Sobolev regularity of multipliers in multidimensional control problems of Dieudonné-Rashevsky type

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1. The regularity result.

In [Wagner 09] and [Wagner 14], it has been shown that global as well as strong local minimizers (x^*, u^*) of multidimensional control problems of Dieudonné-Rashevsky type

$$F(x,u) \;=\; \int_\Omega f(s,x(s),u(s))\,ds \longrightarrow \inf!\,; \quad (x,u) \in W_0^{1,p}(\Omega,\mathbb{R}^n) \,\times\, L^p(\Omega,\mathbb{R}^{nm})\,; \tag{1.1}$$

$$Jx(s) \;=\; \begin{pmatrix} \partial x_1(s)/\partial s_1 & \dots & \partial x_1(s)/\partial s_m \\ \vdots & & \vdots \\ \partial x_n(s)/\partial s_1 & \dots & \partial x_n(s)/\partial s_m \end{pmatrix} - u(s) \;=\; \mathfrak{o}_{L^p}\,; \quad u(s) \in \mathrm{A} \subset \mathbb{R}^{nm} \quad \text{for a. a. } s \in \Omega\,; \tag{1.2}$$

with n, $m \geqslant 2$, $\Omega \subset \mathbb{R}^m$, $m < p < \infty$ and a compact set $\mathrm{A} \subset \mathbb{R}^{nm}$ with nonempty interior satisfy necessary optimality conditions in the form of Pontryagin's principle provided that the data of (1.1) – (1.2) fit into a convex or polyconvex framework, cf. [Wagner 09], p. 549, Theorem 2.2., and [Wagner 14], p. 9, Theorem 4.3., and p. 21, Theorem 5.4. In particular, this set of conditions contains a canonical equation in integrated form, involving multipliers $\lambda_0 \geqslant 0$ and $y^{(1)} \in L^{p/(p-1)}(\Omega,\mathbb{R}^{nm})$, which reads as follows:

$$\lambda_0 \sum_{i=1}^n \int_\Omega \frac{\partial f}{\partial \xi_i}(\,s,x^*(s),u^*(s)\,)\,\varphi_i(s)\,ds \,+\, \sum_{i=1}^n \sum_{j=1}^m \int_\Omega \frac{\partial \varphi_i}{\partial s_j}(s)\,y^{(1)}_{i,j}(s)\,ds \;=\; 0 \quad \forall\,\varphi \in W_0^{1,p}(\Omega,\mathbb{R}^n)\,. \tag{1.3}$$

Inserting vector functions, which differ from $\mathfrak{o}$ in a single component only, we find that (1.3) implies n first-oder PDE's

$$\langle\, \lambda_0 \frac{\partial f}{\partial \xi_i}(\,s,x^*(s),u^*(s)\,)\,,\,\varphi\,\rangle \;=\; -\,\langle\, \sum_{j=1}^m \frac{\partial y^{(1)}_{i,j}}{\partial s_j}\,,\,\varphi\,\rangle \quad \forall\,\varphi \in C_0^\infty(\Omega,\mathbb{R})\,,\; 1 \leqslant i \leqslant n\,, \tag{1.4}$$

in which the partial derivatives of $y^{(1)}_{i,j}$ have to be understood in the sense of Schwartz distributions. With regard to applications of (1.1) – (1.2) in mathematical imaging (see [Brune/Maurer/Wagner 09], [Franek/Franek/Maurer/Wagner 12] and [Wagner 12]), we prove a refinement of the maximum principle, claiming that the multipliers $y^{(1)}$ can be chosen from an appropriate Sobolev space rather than from $L^{p/(p-1)}(\Omega,\mathbb{R}^{nm})$, thus ensuring within (1.4) the separate existence of the generalized derivatives $\partial y^{(1)}_{i,j}/\partial s_j$ as L^r-functions. Since the divergence as their sum may be an element of an L^r-space but the summands do not (cf. [Bourdaud/Wojciechowski 00], p. 326), the proof of this claim is nontrivial. For simplicity, we state the theorem in a special case only, assuming dimensions $n = m = 2$, polyconvexity of the integrand f (cf. [Dacorogna 08], p. 156 f., Definition 5.1.(iii)) and convexity of the restriction set A. The extension of the theorem and its proof to the general case covered in [Wagner 09] and [Wagner 14] is obvious.

