_3(w)\, \mathrm{adj}_2(v) = \sum_{s=1}^{N} \lambda_s\, \mathrm{adj}_3(v_s) + C_3(w) \sum_{s=1}^{N} \lambda_s\, \mathrm{adj}_2(v_s) \;=\; \sum_{s=1}^{N} \lambda_s\, \big(\, \mathrm{adj}_3(v_s) + C_3(w)\, \mathrm{adj}_2(v_s) \big)\,,$$
$$\vdots$$
$$\mathrm{adj}_{(n\wedge m)}(v) + C_{(n\wedge m)}(w)\, \mathrm{adj}_{(n\wedge m)-1}(v) = \sum_{s=1}^{N} \lambda_s\, \big(\, \mathrm{adj}_{(n\wedge m)}(v_s) + C_{(n\wedge m)}(w)\, \mathrm{adj}_{(n\wedge m)-1}(v_s) \big) \,\Big\}$$

$$= \Big\{\, z \in \mathbb{R}^{nm} \;\Big|\; T(z) \;=\; \sum_{s=1}^{N} \lambda_s\, T\big(\, v_s + w \,\big)\,,\; \lambda_s \geqslant 0\,,\; \sum_{s=1}^{N} \lambda_s = 1 \,\Big\} \tag{2.13}$$

$$= \Big\{\, z \in \mathbb{R}^{nm} \;\Big|\; T(z) \in \mathrm{co}\big\{\, T\big(\, v_1 + w \,\big), \ldots, T\big(\, v_N + w \,\big) \,\big\} \,\Big\} \;=\; \mathrm{Pco}\{\, v_1 + w, \ldots, v_N + w \,\} \tag{2.14}$$

by Proposition 2.6., and $\mathrm{P}+w = \mathrm{Pco}\{\, v_1 + w, \ldots, v_N + w \,\}$ forms a polyconvex polytope together with P. ∎

Proposition 2.10. (Minkowski multiplication of polyconvex polytopes) *Assume that dimensions $n,\ m,\ r \geqslant 2$ with $(n \wedge r) \leqslant (n \wedge m) \wedge (m \wedge r)$ are given. If $\mathrm{P}' = \mathrm{Pco}\{\, v_1,\, \dots,\, v_N \,\} \subset \mathbb{R}^{nm}$ and $\mathrm{P}'' = \mathrm{Pco}\{\, w_1,\, \dots,\, w_M \,\} \subset \mathbb{R}^{mr}$ are polyconvex polytopes then its Minkowski product $\mathrm{P}' \cdot \mathrm{P}'' = \{\, v \cdot w \;\big|\; v \in \mathrm{P}',\ w \in \mathrm{P}'' \,\} \subset \mathbb{R}^{nr}$ forms again a polyconvex polytope, which is represented through $\mathrm{P}' \cdot \mathrm{P}'' = \mathrm{Pco}\{\, v_i \cdot w_j \;\big|\; 1 \leqslant i \leqslant N\,,\ 1 \leqslant j \leqslant M \,\}$.*

Proof. By Proposition 2.6., the polyconvex polytopes P', P'' admit the following representations:

$$\mathrm{P}' = \{\, \sum_{i=1}^{N} \lambda_i\, v_i \in \mathbb{R}^{nm} \;\big|\; \sum_{i=1}^{N} \lambda_i = 1\,,\ \lambda_i \geqslant 0\,,\ \mathrm{adj}_k\big(\sum_{i=1}^{N} \lambda_i\, v_i \big) = \sum_{i=1}^{N} \lambda_i\, \mathrm{adj}_k(v_i)\,,\ 1 \leqslant k \leqslant (n \wedge m) \,\}\,; \tag{2.15}$$

$$\mathrm{P}'' = \{\, \sum_{j=1}^{M} \mu_j\, w_j \in \mathbb{R}^{mr} \;\big|\; \sum_{j=1}^{M} \mu_j = 1\,,\ \mu_j \geqslant 0\,,\ \mathrm{adj}_l\big(\sum_{j=1}^{M} \mu_j\, w_j \big) = \sum_{j=1}^{M} \mu_j\, \mathrm{adj}_l(w_j)\,,\ 1 \leqslant l \leqslant (m \wedge r) \,\}\,. \tag{2.16}$$

It follows that $z \in \mathrm{P}' \cdot \mathrm{P}''$ iff

$$z \;=\; \big(\sum_{i=1}^{N} \lambda_i\, v_i \big) \big(\sum_{j=1}^{M} \mu_j\, w_j \big) \;=\; \sum_{i=1}^{N} \sum_{j=1}^{M} \lambda_i\, \mu_j\, (v_i\, w_j)\,,\ \sum_{i=1}^{N} \lambda_i \;=\; 1\,,\ \sum_{j=1}^{M} \mu_j \;=\; 1\,,\ \lambda_i\,,\ \mu_j \geqslant 0 \tag{2.17}$$

$$\Longrightarrow \quad z \;=\; \sum_{i=1}^{N} \sum_{j=1}^{M} \sigma_{ij}\, (v_i\, w_j)\,,\ \sigma_{ij} \;=\; \lambda_i\, \mu_j \geqslant 0\,,\ \sum_{i=1}^{N} \sum_{j=1}^{M} \sigma_{ij} \;=\; \sum_{i=1}^{N} \sum_{j=1}^{M} \lambda_i\, \mu_j \;=\; 1 \tag{2.18}$$

as well as

$$\mathrm{adj}_s(z) \;=\; \mathrm{adj}_s\Big(\big(\sum_{i=1}^{N} \lambda_i\, v_i \big) \big(\sum_{j=1}^{M} \mu_j\, w_j \big) \Big) \;=\; \mathrm{adj}_s\big(\sum_{i=1}^{N} \lambda_i\, v_i \big) \cdot \mathrm{adj}_s\big(\sum_{j=1}^{M} \mu_j\, w_j \big) \tag{2.19}$$

$$=\; \big(\sum_{i=1}^{N} \lambda_i\, \mathrm{adj}_s(v_i) \big) \cdot \big(\sum_{j=1}^{M} \mu_j\, \mathrm{adj}_s(w_j) \big) \;=\; \sum_{i=1}^{N} \sum_{j=1}^{M} \lambda_i\, \mu_j\, \big(\mathrm{adj}_s(v_i) \cdot \mathrm{adj}_s(w_j) \big) \tag{2.20}$$

$$=\; \sum_{i=1}^{N} \sum_{j=1}^{M} \sigma_{ij}\, \mathrm{adj}_s(v_i\, w_j)\,,\ 1 \leqslant s \leqslant (n \wedge m) \wedge (m \wedge r) \tag{2.21}$$

(using (2.15), (2.16) and [Dacorogna 08], p. 257, Proposition 5.66., (i)). Consequently, $\mathrm{P}' \cdot \mathrm{P}''$ may be represented through

$$\begin{aligned} \mathrm{P}' \cdot \mathrm{P}'' \;=\; \{\, \sum_{i=1}^{N} \sum_{j=1}^{M} \sigma_{ij}\, (v_i\, w_j) \;\big|\; \sum_{i=1}^{N} \sum_{j=1}^{M} \sigma_{ij} \;=\; 1\,, \sigma_{ij} \geqslant 0\,,\ \mathrm{adj}_s\big(\sum_{i=1}^{N} \sum_{j=1}^{M} \sigma_{ij}\, (v_i\, w_j) \big) \\ =\; \sum_{i=1}^{N} \sum_{j=1}^{M} \sigma_{ij}\, \mathrm{adj}_s(v_i\, w_j)\,, 1 \leqslant s \leqslant (n \wedge m) \wedge (m \wedge r) \,\} \;=\; \mathrm{Pco}\{\, v_1\, w_1,\, \dots,\, v_N\, w_M \,\} \end{aligned} \tag{2.22}$$

as a polyconvex hull of finitely many points and, consequently, forms a polyconvex polytope. ∎

c) Examples of polyconvex polytopes.

Example 2.11. (Polyconvex hull of a two-point set) [09] *Let $n,\ m \geqslant 2$. For $v_1,\ v_2 \in \mathbb{R}^{nm}$, we have*

$$\mathrm{Pco}\{\, v_1,\, v_2 \,\} = \begin{cases} \{\, (1-\lambda)\, v_1 + \lambda\, v_2 \;\big|\; 0 \leqslant \lambda \leqslant 1 \,\} & |\ \mathrm{Rg}\,(v_1 - v_2) \leqslant 1\,; \\ \{\, v_1,\, v_2 \,\} & |\ \mathrm{Rg}\,(v_1 - v_2) \geqslant 2\,. \end{cases} \tag{2.23}$$

Proof. The first alternative is clear from Lemma 2.3. Assuming now $\mathrm{Rg}\,(v_2 - v_1) \geqslant 2$, we find a rank-two submatrix $\left(\begin{smallmatrix} e & f \\ g & h \end{smallmatrix}\right)$ of $(v_2 - v_1)$ with nonzero determinant. Consequently, denoting by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ the submatrix of v_1 with the corresponding indices, we observe that

$$\varphi(\alpha) \;=\; \det\big(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) + \alpha \left(\begin{smallmatrix} e & f \\ g & h \end{smallmatrix}\right) \big) \;=\; \alpha^2 \det \left(\begin{smallmatrix} e & f \\ g & h \end{smallmatrix}\right) + \alpha \begin{pmatrix} d \\ -c \\ -b \\ a \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} e \\ f \\ g \\ h \end{pmatrix} + \det \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \tag{2.24}$$

[09] Cf. [Dacorogna 08], p. 14.

is a nontrivial quadratic function for $0 \leqslant \alpha \leqslant 1$. Since, by Proposition 2.6., $v_\alpha = (1-\alpha)\, v_1 + \alpha\, v_2$ belongs to $\operatorname{Pco}\{\, v_1,\, v_2 \,\}$ iff $(1-\alpha)\operatorname{adj}_2(v_1) + \alpha \operatorname{adj}_2(v_2) \;=\; \operatorname{adj}_2\big(\, v_1 + \alpha\,(v_2 - v_1)\,\big)$, we see that $v_\alpha \notin \operatorname{Pco}\{\, v_1,\, v_2 \,\}$ for all $0 < \alpha < 1$, and the polyconvex hull consists of the given points v_1, v_2 only. ■

Example 2.12. (Three-point set with nontrivial polyconvex hull) *Consider the three matrices* $v_1 = \left(\begin{smallmatrix} -3/2 & 0 \\ 0 & -2/3 \end{smallmatrix}\right)$, $v_2 = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 1/2 \end{smallmatrix}\right)$ *and* $v_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. *Then for the set* $\mathrm{E} = \{\, v_1,\, v_2,\, v_3 \,\} \subset \mathbb{R}^{2\times 2}$, *it holds that* $\mathrm{E} \subsetneqq \operatorname{Pco}(\mathrm{E}) = \mathrm{E} \cup \big\{\, \left(\begin{smallmatrix} \alpha & 0 \\ 0 & 1/\alpha \end{smallmatrix}\right) \in \mathbb{R}^{2\times 2} \;\big|\; 1 < \alpha < 2 \,\big\} \subsetneqq \operatorname{co}(\mathrm{E}) = \Delta(\, v_1,\, v_2,\, v_3 \,)$.

Proof. Observe first that $\det(v_1) = \det(v_2) = \det(v_3) = 1$. By Proposition 2.6., the polyconvex hull $\operatorname{Pco}\{\, v_1,\, v_2,\, v_3 \,\}$ admits the precise representative $\widetilde{\mathrm{Q}} = \operatorname{co}\{\, T(v_1),\, T(v_2),\, T(v_3) \,\} = \Delta(\, v_1,\, v_2,\, v_3 \,) \times \{\, 1 \,\} \subset \mathbb{R}^5$. In order to determine $\operatorname{Pco}(\mathrm{E})$, we must find all the points $v \in \Delta(\, v_1,\, v_2,\, v_3 \,)$ with $\det(v) = 1$. Obviously, these are precisely the matrices $\left(\begin{smallmatrix} \alpha & 0 \\ 0 & 1/\alpha \end{smallmatrix}\right) \in \Delta(\, v_1,\, v_2,\, v_3 \,)$, which are parametrized by $1 \leqslant \alpha \leqslant 2$. ■

Example 2.13. (Cross-shaped polyconvex polytope) *Consider the matrices* $v_1 = \left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right)$, $v_2 = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$, $v_3 = \left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right)$, $v_4 = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \mathbb{R}^{2\times 2}$ *and denote the* $(2,2)$*-unit matrix by* E_2. *Then* $\operatorname{Pco}\{\, v_1,\, v_2,\, v_3,\, v_4 \,\} = [\, v_1\,,\, E_2 \,] \cup [\, v_2,\,,\, E_2 \,] \cup [\, v_3\,,\, E_2 \,] \cup [\, v_4\,,\, E_2 \,]$.

Proof. Since $\det(v_1) = \det(v_2) = \det(v_3) = \det(v_4) = 1$, we find that $v \in \operatorname{Pco}\{\, v_1,\, v_2,\, v_3,\, v_4 \,\}$ iff $v = \left(\begin{smallmatrix} 1 & b \\ c & 1 \end{smallmatrix}\right)$ with $-1 \leqslant b \leqslant 1$, $-1 \leqslant c \leqslant 1$ and $\det(v) = 1 - bc = 1 \iff bc = 0$. Thus $\operatorname{Pco}\{\, v_1,\, v_2,\, v_3,\, v_4 \,\}$ takes the claimed shape as a union of rank-one segments. ■

Example 2.14. (A convex polytope, which is not a polyconvex polytope) *The rank-two segment* $\mathrm{E} = \big[\, \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)\ \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \,\big] \subset \mathbb{R}^{2\times 2}$ *is a polyconvex set, which forms a convex but not a polyconvex polytope.*

Proof. The precise representative of E is the convex set $\widetilde{\mathrm{Q}} = \operatorname{co}\big\{\, \big(\, \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha \end{smallmatrix}\right),\, \alpha^2 \,\big) \in \mathbb{R}^5 \;\big|\; 0 \leqslant \alpha \leqslant 1 \,\big\}$. Since $\widetilde{\mathrm{Q}}$ is not a convex polytope, Proposition 2.8. implies that E is not a polyconvex polytope. ■

Example 2.15. (A convex polytope, which is at the same time a polyconvex polytope) *The* nm*-dimensional cube* $\mathrm{E} = [\, 0\,,\, 1 \,]^{nm} \subset \mathbb{R}^{nm}$ *is a convex as well as a polyconvex polytope.*

Proof. Consider an arbitrary point $v = \big(\, \lambda_1,\, \lambda_2,\, \lambda_3,\, \dots\,,\, \lambda_{nm} \,\big) \in [\, 0\,,\, 1 \,]^{nm} = \mathrm{E}$. Then $v = v^{(0)}$ is obtained as the rank-one convex combination

$$v^{(0)} = (1-\lambda_1)\big(\, 0,\, \lambda_2, \lambda_3,\, \dots\,,\, \lambda_{nm} \,\big) + \lambda_1 \big(\, 1,\, \lambda_2, \lambda_3,\, \dots\,,\, \lambda_{nm} \,\big) = (1-\lambda_1)\, v_1^{(1)} + \lambda_1\, v_2^{(1)} \quad \text{with} \tag{2.25}$$
$$\det(v^{(0)}) \;=\; (1-\lambda_1)\, \det(v_1^{(1)}) + \lambda_1\, \det(v_2^{(1)}) \tag{2.26}$$

since [Dacorogna 08], p. 254 f., Proposition 5.65., (vi); the points $v_1^{(1)}$, $v_2^{(1)} \in \mathrm{E}$ are obtained as the rank-one convex combinations

$$v_1^{(1)} = (1-\lambda_2)\big(\, 0,\, 0, \lambda_3,\, \dots\,,\, \lambda_{nm} \,\big) + \lambda_2 \big(\, 0,\, 1, \lambda_3,\, \dots\,,\, \lambda_{nm} \,\big) = (1-\lambda_2)\, v_1^{(2)} + \lambda_2\, v_2^{(2)}\,; \tag{2.27}$$
$$v_2^{(1)} = (1-\lambda_2)\big(\, 1,\, 0, \lambda_3,\, \dots\,,\, \lambda_{nm} \,\big) + \lambda_2 \big(\, 1,\, 1, \lambda_3,\, \dots\,,\, \lambda_{nm} \,\big) = (1-\lambda_2)\, v_3^{(2)} + \lambda_2\, v_4^{(2)} \quad \text{with} \tag{2.28}$$
$$\det(v_1^{(1)}) \;=\; (1-\lambda_2)\, \det(v_1^{(2)}) + \lambda_2\, \det(v_2^{(2)})\,; \tag{2.29}$$
$$\det(v_2^{(1)}) \;=\; (1-\lambda_2)\, \det(v_3^{(2)}) + \lambda_2\, \det(v_4^{(2)})\,, \tag{2.30}$$

and the construction may be continued until v and $\det(v)$ are finally represented as convex combinations of the vertices of E and its determinants, respectively. ■

Counterexample 2.16. (Polyconvexity is not preserved under rotations) *For every* $n \geqslant 2$, *there exist a polyconvex polytope* $\mathrm{P} \subset \mathbb{R}^{n\times n}$ *and a rotation* $\mathcal{A}\colon\ \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ *such that* $\mathcal{A}(\mathrm{P})$ *is not polyconvex.*

Proof. Consider the points $v_1 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $v_2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathbb{R}^{2\times 2}$. By Example 2.11., $\mathrm{P} = \mathrm{Pco}\,\{\, v_1,\, v_2 \,\} = \{\, v_1,\, v_2 \,\}$. Define the rotation $\mathcal{A}\colon\ \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ through

$$\mathcal{A}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos(\pi/4) & 0 & 0 & -\sin(\pi/4) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sin(\pi/4) & 0 & 0 & \cos(\pi/4) \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}. \tag{2.31}$$

Then we get $\mathcal{A}(\mathrm{P}) = \{\, \mathcal{A}(v_1),\, \mathcal{A}(v_2) \,\} = \{\, \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right),\, \left(\begin{smallmatrix} 0 & 0 \\ 0 & \sqrt{2} \end{smallmatrix}\right) \,\}$. Since $\mathrm{Rg}\,\big(\,\mathcal{A}(v_2) - \mathcal{A}(v_1)\,\big) = 1$, the image of P under $\mathcal{A}$ fails to be polyconvex. In $\mathbb{R}^{n\times n}$ with $n > 2$, we obtain an analogous result for the points $\left(\begin{smallmatrix} v_1 & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} v_2 & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} \end{smallmatrix}\right)$ ∎

Counterexample 2.17. (Polyconvexity is not preserved under reflections) *For every $n \geqslant 2$, there exist a polyconvex polytope $\mathrm{P} \subset \mathbb{R}^{n\times n}$ and a reflection $\mathcal{A}\colon\ \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ with respect to a hyperplane such that $\mathcal{A}(\mathrm{P})$ is not polyconvex.*

Proof. Consider again the points $v_1 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $v_2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and the polyconvex polytope $\mathrm{P} = \mathrm{Pco}\,\{\, v_1,\, v_2 \,\} = \{\, v_1,\, v_2 \,\}$. We define the hyperplane $\mathrm{H} = \{\, \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbb{R}^{2\times 2} \;\big|\; a + (1-\sqrt{2})\,d = 0 \,\}$. Define the reflection $\mathcal{A}\colon\ \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ with respect to H through

$$\mathcal{A}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \frac{1}{2-\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1-\sqrt{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1-\sqrt{2} & 0 & 0 & 3-2\sqrt{2} \end{pmatrix} \right) \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}. \tag{2.32}$$

As in the previous counterexample, we get $\mathcal{A}(\mathrm{P}) = \{\, \mathcal{A}(v_1),\, \mathcal{A}(v_2) \,\} = \{\, \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right),\, \left(\begin{smallmatrix} 0 & 0 \\ 0 & \sqrt{2} \end{smallmatrix}\right) \,\}$, and the image of P under $\mathcal{A}$ fails to be polyconvex. In higher dimension, we may argue accordingly. ∎

3. Separation properties of polyconvex sets.

a) Support and separation: definitions.

In view of the considerations above, the appropriate objects to be used for separation of polyconvex sets are quasiaffine hypersurfaces, cf. Example 2.4., 3).

Definition 3.1. *Assume that P_1 and $\mathrm{P}_2 \subseteq \mathbb{R}^{nm}$ are nonempty polyconvex sets. Let a vector $\beta \in \mathbb{R}^{\tau(n,m)}$ and a number $\alpha \in \mathbb{R}$ be given.*

1) We say that the quasiaffine hypersurface $\mathrm{S} = \{\, v \in \mathbb{R}^{nm} \;\big|\; \langle\, \beta\,,\, T(v)\,\rangle = \alpha \,\}$ supports the polyconvex set P_1 in a point $v_1 \in \mathrm{P}_1$ iff $v_1 \in \mathrm{S}$ and $\langle\, \beta\,,\, T(v)\,\rangle \geqslant \alpha$ for all $v \in \mathrm{P}_1$.