Theorem 1.1. *Consider the problem* (1.1) – (1.2) *with* $n = m = 2$ *under the assumptions mentioned in* [Wagner 14]*, p. 18, Theorem 4.11., and choose for the polyconvex integrand* $f(s,\xi,v)$ *a convex representative* $g(s,\xi,v,\omega_2)$ *in accordance with these assumptions. Further, let* $\mathrm{A} = \mathrm{K} \subset \mathbb{R}^4$ *be a compact, convex set with* $\mathfrak{o} \in \mathrm{int}\,(\mathrm{K})$*. If* (x^*, u^*) *is a global minimizer of the problem then there exist multipliers* $\lambda_0 > 0$*,*

$y^{(1)} \in W^{p/(p-1),1}(\Omega, \mathbb{R}^4)$ and $y^{(2)} \in L^{p/(p-2)}(\Omega, \mathbb{R})$ such that the maximum condition [Wagner 14]*, p. 9,* (4.6)*, is satisfied together with the canonical equations*

$$(\mathcal{K})_1 \quad \operatorname{div} y_1^{(1)}(s) = \lambda_0 \frac{\partial g}{\partial \xi_1}(\, s, x^*(s), u^*(s), \det u^*(s)\,) \quad \textit{for almost all } s \in \Omega\,; \tag{1.5}$$

$$(\mathcal{K})_2 \quad \operatorname{div} y_2^{(1)}(s) = \lambda_0 \frac{\partial g}{\partial \xi_2}(\, s, x^*(s), u^*(s), \det u^*(s)\,) \quad \textit{for almost all } s \in \Omega\,. \tag{1.6}$$

2. Proof of Theorem 1.1.

Let $q_1 = p/(p-1)$ and $q_2 = p/(p-2)$. The proof of the maximum principle is based on the separation of two convex variational sets C and D within the space $\mathbb{R} \times L^p(\Omega, \mathbb{R}^4) \times L^{p/2}(\Omega, \mathbb{R})$, cf. [Wagner 14] , pp. 10 – 16. The desired gain of regularity for the multiplier $y^{(1)}$ will be obtained by replacing $L^p(\Omega, \mathbb{R}^4)$, the original target space of the state equation (1.2), by the dual space $\big(\, W^{1,q_1}(\Omega, \mathbb{R}^4)\,\big)^* \hookleftarrow \big(\, L^{q_1}(\Omega, \mathbb{R}^4)\,\big)^* \cong L^p(\Omega, \mathbb{R}^4)$. It turns out that the separation argument still works in this extended framework. Recall that any functional $Z \in \big(\, W^{1,q_1}(\Omega, \mathbb{R})\,\big)^*$ may be represented as

$$\langle\, Z\,,\, \psi\,\rangle_{(W^{1,q_1})^* - W^{1,q_1}} = \int_\Omega \big(\, Z_0\, \psi + Z_1 \frac{\partial \psi}{\partial s_1} + Z_2 \frac{\partial \psi}{\partial s_2}\,\big)\, ds \text{ with } Z_0,\, Z_1,\, Z_2 \in \big(\, L^{q_1}(\Omega, \mathbb{R})\,\big)^* \cong L^p(\Omega, \mathbb{R}), \tag{2.1}$$

cf. [Adams/Fournier 07] , p. 62 f., Theorem 3.9.. Consequently, $L^p(\Omega, \mathbb{R}^4)$ may be continuously imbedded into $\big(\, W^{1,q_1}(\Omega, \mathbb{R}^4)\,\big)^* \cong \big(\, \big(\, L^{q_1}(\Omega,\ \mathbb{R}^4)\,\big)^*\,\big)^3 \cong \big(\, L^p(\Omega,\ \mathbb{R}^4)\,\big)^3$ by $z \longmapsto (z, \mathfrak{o}, \mathfrak{o})$. Since $1 < q_1 < \infty$, $\big(\, W^{1,q_1}(\Omega, \mathbb{R}^4)\,\big)^*$ is a reflexive Banach space. In relation to a global minimizer (x^*, u^*) of (1.1) – (1.2), we define the sets