2) We say that the quasiaffine hypersurface $\mathrm{S} = \{\, v \in \mathbb{R}^{nm} \;\big|\; \langle\, \beta\,,\, T(v)\,\rangle = \alpha \,\}$ separates the polyconvex sets P_1 and P_2 weakly iff $\sup_{v\in \mathrm{P}_1} \langle\, \beta\,,\, T(v)\,\rangle \leqslant \inf_{w\in \mathrm{P}_2} \langle\, \beta\,,\, T(w)\,\rangle$. The separation is called proper if, moreover, there exists a point $w_0 \in \mathrm{P}_2$ with $\sup_{v\in \mathrm{P}_1} \langle\, \beta\,,\, T(v)\,\rangle < \langle\, \beta\,,\, T(w_0)\,\rangle$.

3) We say that the quasiaffine hypersurface $\mathrm{S} = \{\, v \in \mathbb{R}^{nm} \;\big|\; \langle\, \beta\,,\, T(v)\,\rangle = \alpha \,\}$ separates the polyconvex sets P_1 and P_2 strongly iff there exists a number $\varepsilon > 0$ such that $\sup_{v\in \mathrm{P}_1} \langle\, \beta\,,\, T(v)\,\rangle + \varepsilon \leqslant \inf_{w\in \mathrm{P}_2} \langle\, \beta\,,\, T(w)\,\rangle$.

Although this definition appears as a natural extension of the separation concept for convex sets, most separation results from convex analysis cannot be generalized within this framework. The basic result concerns support and separation with respect to singletons: every polyconvex set P is supported in any boundary point $v \in \partial \mathrm{P}$ by a quasiaffine hypersurface, and any point $w \notin \mathrm{P}$ can be either strongly or weakly separated from P by a quasiaffine hypersurface depending on whether P is compact or not. [10] This result allows only for a slight generalization (see Propositions 3.3. − 3.5. below). It is already known that the weak and the strong separation theorem of convex analysis have no polyconvex analogues. This fact will be documented by counterexamples.

b) Separation results for polyconvex polytopes.

Lemma 3.2. (Separation of polyconvex polytopes) *The polyconvex polytopes* $\mathrm{P}_1,\ \mathrm{P}_2 \subset \mathbb{R}^{nm}$ *allow for a weak, proper or strong separation by a quasiaffine hypersurface* S *iff its precise representatives* $\widetilde{\mathrm{Q}}_1$, $\widetilde{\mathrm{Q}}_2 \subset \mathbb{R}^{\tau(n,m)}$ *can be weakly, properly or strongly separated by a hyperplane in* $\mathbb{R}^{\tau(n,m)}$.

Proof. Assume that $\mathrm{P}_1 = \mathrm{Pco}\{ v_1, \dots, v_N \}$ and $\mathrm{P}_2 = \mathrm{Pco}\{ w_1, \dots, w_M \}$ are weakly separated by the quasiaffine hypersurface $\mathrm{S} = \{ v \in \mathbb{R}^{nm} \mid \langle \beta , T(v) \rangle = \alpha \}$. This implies

$$\sup \{ \langle \beta , T(v_i) \rangle \mid 1 \leqslant i \leqslant N \} \leqslant \inf \{ \langle \beta , T(w_j) \rangle \mid 1 \leqslant j \leqslant M \} \quad \Longrightarrow \tag{3.1}$$

$$\sup \{ \langle \beta , V \rangle \mid V \in \widetilde{\mathrm{Q}}_1 \} = \sup \Big\{ \Big\langle \beta , \sum_{i=1}^{N} \lambda_i T(v_i) \Big\rangle \Bigm| \sum_{i=1}^{N} \lambda_i = 1 ,\ 0 \leqslant \lambda_i \leqslant 1 ,\ 1 \leqslant i \leqslant N \Big\}$$

$$\leqslant \inf \Big\{ \Big\langle \beta , \sum_{j=1}^{M} \mu_j T(w_j) \Big\rangle \Bigm| \sum_{j=1}^{M} \mu_j = 1 ,\ 0 \leqslant \mu_j \leqslant 1 ,\ 1 \leqslant j \leqslant M \Big\} = \inf \{ \langle \beta , W \rangle \mid W \in \widetilde{\mathrm{Q}}_2 \} , \tag{3.2}$$

and the hyperplane $\mathrm{H} = \{ V \in \mathbb{R}^{\tau(n,m)} \mid \langle \beta , V \rangle = \alpha \}$ separates the convex polytopes $\widetilde{\mathrm{Q}}_1$ and $\widetilde{\mathrm{Q}}_2$ weakly. Reversely, if $\widetilde{\mathrm{Q}}_1$ and $\widetilde{\mathrm{Q}}_2$ are weakly separated by H then (3.2) implies particularly that

$$\sup \Big\{ \langle \beta , T(v) \rangle \Bigm| T(v) = \sum_{i=1}^{N} \lambda_i T(v_i) ,\ \sum_{i=1}^{N} \lambda_i = 1 ,\ 0 \leqslant \lambda_i \leqslant 1 ,\ 1 \leqslant i \leqslant N \Big\} \tag{3.3}$$

$$\leqslant \inf \Big\{ \langle \beta , T(w) \rangle \Bigm| T(w) = \sum_{j=1}^{M} \mu_j T(w_j) ,\ \sum_{j=1}^{M} \mu_j = 1 ,\ 0 \leqslant \mu_j \leqslant 1 ,\ 1 \leqslant j \leqslant M \Big\} \quad \Longleftrightarrow$$

$$\sup \{ \langle \beta , T(v) \rangle \mid v \in \mathrm{P}_1 \} \leqslant \inf \{ \langle \beta , T(w) \rangle \mid w \in \mathrm{P}_2 \} , \tag{3.4}$$

and the quasiaffine hypersurface S separates P_1 and P_2 weakly. The cases of proper and strong separation can be treated analogously. ∎

Proposition 3.3. (Separation of a polyconvex polytope and a rank-one segment) *Assume that* $\mathrm{P} \subset \mathbb{R}^{nm}$ *is a polyconvex polytope and* $\mathrm{R} = \{ (1-\lambda) w_0 + \lambda w_1 \in \mathbb{R}^{nm} \mid 0 \leqslant \lambda \leqslant 1 \}$ *is a segment with* $\mathrm{Rg}\,(w_1 - w_0) = 1$.

1) If $\mathrm{P} \cap \mathrm{R} = \{ w_0 \}$ *then* P *and* R *can be properly separated by a quasiaffine hypersurface* $\mathrm{S} \subset \mathbb{R}^{nm}$.

2) If $\mathrm{P} \cap \mathrm{R}$ *is empty then* P *and* R *can be strongly separated by a quasiaffine hypersurface* $\mathrm{S} \subset \mathbb{R}^{nm}$.

Proof. 1) The precise representatives of $\mathrm{P} = \mathrm{Pco}\{ v_1, \dots, v_N \}$ and R are the sets $\widetilde{\mathrm{Q}}_1 = \mathrm{co}\{ T(v_1), \dots, T(v_N) \}$ and $\widetilde{\mathrm{Q}}_2 = \{ (1-\lambda_0) T(w_0) + \lambda_0 T(w_1) \mid 0 \leqslant \lambda_0 \leqslant 1 \}$. Assume that $V \in \widetilde{\mathrm{Q}}_1 \cap \widetilde{\mathrm{Q}}_2$. Then there exist $\lambda_0, \lambda_1, \dots , \lambda_N \in [0 , 1]$ with $\sum_{1 \leqslant i \leqslant N} \lambda_i = 1$ such that

$$V = \sum_{i=1}^{N} \lambda_i T(v_i) = (1-\lambda_0) T(w_0) + \lambda_0 T(w_1) = T\big((1-\lambda_0) w_0 + \lambda_0 w_1 \big) . \tag{3.5}$$

Consequently, $V = T(v)$ with $v \in \mathrm{P} \cap \mathrm{R}$, and we get $v = w_0$. Thus the convex sets $\mathrm{relint}\,(\widetilde{\mathrm{Q}}_1) \subseteq \widetilde{\mathrm{Q}}_1$ and $\mathrm{relint}\,(\widetilde{\mathrm{Q}}_2)$ are disjoint. Consequently, there exists a hyperplane $\widetilde{\mathrm{H}} = \{ V \in \mathbb{R}^{\tau(n,m)} \mid \langle \beta , V \rangle = \alpha \}$,

[10] [Dacorogna 08] , p. 321, Theorem 7.9.

which separates $\widetilde{\mathrm{Q}}_1$ and $\widetilde{\mathrm{Q}}_2$ properly, [11] and the quasiaffine hypersurface $\mathrm{S} = \{ v \in \mathbb{R}^{nm} \mid \langle \beta , T(v) \rangle = \alpha \}$ separates P and R weakly. If there exists a point $W_0 \in \widetilde{\mathrm{Q}}_2$ with $\sup_{V \in \tilde{\mathrm{Q}}_1} \langle \beta , V \rangle < \langle \beta , W_0 \rangle$ then $W_0 = T\big((1-\lambda_0)\, w_0 + \lambda_0\, w_1 \big)$ with some $0 \leqslant \lambda_0 \leqslant 1$, and S separates P and R properly. Otherwise, let us assume that $\sup_{V \in \tilde{\mathrm{Q}}_1} \langle \beta , V \rangle = \langle \beta , W \rangle$ for all $W \in \widetilde{\mathrm{Q}}_2$. If, moreover, all $v \in \mathrm{P}$ satisfy $\langle \beta , T(v) \rangle = \sup_{V \in \tilde{\mathrm{Q}}_1} \langle \beta , V \rangle$ then we would obtain $\langle \beta , V_0 \rangle = \sup_{V \in \tilde{\mathrm{Q}}_1} \langle \beta , V \rangle$ for all $V_0 \in \widetilde{\mathrm{Q}}_1$, and we get a contradiction to the fact that the separation of $\widetilde{\mathrm{Q}}_1$ and $\widetilde{\mathrm{Q}}_2$ was proper. We conclude the existence of a point $v_0 \in \mathrm{P}$ such that $\langle \beta , T(v_0) \rangle < \sup_{V \in \tilde{\mathrm{Q}}_1} \langle \beta , V \rangle$, and this confirms again the proper separation of P and R by the quasiaffine hypersurface $\mathrm{S} = \{ v \in \mathbb{R}^{nm} \mid \langle \beta , T(v) \rangle = \alpha \}$.

2) We repeat the arguments of Part 1) in order to confirm that $\widetilde{\mathrm{Q}}_1 \cap \widetilde{\mathrm{Q}}_2 = \emptyset$. As an image under a continuous function, the set $\{ T(v_1), \dots , T(v_N) \} \subset \mathbb{R}^{\tau(n,m)}$ is compact together with P, and the compactness is carried over to its convex hull $\widetilde{\mathrm{Q}}_1$. Since $\widetilde{\mathrm{Q}}_2$ is compact as well, the sets may be strongly separated by a hyperplane $\widetilde{\mathrm{H}} = \{ V \in \mathbb{R}^{\tau(n,m)} \mid \langle \beta , V \rangle = \alpha \}$. This implies strong separation of P and R by the quasiaffine hypersurface $\mathrm{S} = \{ v \in \mathbb{R}^{nm} \mid \langle \beta , T(v) \rangle = \alpha \}$. ∎

Proposition 3.4. (Separation of a polyconvex polytope and a rank-one ray) *Proposition 3.3. remains true if the segment is replaced by a ray* $\mathrm{R} = \{ w_0 + \mu\,(w_1 - w_0) \in \mathbb{R}^{nm} \mid \mu \geqslant 0 \}$ *with* $\mathrm{Rg}\,(w_1 - w_0) = 1$.

Proof. Since the precise representative of R is the set $\widetilde{\mathrm{Q}}_2 = \{ T(w_0) + \mu \big(T(w_1) - T(w_0) \big) \mid \mu \geqslant 0 \} = \{ T\big((1-\mu)\, w_0 + \mu\,(w_1 - w_0) \big) \mid \mu \geqslant 0 \}$, the proof of Proposition 3.3. can be repeated accordingly. ∎

Proposition 3.5. *1) For a general nonempty polyconvex set* $\mathrm{P} \subset \mathbb{R}^{nm}$*, Proposition 3.3., 1) remains true if either* $\mathrm{P} \cap \mathrm{R}$ *is empty or* $\mathrm{P} \cap \mathrm{R} = \{ w_0 \}$.

2) For a general compact polyconvex set $\mathrm{P} \subset \mathbb{R}^{nm}$*, Proposition 3.3., 2) remains true.*

3) Assertions 1) and 2) remain true if the segment is replaced by a ray $\mathrm{R} = \{ w_0 + \mu\,(w_1 - w_0) \in \mathbb{R}^{nm} \mid \mu \geqslant 0 \}$ *with* $\mathrm{Rg}\,(w_1 - w_0) = 1$.

Proof. Denote again by $\widetilde{\mathrm{Q}}_1$ and $\widetilde{\mathrm{Q}}_2$ the precise representatives of P and R. Now $V \in \widetilde{\mathrm{Q}}_1 \cap \widetilde{\mathrm{Q}}_2$ implies the existence of $\lambda_0, \lambda_1, \dots , \lambda_{\tau(n,m)+1} \in [\,0\,,\,1\,]$ with $\sum_{1 \leqslant i \leqslant \tau(n,m)+1} \lambda_i = 1$ such that

$$V \;=\; \sum_{i=1}^{\tau(n,m)+1} \lambda_i\, T(v_i) \;=\; (1-\lambda_0)\, T(w_0) + \lambda_0\, T(w_1) \;=\; T\big((1-\lambda_0)\, w_0 + \lambda_0\, w_1 \big)\,, \tag{3.6}$$

and the proof can be continued as above. ∎

c) Counterexamples for polyconvex separation.

By the first example, the possible extension of the weak as well as of the strong separation theorem to polyconvex sets is ruled out.

Counterexample 3.6. (A pair of disjoint polyconvex polytopes, which cannot be weakly separated) *Consider the points* $v_1 = \begin{pmatrix} 0 & 1 \\ 1 & -0.5 \end{pmatrix}$ *and* $v_2 = \begin{pmatrix} 0.5 & 0 \\ 0 & 2 \end{pmatrix}$. *The four-dimensional cube* $\mathrm{P}_1 = [\,0\,,\,1\,]^4 \subset \mathbb{R}^{2\times 2}$ *and the set* $\mathrm{P}_2 = \{ v_1, v_2 \} \subset \mathbb{R}^{2\times 2}$ *are polyconvex polytopes with* $\mathrm{int}\,(\mathrm{P}_1) \neq \emptyset$ *and* $\mathrm{P}_1 \cap \mathrm{co}\,(\mathrm{P}_2) \neq \emptyset$ *but* $\mathrm{P}_1 \cap \mathrm{P}_2 = \emptyset$. *Nevertheless, there exists no quasiaffine hypersurface* $\mathrm{S} = \{ v \in \mathbb{R}^{2\times 2} \mid \langle \beta , T(v) \rangle = \alpha \}$, *which separates* P_1 *and* P_2 *weakly.*

Proof. Example 2.15. shows that the convex cube P_1 is at the same time a polyconvex polytope. Obviously, $\mathrm{int}\,(\mathrm{P}_1) = (\,0\,,\,1\,)^4 \subset \mathbb{R}^{2\times 2}$ is nonempty. Since $\mathrm{Rg}\,(v_2 - v_1) = \mathrm{Rg} \begin{pmatrix} 0.5 & -1 \\ -1 & 2.5 \end{pmatrix} = 2$, we know from Example

---

11) [Schneider 93], p. 14, Theorem 1.3.8.

2.11. that $\{ v_1, v_2 \} = \mathrm{Pco}\{ v_1, v_2 \}$ forms a polyconvex polytope as well. It is clear that $\mathrm{P}_1 \cap \mathrm{P}_2 = \emptyset$ while $v_0 = 0.5\, v_1 + 0.5\, v_2 = \left(\begin{smallmatrix} 0.25 & 0.5 \\ 0.5 & 0.75 \end{smallmatrix}\right) \in \mathrm{co}\,(\mathrm{P}_1) \cap \mathrm{co}\,(\mathrm{P}_2)$. Observe now that $\det(v_1) = -1$ and $\det(v_2) = 1$. Consequently, the set

$$\widetilde{\mathrm{Q}}_2 = \Big\{ (1-\lambda) \begin{pmatrix} \left(\begin{smallmatrix} 0 & 1 \\ 1 & -0.5 \end{smallmatrix}\right) \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} \left(\begin{smallmatrix} 0.5 & 0 \\ 0 & 2 \end{smallmatrix}\right) \\ 1 \end{pmatrix} \in \mathbb{R}^{2\times 2} \times \mathbb{R} \;\Big|\; 0 \leqslant \lambda \leqslant 1 \Big\}, \tag{3.7}$$

which is the precise representative of P_2, contains the point $V_0 = \begin{pmatrix} \left(\begin{smallmatrix} 0.25 & 0.5 \\ 0.5 & 0.75 \end{smallmatrix}\right) \\ 0 \end{pmatrix} = 0.5\, V_1 + 0.5\, V_2$ with $V_1 = \begin{pmatrix} \left(\begin{smallmatrix} 0 & 1 \\ 1 & -0.5 \end{smallmatrix}\right) \\ -1 \end{pmatrix}$ and $V_2 = \begin{pmatrix} \left(\begin{smallmatrix} 0.5 & 0 \\ 0 & 2 \end{smallmatrix}\right) \\ 1 \end{pmatrix}$. On the other hand, the precise representative $\widetilde{\mathrm{Q}}_1$ of $\mathrm{P}_1 = \mathrm{Pco}\{ \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right), \ldots, \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right) \}$ contains the points $W_1 = \begin{pmatrix} \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right) \\ 0 \end{pmatrix} W_4 = \begin{pmatrix} \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \\ 0 \end{pmatrix}$, $W_2 = \begin{pmatrix} \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \\ 0 \end{pmatrix}$, $W_3 = \begin{pmatrix} \left(\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}\right) \\ 0 \end{pmatrix}$ and $W_4 = \begin{pmatrix} \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \\ 0 \end{pmatrix}$ with $V_0 = 0.25\,\big(W_1 + W_2 + W_3 + W_4 \big)$. Denoting $\mathrm{co}\{ W_1, W_2, W_3, W_4 \} = \widetilde{\mathrm{Q}}_0 \subset \widetilde{\mathrm{Q}}_1$, we conclude that $\mathrm{relint}\,(\widetilde{\mathrm{Q}}_0) \cap \mathrm{relint}\,(\widetilde{\mathrm{Q}}_2) \neq \emptyset$, and a proper separation of $\widetilde{\mathrm{Q}}_0$ and $\widetilde{\mathrm{Q}}_2$ within the space $\mathbb{R}^5$ is impossible. Consequently, for a weakly separating hyperplane $\langle \beta, V \rangle = \alpha$, $\beta \in \mathbb{R}^5$ and $\alpha \in \mathbb{R}$ must satisfy the linear system $\langle \beta, W_1 \rangle = \langle \beta, W_2 \rangle = \langle \beta, W_3 \rangle = \langle \beta, W_4 \rangle = \langle \beta, V_2 \rangle = \alpha$,[12] which is uniquely solved by $\beta^* = \big(-\alpha, \alpha, 0, \alpha, -0.5\,\alpha \big)$, $\alpha \neq 0$. However, for $W_5 = \begin{pmatrix} \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \\ 1 \end{pmatrix} \in \widetilde{\mathrm{Q}}_1$, $W_6 = \begin{pmatrix} \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \\ -1 \end{pmatrix} \in \widetilde{\mathrm{Q}}_1$, we obtain $\langle \beta^*, W_5 \rangle = -0.5\,\alpha$ and $\langle \beta^*, W_6 \rangle = 1.5\,\alpha$. Consequently, there exists not even a weakly separating hyperplane for $\widetilde{\mathrm{Q}}_1$ and $\widetilde{\mathrm{Q}}_2$. From Lemma 3.2., we conclude the non-existence of a weakly separating quasiaffine hypersurface S for P_1 and P_2. ∎

Our second example shows that the rank-one assumption in Propositions 3.3., 2) and 3.5., 2) is essential. In fact, we see that the strong separation of a pair of disjoint, compact and even arcwise connected polyconvex sets is, in general, impossible.