$$\mathrm{C} \,=\, \Big\{\, \big(\, \varrho,\, z_1,\, z_2\,\big) \in \mathbb{R} \times \Big(\, L^p(\Omega, \mathbb{R}^4) \,\cap\, \big(\, W^{1,q_1}(\Omega, \mathbb{R}^4)\,\big)^*\,\Big) \times L^{p/2}(\Omega, \mathbb{R}) \quad \text{with} \tag{2.2}$$

$$\varrho = \varepsilon + D_x G(x^*, u^*, w^*)(x - x^*) + D_u G(x^*, u^*, w^*)(u - u^*) + D_w G(x^*, u^*, w^*)(w - w^*)\,; \tag{2.3}$$

$$z_1 = Jx - Jx^* - (u - u^*)\,; \quad z_2 = (w_2 - w_2^*) - D_u \det(u^*)(u - u^*)\,; \tag{2.4}$$

$$\varepsilon \geqslant 0\,,\ x \in W_0^{1,p}(\Omega, \mathbb{R}^n)\,,\ u \in \mathrm{U}\,,\ w_2 \in L^{p/2}(\Omega, \mathbb{R})\,\Big\}\,; \tag{2.5}$$

$$\widetilde{\mathrm{C}}_\eta \,=\, \Big\{\, \big(\, \varrho,\, z_1,\, z_2\,\big) \in \mathbb{R} \times \big(\, W^{1,q_1}(\Omega, \mathbb{R}^4)\,\big)^* \times \big(\, W^{1,q_2}(\Omega, \mathbb{R})\,\big)^* \quad \text{with} \tag{2.6}$$

$$z_1 = Jx - Jx^* - (u - u^*)\,,\ \|\, z_1\,\|_{(W^{1,q_1})^*} \leqslant \eta\,; \tag{2.7}$$

$$z_2 = (w_2 - w_2^*) - \widetilde{D}_u \det(u^*)(u - u^*)\,,\ \|\, z_2\,\|_{(W^{1,q_2})^*} \leqslant \eta\,; \tag{2.8}$$

$$x \in W_0^{1,p}(\Omega, \mathbb{R}^2)\,,\ u \in \mathrm{U}_0 + \mathrm{K}(\mathfrak{o}, \eta) \subset \big(\, W^{1,q_1}(\Omega, \mathbb{R}^4)\,\big)^*\,,\ w_2 \in \big(\, W^{1,q_1)}(\Omega, \mathbb{R})\,\big)^*\,\Big\}\,, \quad \eta \geqslant 0\,; \tag{2.9}$$

$$\widetilde{\mathrm{D}} \,=\, \Big\{\, \big(\, \varrho,\, z_1,\, z_2\,\big) \in \mathbb{R} \times \big(\, W^{1,q_1}(\Omega, \mathbb{R}^4)\,\big)^* \times L^{p/2}(\Omega, \mathbb{R}) \text{ with} \tag{2.10}$$

$$\varrho < 0\,;\ \ z_1 \in \mathrm{K}(\mathfrak{o}, \tfrac{1}{2}\,|\,\varrho\,|/K_0) \subset \big(\, W^{1,q_1}(\Omega, \mathbb{R}^4)\,\big)^*\,;\ \ z_2 \in \mathrm{K}(\mathfrak{o}, \tfrac{1}{2}\,|\,\varrho\,|/K_0) \subset L^{p/2}(\Omega, \mathbb{R})\,\Big\} \tag{2.11}$$

where

$$\mathrm{U} \,=\, \{\, z_1 \in L^p(\Omega, \mathbb{R}^4) \,\big|\, z_1(s) \in \mathrm{A} = \mathrm{K} \text{ for almost all } s \in \Omega\,\}\,, \tag{2.12}$$

$$\mathrm{U}_0 \,=\, \mathrm{U} \cap \{\, z_1 \in L^p(\Omega, \mathbb{R}^4) \,\big|\, \exists\, x \in W_0^{1,p}(\Omega, \mathbb{R}^2) \text{ such that } z_1 = Jx\,\} \tag{2.13}$$