Counterexample 3.7. (A pair of disjoint, compact and arcwise connected polyconvex sets, which cannot be strongly separated) *Define the points* $v_1 = \left(\begin{smallmatrix} 1+a & 0.4 \\ 0.4 & 1+d \end{smallmatrix}\right)$ *and* $v_2 = \left(\begin{smallmatrix} 1-a & 0.4 \\ 0.4 & 1-d \end{smallmatrix}\right)$ *with numbers* $a = d = 0.4$. *The polyconvex polytope* $\mathrm{P}_1 = \mathrm{Pco}\{ \left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \}$ *and the convex polytope* $\mathrm{P}_2 = \{ (1-\lambda)\, v_1 + \lambda\, v_2 \;|\; 0 \leqslant \lambda \leqslant 1 \}$ *are compact, arcwise connected subsets of* $\mathbb{R}^{2\times 2}$ *with* $\mathrm{co}\,(\mathrm{P}_1) \cap \mathrm{P}_2 \neq \emptyset$ *but* $\mathrm{P}_1 \cap \mathrm{P}_2 = \emptyset$. *Nevertheless, there exists no quasiaffine hypersurface* $\mathrm{S} = \{ v \in \mathbb{R}^{2\times 2} \;|\; \langle \beta, T(v) \rangle = \alpha \}$, *which separates* P_1 *and* P_2 *strongly.*

Proof. From Example 2.13. we know that P_1 is an arcwise connected polyconvex polytope, thus being compact. The rank-two segment P_2 is polyconvex and compact as well. Note first that $v_0 = \left(\begin{smallmatrix} 1 & 0.4 \\ 0.4 & 1 \end{smallmatrix}\right)$ belongs to $\big(\mathrm{co}\,(\mathrm{P}_1) \setminus \mathrm{P}_1 \big) \cap \mathrm{P}_2$ since $v_0 = 0.5 \left(\begin{smallmatrix} 1 & 0.8 \\ 0 & 1 \end{smallmatrix}\right) + 0.5 \left(\begin{smallmatrix} 1 & 0 \\ 0.8 & 1 \end{smallmatrix}\right)$ since $\left(\begin{smallmatrix} 1 & 0.8 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0.8 & 1 \end{smallmatrix}\right) \in \mathrm{P}_1$, $v_0 = 0.5\, v_1 + 0.5\, v_2$ and $\mathrm{P}_1 \cap \mathrm{P}_2 \subseteq \{ \left(\begin{smallmatrix} 1 & 0.4 \\ 0.4 & 1 \end{smallmatrix}\right) \}$. A simple calculation yields that $\det(v_1) = 1.8$ and $\det(v_2) = 0.2$. The precise representative of P_1 is the set $\widetilde{\mathrm{Q}}_1 = \{ \begin{pmatrix} \left(\begin{smallmatrix} 1 & b \\ c & 1 \end{smallmatrix}\right) \\ 1 \end{pmatrix} \in \mathbb{R}^5 \;|\; |\, b + c\,| \leqslant 1 \}$ while the precise representative $\widetilde{\mathrm{Q}}_2$ of P_2 contains the segment $\{ (1-\lambda) \begin{pmatrix} v_1 \\ 1.8 \end{pmatrix} + \lambda \begin{pmatrix} v_2 \\ 0.2 \end{pmatrix} \;|\; 0 \leqslant \lambda \leqslant 1 \}$, which intersects $\widetilde{\mathrm{Q}}_1$ in the point $V_0 = \begin{pmatrix} v_0 \\ 1 \end{pmatrix} \in \mathrm{relint}\,(\widetilde{\mathrm{Q}}_1)$. Consequently, there exists no hyperplane $\widetilde{\mathrm{H}} = \{ V \in \mathbb{R}^{\tau(n,m)} \;|\; \langle \beta, V \rangle = \alpha \}$, which separates $\widetilde{\mathrm{Q}}_1$ and $\widetilde{\mathrm{Q}}_2$ strongly.[13] This implies the non-existence of a quasiaffine hypersurface $\mathrm{S} = \{ v \in \mathbb{R}^{2\times 2} \;|\; \langle \beta, T(v) \rangle = \alpha \}$, which separates P_1 and P_2 strongly. ∎

---

[12] The equation $\langle \beta, V_1 \rangle = \alpha$ is redundant.

[13] [Schneider 93], p. 13, Lemma 1.3.6.

4. Approximation of polyconvex sets by polyconvex polytopes.

a) Statement of the approximation theorem.

Definition 4.1. (Hausdorff distance for subsets of $\mathbb{R}^{nm}$) [14] *For nonempty, compact subsets* $\mathrm{A}', \mathrm{A}'' \subset \mathbb{R}^{nm}$*, we define the Hausdorff distance through*

$$\mathcal{H}(\mathrm{A}', \mathrm{A}'') = \mathrm{Max}\Big(\mathop{\mathrm{Max}}_{v' \in \mathrm{A}'} \mathrm{Dist}(v', \mathrm{A}''), \mathop{\mathrm{Max}}_{v'' \in \mathrm{A}''} \mathrm{Dist}(v'', \mathrm{A}') \Big). \tag{4.1}$$

Equivalently, it holds that $\mathcal{H}(\mathrm{A}', \mathrm{A}'') \leqslant \varepsilon \iff$ *for every* $v' \in \mathrm{A}'$ *there exists* $v'' \in \mathrm{A}''$ *with* $\| v' - v'' \| \leqslant \varepsilon$*, and for every* $v'' \in \mathrm{A}''$ *there exists* $v' \in \mathrm{A}'$ *with* $\| v'' - v' \| \leqslant \varepsilon$*, i. e.* $\mathrm{A}' \subseteq \mathrm{A}'' + \mathrm{K}(\mathfrak{o}, \varepsilon)$ *and* $\mathrm{A}'' \subseteq \mathrm{A}' + \mathrm{K}(\mathfrak{o}, \varepsilon)$*.*

Our goal is to find within the framework of polyconvex analysis an analogue to the following well-known theorem about convex bodies.

Theorem 4.2. (Approximation of convex bodies by convex polytopes) [15] *Given a convex body* $\mathrm{C} \subset \mathbb{R}^{nm}$*. Then for every* $\varepsilon > 0$ *there exist convex polytopes* $\mathrm{C}', \mathrm{C}'' \subset \mathbb{R}^{nm}$ *with* $\mathrm{C}' \subseteq \mathrm{C} \subseteq \mathrm{C}''$ *and Hausdorff distances* $\mathcal{H}(\mathrm{C}', \mathrm{C}) \leqslant \varepsilon$ *and* $\mathcal{H}(\mathrm{C}, \mathrm{C}'') \leqslant \varepsilon$*.*

Concerning compact polyconvex sets, we will prove the following assertion:

Theorem 4.3. (Approximation of compact polyconvex sets by polyconvex polytopes) *Assume that a set* $\mathrm{P} \subset \mathbb{R}^{nm}$ *is nonempty and compact. Then* P *is polyconvex iff for every* $\varepsilon > 0$ *there exist polyconvex polytopes* $\mathrm{P}', \mathrm{P}'' \subset \mathbb{R}^{nm}$ *with* $\mathrm{P}' \subseteq \mathrm{P} \subseteq \mathrm{P}''$ *and Hausdorff distances* $\mathcal{H}(\mathrm{P}', \mathrm{P}) \leqslant \varepsilon$ *and* $\mathcal{H}(\mathrm{P}, \mathrm{P}'') \leqslant \varepsilon$*.*

b) Proof of Theorem 4.3.

• **Step 1.** *Polyconvexity of a set, which admits inner and outer approximations by polyconvex polytopes.* Assume first that $\mathrm{P} \subset \mathbb{R}^{nm}$ is a compact set such that for every $\varepsilon > 0$ there exist polyconvex polytopes $\mathrm{P}'(\varepsilon)$, $\mathrm{P}''(\varepsilon) \subset \mathbb{R}^{nm}$ with $\mathrm{P}'(\varepsilon) \subseteq \mathrm{P} \subseteq \mathrm{P}''(\varepsilon)$ and Hausdorff distances $\mathcal{H}(\mathrm{P}'(\varepsilon), \mathrm{P}) \leqslant \varepsilon$ and $\mathcal{H}(\mathrm{P}, \mathrm{P}''(\varepsilon)) \leqslant \varepsilon$. Then the sets $\mathrm{P}'(\varepsilon)$, $\mathrm{P}''(\varepsilon)$, $0 < \varepsilon \leqslant 1$, and P are uniformly bounded by $R > 0$. Denote by $\mathcal{A}$ the system of all nonempty, closed subsets of $\mathrm{K}(\mathfrak{o}, R)$. Together with the Hausdorff distance, $\mathcal{A}$ forms a compact metric space, cf. [Schneider 93], p. 49, Theorems 1.8.2. and 1.8.3. From our assumption $\mathrm{P}'(\varepsilon) \subseteq \mathrm{P} \subseteq \mathrm{P}''(\varepsilon)$ for all $0 < \varepsilon \leqslant 1$, it follows that $\mathrm{Pco}(\mathrm{P}'(\varepsilon)) = \mathrm{P}'(\varepsilon) \subseteq \mathrm{P} \subseteq \mathrm{Pco}(\mathrm{P}) \subseteq \mathrm{Pco}(\mathrm{P}''(\varepsilon)) = \mathrm{P}''(\varepsilon)$ for all $0 < \varepsilon \leqslant 1$. Since $\mathrm{Pco}(\mathrm{P})$ is compact together with P, cf. (2.1), this implies the relations $\mathcal{H}(\mathrm{P}, \mathrm{Pco}(\mathrm{P})) \leqslant \mathcal{H}(\mathrm{P}'(\varepsilon), \mathrm{P}''(\varepsilon)) \leqslant 2\,\varepsilon$ for all $0 < \varepsilon \leqslant 1$. Consequently, we get $\mathcal{H}(\mathrm{P}, \mathrm{Pco}(\mathrm{P})) = 0$ and $\mathrm{P} = \mathrm{Pco}(\mathrm{P})$, and the set P is polyconvex.

• **Step 2.** *Construction of the inner approximation* P' *for a given polyconvex set.* Let now a nonempty, compact, polyconvex set $\mathrm{P} \subset \mathbb{R}^{nm}$ be given and fix $\varepsilon > 0$. Since P is compact, the covering $\{\, \mathrm{int}(\mathrm{K}(v, \varepsilon)) \,\}_{v \in \mathrm{P}}$ of P with open balls contains a finite subcovering $\mathrm{P} \subset \mathrm{int}(\mathrm{K}(v_1, \varepsilon)) \cup \ldots \cup \mathrm{int}(\mathrm{K}(v_N, \varepsilon)) \subset \mathrm{K}(v_1, \varepsilon) \cup \ldots \cup \mathrm{K}(v_N, \varepsilon)$. Define the polyconvex polytope $\mathrm{P}' = \mathrm{Pco}\{ v_1, \ldots, v_N \}$. Then it follows that

$$\{ v_1, \ldots, v_N \} \subseteq \mathrm{P} \subseteq \{ v_1, \ldots, v_N \} + \mathrm{K}(\mathfrak{o}, \varepsilon) \quad \Longrightarrow \tag{4.2}$$

$$\mathrm{P}' = \mathrm{Pco}\{ v_1, \ldots, v_N \} \subseteq \mathrm{Pco}(\mathrm{P}) = \mathrm{P} \subseteq \{ v_1, \ldots, v_N \} + \mathrm{K}(\mathfrak{o}, \varepsilon) \tag{4.3}$$

$$\subseteq \mathrm{Pco}\{ v_1, \ldots, v_N \} + \mathrm{K}(\mathfrak{o}, \varepsilon) = \mathrm{P}' + \mathrm{K}(\mathfrak{o}, \varepsilon) \subseteq \mathrm{P} + \mathrm{K}(\mathfrak{o}, \varepsilon) \tag{4.4}$$

and, consequently, $\mathcal{H}(\mathrm{P}', \mathrm{P}) \leqslant \varepsilon$.

[14] [Rockafellar/Wets 98], p. 117, Example 4.13.

[15] [Schneider 93], p. 54 f., Theorem 1.8.13., together with [Grünbaum 03], p. 316.

• **Step 3.** *Two lemmata about Minkowski multiplication with cubes.* We distinguish the cases $n \leqslant m$ (right multiplication) and $n > m$ (left multiplication).

Lemma 4.4. *Let $n \leqslant m$. Assume that a compact set $\mathrm{S} \subset \mathbb{R}^{nm}$ is given such that for all $v = \big(b,\, c\big) \in \big(\mathbb{R}^{n\times n} \times \mathbb{R}^{n\times(m-n)}\big) \cap \mathrm{S}$, the left (n,n)-submatrix b of v is regular. For $0 < \delta < \frac{1}{2}$, define the cube $\mathrm{W}_\delta \subset \mathbb{R}^{m\times m}$ by*

$$\mathrm{W}_\delta \;=\; \big\{\, E_m + w \in \mathbb{R}^{m\times m} \;\big|\; w_{1,1},\, \dots,\, w_{m,m} \in [\,\delta\,,\,\delta\,]\,\big\} \tag{4.5}$$

with the (m,m)-unit matrix E_m. Then we have

$$\mathrm{K}(v, C_1\,\delta) \subseteq v\,\mathrm{W}_\delta \;=\; \{\, v\cdot w \in \mathbb{R}^{nm} \;\big|\; w \in \mathrm{W}_\delta\,\} \;\subseteq\; \mathrm{K}(v, C_2\,\delta) \subset \mathbb{R}^{nm} \tag{4.6}$$

for all $v \in \mathrm{S}$ with constants $C_2 \geqslant C_1 > 0$ depending on S only.

Proof. We use the matrix norm $\|\,v\,\|^2 = \sum_{i,j}(\,v_{ij}\,)^2$, which is compatible with the Euclidean vector norm, cf. [Maess 84] , p. 51. Our assumption about S implies that $\mathfrak{o} \neq \mathrm{S}$. Consequently, there exist numbers R_1, R_2, S_1, $S_2 \in \mathbb{R}$ such that

$$0 \;<\; R_1 \;\leqslant\; \|\,b^{-1}\,\| \;\leqslant\; R_2 \quad \forall\, v = \big(b,\, c\big) \in \big(\mathbb{R}^{n\times n} \times \mathbb{R}^{n\times(m-n)}\big) \cap \mathrm{S}\,; \tag{4.7}$$

$$0 \;<\; S_1 \;\leqslant\; \|\,v\,\| \;\leqslant\; S_2 \quad \forall\, v \in \mathrm{S}\,. \tag{4.8}$$

For every $z \in \mathbb{R}^{nm}$, the system $v\,(E_m + w) = z \iff b\,w_1 + c\,w_2 = z - v$ admits a solution of the shape $w = \big(w_1,\, \mathfrak{o}\big) \in \mathbb{R}^{nm} \times \mathbb{R}^{(m-n)\times m}$. Thus we have $w_1 = b^{-1}\,(z - v)$ and

$$\|\,w\,\| \;=\; \|\,w_1\,\| \;=\; \|\,b^{-1}\,(z-v)\,\| \;\leqslant\; \|\,b^{-1}\,\|\;\|\,z - v\,\|\,, \tag{4.9}$$

and for every $z \in \mathbb{R}^{nm}$ satisfying $\|\,z - v\,\| \leqslant \delta/R_2 \leqslant \delta/\,\|\,b^{-1}\,\|$, we obtain a solution w such that $(E_m + w) \in \mathrm{W}_\delta$. Consequently, we get $\mathrm{K}(v, C_1\,\delta) \subseteq v\,\mathrm{W}_\delta$ for all $v \in \mathrm{S}$ where $C_1 = 1/R_2$. In order to establish the second relation, consider $w \in \mathbb{R}^{m\times m}$ with $-\delta \leqslant w_{kj} \leqslant \delta$, $1 \leqslant k,\, j \leqslant m$, and calculate

$$\|\,v - v\,(E_m + w)\,\|^2 = \|\,v\,w\,\|^2 = \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{m}\big(v_{ik}\,w_{kj}\big)^2 \leqslant \delta^2\cdot\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{m}\big(v_{ik}\big)^2 \;\leqslant\; m^2\,\delta^2\|\,v\,\|^2 \;\leqslant\; m^2\,S_2^2\,\delta^2\,.$$

Thus the second inequality $v\,\mathrm{W}_\delta \subseteq \mathrm{K}(v, C_2\,\delta)$ holds for all $v \in \mathrm{S}$ where $C_2 = m\,S_2$. ∎ (4.10)

In a completely analogous way, we may establish

Lemma 4.5. *Let $n > m$. Assume that a compact set $\mathrm{S} \subset \mathbb{R}^{nm}$ is given such that for all $v = \big(b,\, c\big)^{\mathrm{T}} \in \big(\mathbb{R}^{m\times m} \times \mathbb{R}^{(n-m)\times m}\big) \cap \mathrm{S}$, the upper (m,m)-submatrix b of v is regular. For $0 < \delta < \frac{1}{2}$, define the cube $\mathrm{W}_\delta \subset \mathbb{R}^{n\times n}$ by*

$$\mathrm{W}_\delta \;=\; \big\{\, E_n + w \in \mathbb{R}^{n\times n} \;\big|\; w_{1,1},\, \dots,\, w_{n,n} \in [\,\delta\,,\,\delta\,]\,\big\} \tag{4.11}$$

with the (n,n)-unit matrix E_n. Then we have

$$\mathrm{K}(v, C_1\,\delta) \;\subseteq\; W_\delta\, v \;=\; \{\, w\cdot v \in \mathbb{R}^{nm} \;\big|\; w \in \mathrm{W}_\delta\,\} \;\subseteq\; \mathrm{K}(v, C_2\,\delta) \subset \mathbb{R}^{nm} \tag{4.12}$$

for all $v \in \mathrm{S}$ with constants $C_2 \geqslant C_1 > 0$ depending on S only. ∎

• **Step 4.** *Construction of the outer approximation P'' for a given polyconvex set.* In order to apply the lemmata for Step 3, we translate P by an appropriate vector $v_0 \in \mathbb{R}^{nm}$ such that the respective submatrices of the elements of $\mathrm{P}+v_0$ become diagonally dominant. Without loss of generality, let us assume that $n \leqslant m$. In order to determine v_0, we invoke the following well-known lemma.

Lemma 4.6. [16] *If a matrix $b \in \mathbb{R}^{n\times n}$ satisfies the relations $|\,b_{ii}\,| > \sum_{k\neq i} |\,b_{ik}\,|$ for all $1 \leqslant i \leqslant n$ then b is regular with* $\det(b) > 0$.

We put $v_0 = \big(C_0\,E_n,\ \mathfrak{o}\big) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times(m-n)}$ where the constant $C_0 > 0$ is chosen such that

$$C_0^2 \;>\; n\cdot \mathrm{Max}\big(n,\, S_2^2\big) \quad\Longrightarrow\quad C_0^2 \;>\; n \sum_{i=1}^{n}\sum_{j=1}^{m} |\,v_{ij}\,|^2 \quad \forall\, v \in \mathrm{P}\,. \tag{4.13}$$

This implies

$$C_0^2 \;>\; n \sum_{i=1}^{n} |\,a_{1i}\,|^2 \;\geqslant\; \Big(\sum_{i=1}^{n} |\,a_{1i}\,|\Big)^2 \;\Longrightarrow\; a_{11} + C_0 \;>\; \sum_{j\neq 1} |\,a_{1j}\,| \quad \forall\, v = \big(a,\, d\big) \in \mathrm{P}\,; \tag{4.14}$$

$$\vdots$$

$$C_0^2 \;>\; n \sum_{i=1}^{n} |\,a_{ni}\,|^2 \;\geqslant\; \Big(\sum_{i=1}^{n} |\,a_{ni}\,|\Big)^2 \;\Longrightarrow\; a_{nn} + C_0 \;>\; \sum_{j\neq 1} |\,a_{nj}\,| \quad \forall\, v = \big(a,\, d\big) \in \mathrm{P}\,. \tag{4.15}$$

Consequently, for all $v = \big(a,\, d\big)$, the matrices $b = a + C_0\,E_n$ are regular by Lemma 4.6., and Lemma 4.4. may be applied to the set $\mathrm{S} = \mathrm{P} + v_0$. In relation to $\mathrm{P}+v_0$, we define the numbers $R_2,\, S_2 \in \mathbb{R}$ as in the proof of Lemma 4.4.

Fixing $\varepsilon > 0$, we define $\delta = \varepsilon/C_2 = \varepsilon/(\,m\,S_2\,)$. Since the set $\mathrm{P}+v_0$ is compact, the open covering $\{\,\mathrm{int}\,(K(v+v_0, C_1\,\delta))\,\}_{\,v\,\in\,\mathrm{P}}$ with $C_1\,\delta = \varepsilon/(m\,S_2\,R_2\,)$ contains a finite subcovering $\mathrm{P}+v_0 \subseteq \mathrm{int}\,(\mathrm{K}(v_1 + v_0, C_1\,\delta)) \cup \ldots \cup \mathrm{int}\,(\mathrm{K}(v_M + v_0, C_1\,\delta)) \subset \mathrm{K}(v_1+v_0, C_1\,\delta) \cup \ldots \cup \mathrm{K}(v_M+v_0, C_1\,\delta)$. From Lemma 4.4., we get

$$\mathrm{P}+v_0 \;\subset\; \mathrm{K}(v_1+v_0, C_1\,\delta) \cup \ldots \cup \mathrm{K}(v_M+v_0, C_1\,\delta) \;\subseteq\; (v_1+v_0)\,\mathrm{W}_\delta \cup \ldots \cup (v_M+v_0)\,\mathrm{W}_\delta \tag{4.16}$$

$$=\; \{\,v_1+v_0,\, \ldots,\, v_M+v_0\,\}\,\mathrm{W}_\delta \;\subseteq\; \mathrm{Pco}\,\{\,v_1+v_0,\, \ldots,\, v_M+v_0\,\}\,\mathrm{W}_\delta = \widetilde{\mathrm{P}}''\,. \tag{4.17}$$

By Example 2.15. and Proposition 2.9., W_δ is a polyconvex polytope, and by Proposition 2.10., $\widetilde{\mathrm{P}}''$ is a polyconvex polytope as well. Further, from the proof of Proposition 2.9. we get

$$\mathrm{Pco}\,\{\,v_1+v_0,\, \ldots,\, v_M+v_0\,\} \;=\; \mathrm{Pco}\,\{\,v_1,\, \ldots,\, v_M\,\} + v_0 \;\subseteq\; \mathrm{P}+v_0 \quad\Longrightarrow \tag{4.18}$$

$$\widetilde{\mathrm{P}}'' \;\subseteq\; \big(\,\mathrm{P}+v_0\,\big)\,\mathrm{W}_\delta \;\subseteq (\,\mathrm{P}+v_0\,) + \mathrm{K}(\mathfrak{o}, C_2\,\delta) \;=\; (\,\mathrm{P}+v_0\,) + \mathrm{K}(\mathfrak{o}, \varepsilon) \;\subset\; \widetilde{\mathrm{P}}'' + \mathrm{K}(\mathfrak{o}, \varepsilon)\,. \tag{4.19}$$

Thus we observe that $\mathcal{H}\,\big(\,\mathrm{P}+v_0, \widetilde{\mathrm{P}}''\,\big) \leqslant \varepsilon$. Since the Hausdorff distance is translation invariant, this implies the relation $\mathcal{H}\,(\mathrm{P}, \mathrm{P}'') \leqslant \varepsilon$ for $\mathrm{P}'' = \widetilde{\mathrm{P}}'' - v_0$. By Proposition 2.9., P'' is a polyconvex polytope together with $\widetilde{\mathrm{P}}''$. The case $n > m$ can be treated in complete analogy. ∎

c) An analogue to the Blaschke selection theorem.