and $\widetilde{D}_u \det(u^*)\colon \big(\, W^{1,q_1}(\Omega, \mathbb{R}^4)\,\big)^* \to \big(\, W^{1,q_2}(\Omega, \mathbb{R}^4)\,\big)^*$ is the natural extension of the linear, continuous Gâteaux derivative operator $D_u \det(u^*)\colon L^p(\Omega, \mathbb{R}^4) \to L^{p/2}(\Omega, \mathbb{R})$. For given $Z \in \big(\, W^{1,q_1}(\Omega,\ \mathbb{R}^4)\,\big)^*$, $Z \cong (Z_0, Z_1, Z_2) \in \big(\, L^p(\Omega,\ \mathbb{R}^4)\,\big)^3$, the output of $\widetilde{D}_u \det(u^*)\,(Z)$ acts as a linear, continuous functional on $\psi \in W^{1,q_2}(\Omega, \mathbb{R}^4)$ in the following way:

$$\langle\, \widetilde{D}_u T_2(u^*)\,(Z)\,,\,\psi\,\rangle_{(\,W^{1,q_2}\,)^*-W^{1,q_2}} \;=\; \langle\, \widetilde{D}_u T_2(u^*)\big(\,Z_0,\,Z_1,\,Z_2\,\big)\,,\,\psi\,\rangle_{(\,W^{1,q_2}\,)^*-W^{1,q_2}} \tag{2.14}$$

$$= \int_\Omega \Big(\, u_4^*\,Z_{0,1}\,\psi_1 - u_3^*\,Z_{0,2}\,\psi_2 - u_2^*\,Z_{0,3}\,\psi_3 + u_1^*\,Z_{0,4}\,\psi_4\,\Big)\,ds \;+\; \int_\Omega \Big(\, u_4^*\,Z_{1,1}\,\frac{\partial\psi_1}{\partial s_1} - u_3^*\,Z_{1,2}\,\frac{\partial\psi_2}{\partial s_1}$$

$$-\; u_2^*\,Z_{1,3}\,\frac{\partial\psi_3}{\partial s_1} + u_1^*\,Z_{1,4}\,\frac{\partial\psi_4}{\partial s_1}\,\Big)\,ds \;+\; \int_\Omega \Big(\, u_4^*\,Z_{2,1}\,\frac{\partial\psi_1}{\partial s_2} - u_3^*\,Z_{2,2}\,\frac{\partial\psi_2}{\partial s_2} - u_2^*\,Z_{2,3}\,\frac{\partial\psi_3}{\partial s_2} + u_1^*\,Z_{2,4}\,\frac{\partial\psi_4}{\partial s_2}\,\Big)\,ds\,.$$

$K_0 > 0$ is chosen according to the following assertion (ii). It holds still true that (i) C and $\widetilde{\mathrm{D}}$ are nonempty, convex variational sets with $\mathrm{int}\,(\mathrm{D}) \neq \emptyset$, cf. [Wagner 14] , p. 10, Proposition 4.5.; (ii) $\big(\,\varrho,\,z_1,\,z_2\,\big) \in \mathrm{C} \cap \widetilde{\mathrm{C}}_\eta$ implies that $\varrho \geqslant -K_0\,\eta$ where $K_0 > 0$ is a constant independent of η, cf. [Wagner 14] , p. 11, Proposition 4.7., and (iii) $\widetilde{\mathrm{D}}$ is a subset of $\widetilde{\mathrm{C}}_\eta$ with $\eta = \frac{1}{2}\,|\,\varrho\,|/K_0$, cf. [Wagner 14] , p. 11, Proposition 4.6.
Consequently, $\mathrm{C} \cap \widetilde{\mathrm{D}} = \emptyset$, and C and $\widetilde{\mathrm{D}}$ may be weakly separated within the space $\mathbb{R} \times \big(\,W^{1,q_1}(\Omega,\mathbb{R}^4)\,\big)^* \times L^{p/2}(\Omega,\mathbb{R})$ by a nontrivial linear, continuous functional $(\lambda_0, y^{(1)}, y^{(2)})$. Thus $y^{(1)}$ gains the claimed Sobolev regularity. ∎

Remark 2.1. The nonexistence case from [Bourdaud/Wojciechowski 00] cannot occur since the assumed growth condition [Wagner 14] , p. 17, (4.96), guarantees that $\partial g(s, x^*, u^*, \det u^*\,)/\partial\xi_i \in L^{p/(p-1)}(\Omega, \mathbb{R})$, $1 \leqslant i \leqslant 2$, with $1 < p/(p-1) < \infty$.

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