Even the Blaschke theorem finds an analogue within the framework of polyconvex analysis.

Theorem 4.7. (Blaschke selection theorem for convex bodies) [17] *Assume that $\{\,\mathrm{C}_k\,\}$ is a uniformly bounded sequence of convex bodies $\mathrm{C}_k \subset \mathbb{R}^{nm}$. Then $\{\,\mathrm{C}_k\,\}$ contains a subsequence, which converges to a convex body $\mathrm{C} \subset \mathbb{R}^{nm}$ in Hausdorff distance.*

---

[16] [Taussky 49] , p. 674, Theorem IV.

[17] [Schneider 93] , p. 50, Theorem 1.8.6.

Theorem 4.8. (Compactness theorem for sequences of compact polyconvex sets) *Every uniformly bounded sequence* $\{\mathrm{P}_k\}$ *of nonempty, compact polyconvex sets* $\mathrm{P}_k \subset \mathbb{R}^{nm}$ *contains a subsequence, which converges to a nonempty compact polyconvex set* $\mathrm{P} \subset \mathbb{R}^{nm}$ *in Hausdorff distance.*

Proof. • **Step 1.** *Construction of approximating sequences of polyconvex polytopes and the limit element.* Using Theorem 4.3., we approximate every set P_k by polyconvex polytopes $\mathrm{P}'_k \subseteq \mathrm{P}_k \subseteq \mathrm{P}''_k$ with the distances $\mathcal{H}(\mathrm{P}'_k, \mathrm{P}_k) \leqslant 1/k$ and $\mathcal{H}(\mathrm{P}_k, \mathrm{P}''_k) \leqslant 1/k$, $k \in \mathbb{N}$. By assumption, there exists a uniform bound $R > 0$ such that $\mathrm{P}'_k \subseteq \mathrm{P}_k \subseteq \mathrm{P}''_k \subset \mathrm{K}(\mathfrak{o}, R)$ for all $k \in \mathbb{N}$. Together with the Hausdorff distance, the system $\mathcal{A}$ of all nonempty, compact subsets of $\mathrm{K}(\mathfrak{o}, R)$ forms a compact metric space. Consequently, $\{\mathrm{P}_k\}$ contains a subsequence (we will further use the index k), which converges to a set $\mathrm{P} \in \mathcal{A}$ in Hausdorff distance. Within this subsequence, we may further assume that $\mathcal{H}(\mathrm{P}_k, \mathrm{P}) \leqslant 1/k$. Then it follows that $\mathcal{H}(\mathrm{P}'_k, \mathrm{P}) \leqslant \mathcal{H}(\mathrm{P}'_k, \mathrm{P}_k) + \mathcal{H}(\mathrm{P}_k, \mathrm{P}) \leqslant 2/k$ as well as $\mathcal{H}(\mathrm{P}''_k, \mathrm{P}) \leqslant \mathcal{H}(\mathrm{P}''_k, \mathrm{P}_k) + \mathcal{H}(\mathrm{P}_k, \mathrm{P}) \leqslant 2/k$ for all $k \in \mathbb{N}$, and the subsequences $\{\mathrm{P}'_k\}$ and $\{\mathrm{P}''_k\}$ converge to P as well.

• **Step 2. Lemma 4.9.** *Consider two polyconvex polytopes* $\mathrm{P}', \mathrm{P}'' \in \mathcal{A}$*, the sets* $\mathrm{S}' = \{ T(v) \in \mathbb{R}^{\tau(n,m)} \mid v \in \mathrm{P}' \}$*,* $\mathrm{S}'' = \{ T(v) \in \mathbb{R}^{\tau(n,m)} \mid v \in \mathrm{P}'' \}$ *and the precise representatives* $\mathrm{co}(\mathrm{S}') = \widetilde{\mathrm{Q}}' \subset \mathbb{R}^{\tau(n,m)}$*,* $\mathrm{co}(\mathrm{S}'') = \widetilde{\mathrm{Q}}'' \subset \mathbb{R}^{\tau(n,m)}$*. Then it holds that* $\mathcal{H}(\mathrm{S}', \mathrm{S}'') \leqslant L \cdot \mathcal{H}(\mathrm{P}', \mathrm{P}'')$ *and* $\mathcal{H}(\widetilde{\mathrm{Q}}', \widetilde{\mathrm{Q}}'') \leqslant L \cdot \mathcal{H}(\mathrm{P}', \mathrm{P}'')$ *with a constant* $L > 0$.

Proof. The restriction of the mapping $T\colon \mathbb{R}^{nm} \to \mathbb{R}^{\tau(n,m)}$ to $\mathrm{K}(\mathfrak{o}, R)$ is uniformly Lipschitz with constant $L > 0$. For given $\varepsilon > 0$, $\mathcal{H}(\mathrm{P}', \mathrm{P}'') \leqslant \varepsilon$ implies that for a given point $V \in \mathrm{S}'$ with $V = T(v)$, $v \in \mathrm{P}'$, we find a point $w(v) \in \mathrm{P}''$ with $|\,w(v) - v\,| \leqslant \varepsilon \Longrightarrow |\,T(w(v)) - T(v)\,| \leqslant L\,\varepsilon$, and we get the relation $\mathrm{S}' \subseteq \mathrm{S}'' + \mathrm{K}(\mathfrak{o}, L\,\varepsilon)$. In the same way, for $W \in \mathrm{S}''$ with $W = T(w)$, $w \in \mathrm{P}''$, we find a point $v(w) \in \mathrm{P}'$ with $|\,v(w) - w\,| \leqslant \varepsilon \Longrightarrow |\,T(v(w)) - T(w)\,| \leqslant L\,\varepsilon$, and the relation $\mathrm{S}'' \subseteq \mathrm{S}' + \mathrm{K}(\mathfrak{o}, L\,\varepsilon)$ is true as well. Thus the relation $\mathcal{H}(\mathrm{S}', \mathrm{S}'') \leqslant L \cdot \mathcal{H}(\mathrm{P}', \mathrm{P}'')$ is confirmed.
Let us write $\mathrm{P}' = \mathrm{Pco}\{ v_1, \dots, v_N \}$ and $\mathrm{P}'' = \mathrm{Pco}\{ w_1, \dots, w_M \}$. $\mathrm{P}' \subseteq \mathrm{P}'' + \mathrm{K}(\mathfrak{o}, \varepsilon)$. If $\mathcal{H}(\mathrm{P}', \mathrm{P}'') \leqslant \varepsilon$ then we find a point $w(v_i) \in \mathrm{P}''$ with $|\,w(v_i) - v_i\,| \leqslant \varepsilon$ for every v_i, $1 \leqslant i \leqslant N$. Since a given point $V \in \widetilde{\mathrm{Q}}'$ may be written as a convex combination

$$V = \sum_{i=1}^{N} \lambda_i\, T(v_i) = \sum_{i=1}^{N} \lambda_i\, \big(T(v_i) - T(w(v_i)) \big) + \sum_{i=1}^{N} \lambda_i\, T(w(v_i)) \quad \Longrightarrow \tag{4.20}$$

$$\Big|\, V - \sum_{i=1}^{N} \lambda_i\, T(w(v_i)) \,\Big| \leqslant \sum_{i=1}^{N} \lambda_i \,\big|\, T(v_i) - T(w(v_i)) \,\big| \leqslant \sum_{i=1}^{N} \lambda_i\, L \,\big|\, v_i - w(v_i) \,\big| \leqslant L\,\varepsilon\,, \tag{4.21}$$

we obtain the relation $\widetilde{\mathrm{Q}}' \subseteq \widetilde{\mathrm{Q}}'' + \mathrm{K}(\mathfrak{o}, L\,\varepsilon)$. On the other hand, $\mathcal{H}(\mathrm{P}', \mathrm{P}'') \leqslant \varepsilon$ implies that $\mathrm{P}'' \subseteq \mathrm{P}' + \mathrm{K}(\mathfrak{o}, \varepsilon)$, and we find a point $v(w_j) \in \mathrm{P}'$ with $|\,v(w_j) - w_j\,| \leqslant \varepsilon$ for every w_j, $1 \leqslant j \leqslant M$. Thus any given point $W \in \widetilde{\mathrm{Q}}''$ may be written as a convex combination

$$W = \sum_{j=1}^{M} \mu_j\, T(w_j) = \sum_{j=1}^{M} \mu_j\, \big(T(w_j) - T(v(w_j)) \big) + \sum_{j=1}^{M} \mu_j\, T(v(w_j)) \quad \Longrightarrow \tag{4.22}$$

$$\Big|\, W - \sum_{j=1}^{M} \mu_j\, T(v(w_j)) \,\Big| \leqslant \sum_{j=1}^{M} \mu_j \,\big|\, T(w_j) - T(v(w_j)) \,\big| \leqslant \sum_{j=1}^{M} \mu_j\, L \,\big|\, w_j - v(w_j) \,\big| \leqslant L\,\varepsilon\,, \tag{4.23}$$

and the reverse inclusion $\widetilde{\mathrm{Q}}'' \subseteq \widetilde{\mathrm{Q}}' + \mathrm{K}(\mathfrak{o}, L\,\varepsilon)$ holds as well. Summing up, we obtain the claimed relation $\mathcal{H}(\widetilde{\mathrm{Q}}', \widetilde{\mathrm{Q}}'') \leqslant L \cdot \mathcal{H}(\mathrm{P}', \mathrm{P}'')$. ∎

Let us define $\mathrm{S}'_k = \{ T(v) \in \mathbb{R}^{\tau(n,m)} \mid v \in \mathrm{P}'_K \}$ and $\mathrm{S}''_k = \{ T(v) \in \mathbb{R}^{\tau(n,m)} \mid v \in \mathrm{P}''_k \}$. Consequently, we have $\mathrm{S}'_k \subseteq \mathrm{S}''_k$, $\widetilde{\mathrm{Q}}'_k = \mathrm{co}(\mathrm{S}'_k)$ and $\widetilde{\mathrm{Q}}''_k = \mathrm{co}(\mathrm{S}''_k)$ for all $k \in \mathbb{N}$.

• **Step 3.** *Convergent subsequences of* $\{\,\mathrm{S}'_k\,\}$, $\{\,\mathrm{S}''_k\,\}$, $\{\,\widetilde{\mathrm{Q}}'_k\,\}$ *and* $\{\,\widetilde{\mathrm{Q}}''_k\,\}$. Together with $\{\,\mathrm{P}'_k\,\}$ and $\{\,\mathrm{P}''_k\,\}$, the sequences $\{\,\mathrm{S}'_k\,\}$, $\{\,\mathrm{S}''_k\,\}$, $\{\,\widetilde{\mathrm{Q}}'_k\,\}$ and $\{\,\widetilde{\mathrm{Q}}''_k\,\}$ are uniformly bounded in $\mathbb{R}^{\tau(n,m)}$. Moreover, by Lemma 4.9., they are Cauchy sequences with respect to the Hausdorff distance on $\mathbb{R}^{\tau(n,m)}$. Consequently, we get subsequences, which converge to sets S', S'', C', $\mathrm{C}'' \subset \mathbb{R}^{\tau(n,m)}$, respectively (we will not change the index k). From the Blaschke selection theorem (Theorem 4.7.), we deduce that C' and C'' are convex bodies. Since

$$\mathcal{H}\,(\mathrm{S}'_k, \mathrm{S}''_k) \leqslant L \cdot \mathcal{H}\,(\mathrm{P}'_k, \mathrm{P}''_k) \leqslant 2\,L/k \quad \forall\, k \in \mathbb{N} \quad \text{and} \tag{4.24}$$

$$\mathcal{H}\,(\widetilde{\mathrm{Q}}'_k, \widetilde{\mathrm{Q}}''_k) \leqslant L \cdot \mathcal{H}\,(\mathrm{P}'_k, \mathrm{P}''_k) \leqslant 2\,L/k \quad \forall\, k \in \mathbb{N}\,, \tag{4.25}$$

we find $\mathrm{S}' = \mathrm{S}'' = \mathrm{S}$ and $\mathrm{C}' = \mathrm{C}'' = \mathrm{C}$. By [Rockafellar/Wets 98], p. 128, Proposition 4.30. (b), the convergence relation $\mathrm{S}'_k \to \mathrm{S}$ implies $\mathrm{co}\,(\mathrm{S}'_k) = \widetilde{\mathrm{Q}}'_k \to \mathrm{co}\,(\mathrm{S})$, and $\mathrm{C} = \mathrm{co}\,(\mathrm{S})$.

• **Step 4.** *The set* C *is a convex representative for* P. We abbreviate $\mathrm{A} = \{\, v \in \mathbb{R}^{nm} \;\big|\; T(v) \in \mathrm{C}\,\}$. Consider a point $v \in \mathrm{P}$. By definition of P, v is obtained as the limit of a sequence $\{\, v_k \,\}$ with $v_k \in \mathrm{P}'_k$ and $|\, v_k - v\,| \leqslant 1/k$ for all $k \in \mathbb{N}$. By Lipschitz continuity of T, we get $|\, T(v_k) - T(v)\,| \leqslant L \cdot |\, v_k - v\,| \leqslant L/k$ and $T(v_k) \in \widetilde{\mathrm{Q}}'_k$ for all $k \in \mathbb{N}$. Consequently, we get $T(v) \in \mathrm{C}$, $v \in \mathrm{A}$ and $\mathrm{P} \subseteq \mathrm{A}$. On the other hand, assume that $v \in \mathrm{A} \iff T(v) \in \mathrm{C} \iff T(v) \in \mathrm{S}$ by Step 3. By definition of S, $T(v)$ is obtained as the limit of a sequence $\{\, V_k \,\}$ with $V_k \in \mathrm{S}'_k \iff V_k = T(v_k)$ with $v_k \in \mathrm{P}'_k$ and $|\, V_k - T(v)\,| = |\, T(v_k) - T(v)\,| \leqslant 1/k$. Since the projection T^{-1} is Lipschitz with constant 1, this implies $|\, v_k - v\,| \leqslant 1/k$, and v is the limit of a sequence $\{\, v_k \,\}$ with $v_k \in \mathrm{P}'_k$ for all $k \in \mathbb{N}$. Consequently, v belongs to P, and $\mathrm{A} \subseteq \mathrm{P}$. Summing up, P is polyconvex, admitting C as its convex representative, and the proof is complete. ∎

Corollary 4.10. (Transformation of arbitrary compact polyconvex sets) *If* $\mathrm{P} \subset \mathbb{R}^{nm}$ *is a nonempty, compact polyconvex set then the sets* $\mu\,\mathrm{P}$ *and* $\mathrm{P} + w$ *are compact and polyconvex for all* $\mu \geqslant 0$ *and* $w \in \mathbb{R}^{nm}$.

Proof. By Theorem 4.3., we find for every $k \in \mathbb{N}$ polyconvex polytopes P'_k, $\mathrm{P}''_k \subset \mathbb{R}^{nm}$ with $\mathrm{P}'_k \subseteq \mathrm{P} \subseteq \mathrm{P}''_K$, $\mathcal{H}\,(\mathrm{P}'_k, \mathrm{P}) \leqslant 1/k$ and $\mathcal{H}\,(\mathrm{P}, \mathrm{P}''_k) \leqslant 1/k$ for all $k \in \mathbb{N}$. Obviously, we have $\mathrm{P}'_k \to \mathrm{P}$ and $\mathrm{P}''_k \to \mathrm{P}$ in Hausdorff distance. Since convergence of uniformly bounded sequences in Hausdorff distance is equivalent to convergence in the sense of Painlevé-Kuratowski, we deduce that $\mu\,\mathrm{P}'_k \to \mu\,\mathrm{P}$ and $\mathrm{P}'_k + w \to \mathrm{P} + w$. By Proposition 2.9., $\mu\,\mathrm{P}'_k$ and $\mathrm{P}'_k + w$ are polyconvex polytopes together with P'_k. Now Theorem 4.8. ensures the polyconvexity of the compact limit sets $\mu\,\mathrm{P}$ and $\mathrm{P} + w$. ∎

Acknowledgements.

I am indebted to Prof. B. Dacorogna, who invited me to Lausanne for a post-doc stay in 2007/08 and strongly directed my attention to the investigation of polyconvex geometry during this time. The present work has been finished in 2014 within the project “Relaxation theorems and necessary optimality conditions for semiconvex multidimensional control problems”, which has been supported by the German Research Council.

References.

01. [Ball 77] Ball, J. M.: *Convexity conditions and existence theorems in nonlinear elasticity.* Arch. Rat. Mech. Anal. **63** (1977), 337 – 403

02. [Bevan 06] Bevan, J.: *On convex representatives of polyconvex functions.* Proc. Roy. Soc. Edinburgh Sect. A **126** (2006), 23 – 51

03. [Dacorogna 08] Dacorogna, B.: *Direct Methods in the Calculus of Variations.* Springer; New York etc. 2008, 2nd ed.

04. [DACOROGNA/RIBEIRO 06] Dacorogna, B.; Ribeiro, A. M.: *On some definitions and properties of generalized convex sets arising in the calculus of variations.* In: Chipot, M.; Ninomiya, H. (Eds.): Recent Advances on Elliptic and Parabolic Issues. Proceedings of the 2004 Swiss-Japanese Seminar: Zurich, Switzerland, 6-10 December 2004. World Scientific; New Jersey etc. 2006, 103 – 128

05. [GRÜNBAUM 03] Grünbaum, B.: *Convex Polytopes.* Springer; New York etc. 2003, 2nd ed.

06. [MAESS 84] Maess, G.: *Vorlesungen über numerische Mathematik I.* Akademie-Verlag; Berlin 1984

07. [MORREY 52] Morrey, C. B.: *Quasi-convexity and the lower semicontinuity of multiple integrals.* Pacific J. Math. **2** (1952), 25 – 53

08. [MÜLLER 99] Müller, S.: *Variational models for microstructure and phase transitions.* In: Bethuel, F.; Huisken, G.; Müller, S.; Steffen, K.: Calculus of Variations and Geometric Evolution Problems. Springer; Berlin - ... 1999 (Lecture Notes in Mathematics 1713), 85 – 210

09. [ROCKAFELLAR/WETS 98] Rockafellar, R. T.; Wets, R. J.-B.: *Variational Analysis.* Springer; Berlin etc. 1998 (Grundlehren 317)

10. [SCHNEIDER 93] Schneider, R.: *Convex Bodies: The Brunn-Minkowski Theory.* Cambridge University Press; Cambridge 1993

11. [TAUSSKY 49] Taussky, O.: *A recurring theorem on determinants.* Amer. Math. Monthly **56** (1949), 672 – 676

12. [WAGNER 10] Wagner, M.: *Elastic image registration in presence of polyconvex constraints.* Karl-Franzens-Universität Graz, SFB-Report No. 2010-033. Status: accepted (Proceedings of the International Workshop on Optimal Control in Image Processing, Heidelberg, Germany, May 31 - June 1, 2010)

13. [WAGNER 11] Wagner, M.: *Quasiconvex relaxation of multidimensional control problems with integrands $f(t, \xi, v)$.* ESAIM: Control, Optimisation and Calculus of Variations **17** (2011), 190 – 221

14. [WAGNER 12] Wagner, M.: *A direct method for the solution of an optimal control problem arising from image registration.* Numerical Algebra, Control and Optimization **2** (2012), 487 – 510

15. [WAGNER 14] Wagner, M.: *Pontryagin's principle for Dieudonné-Rashevsky type problems with polyconvex data.* Universität Leipzig, Preprint-Reihe des Mathematischen Instituts, Preprint Nr. 01/2014. To appear: Houston J. Math.

16. [ZHANG 98] Zhang, K.: *On various semiconvex hulls in the calculus of variations.* Calc. Var. Part. Diff. Eq. **6** (1998), 143 – 160

Last modification: 29. 06. 2014

Author's address. *Marcus Wagner:* University of Leipzig, Department of Mathematics, P. O. B. 10 09 20, D-04009 Leipzig, Germany. Homepage / e-mail: `www.thecitytocome.de` / `marcus.wagner@math.uni-leipzig.de`