Sobolev regularity of multipliers in multidimensional control problems of Dieudonné-Rashevsky type

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1. The regularity result.

In [Wagner 09] and [Wagner 14], it has been shown that global as well as strong local minimizers \((x^*, u^*)\) of multidimensional control problems of Dieudonné-Rashevsky type

\[
F(x, u) = \int_{\Omega} f(s, x(s), u(s)) \, ds \longrightarrow \inf; \quad (x, u) \in W_0^{1,p}(\Omega, \mathbb{R}^n) \times L^p(\Omega, \mathbb{R}^{nm});
\]

\[
J x(s) = \left( \frac{\partial x_1(s)}{\partial s_1}, \ldots, \frac{\partial x_n(s)}{\partial s_n} \right) - u(s) = 0; \quad u(s) \in A \subset \mathbb{R}^{nm} \quad \text{for a. a. } s \in \Omega;
\]

with \(n, m \geq 2, \Omega \subset \mathbb{R}^m, m < p < \infty\) and a compact set \(A \subset \mathbb{R}^{nm}\) with nonempty interior satisfy necessary optimality conditions in the form of Pontryagin’s principle provided that the data of (1.1) – (1.2) fit into a convex or polyconvex framework, cf. [Wagner 09], p. 549, Theorem 2.2., and [Wagner 14], p. 9, Theorem 4.3., and p. 21, Theorem 5.4. In particular, this set of conditions contains a canonical equation in integrated form, involving multipliers \(\lambda_0 \geq 0\) and \(y^{(1)} \in L^{p/(p-1)}(\Omega, \mathbb{R}^{nm})\), which reads as follows:

\[
\lambda_0 \sum_{i=1}^n \int_{\Omega} \frac{\partial f}{\partial x_i} (s, x^*(s), u^*(s)) \varphi_i(s) \, ds + \sum_{i=1}^n \sum_{j=1}^m \int_{\Omega} \frac{\partial \varphi_i}{\partial s_j} (s) y^{(1)}_{i,j}(s) \, ds = 0 \quad \forall \varphi \in W_0^{1,p}(\Omega, \mathbb{R})^n.
\]

Inserting vector functions, which differ from \(\varphi\) in a single component only, we find that (1.3) implies \(n\) first-oder PDE’s

\[
(\lambda_0 \frac{\partial f}{\partial x_i} (s, x^*(s), u^*(s)), \varphi) = - \left( \sum_{j=1}^m \frac{\partial y^{(1)}_{i,j}}{\partial s_j}, \varphi \right) \quad \forall \varphi \in C^\infty_0(\Omega, \mathbb{R}), 1 \leq i \leq n,
\]

in which the partial derivatives of \(y^{(1)}_{i,j}\) have to be understood in the sense of Schwartz distributions.

With regard to applications of (1.1) – (1.2) in mathematical imaging (see [Bourdaud/Wojciechowski 00], [Franek/Franek/Maurer/Wagner 12] and [Wagner 12]), we prove a refinement of the maximum principle, claiming that the multipliers \(y^{(1)}\) can be chosen from an appropriate Sobolev space rather than from \(L^{p/(p-1)}(\Omega, \mathbb{R}^{nm})\), thus ensuring within (1.4) the separate existence of the generalized derivatives \(\partial y^{(1)}_{i,j}/\partial s_j\) as \(L^\cdot\)-functions. Since the divergence as their sum may be an element of an \(L^\cdot\)-space but the summands do not (cf. [Bourdaud/Wojciechowski 00], p. 326), the proof of this claim is nontrivial. For simplicity, we state the theorem in a special case only, assuming dimensions \(n = m = 2\), polyconvexity of the integrand \(f\) (cf. [Dacorogna 08], p. 156 f., Definition 5.1.(iii)) and convexity of the restriction set \(A\). The extension of the theorem and its proof to the general case covered in [Wagner 09] and [Wagner 14] is obvious.

**Theorem 1.1.** Consider the problem (1.1) – (1.2) with \(n = m = 2\) under the assumptions mentioned in [Wagner 14], p. 18, Theorem 4.11., and choose for the polyconvex integrand \(f(s, \xi, v)\) a convex representative \(g(s, \xi, v, \omega_2)\) in accordance with these assumptions. Further, let \(A = K \subset \mathbb{R}^4\) be a compact, convex set with \(\sigma \in \text{int}(K)\). If \((x^*, u^*)\) is a global minimizer of the problem then there exist multipliers \(\lambda_0 > 0\),
\( y^{(1)} \in W^{p/(p-1),1}(\Omega, \mathbb{R}^4) \) and \( y^{(2)} \in L^{p/(p-2)}(\Omega, \mathbb{R}) \) such that the maximum condition [WAGNER 14, p. 9, (4.6), is satisfied together with the canonical equations

\[
\begin{align*}
(\mathcal{X}_1) & \quad \text{div} \, y_{(1)}^{(1)}(s) = \lambda_0 \frac{\partial g}{\partial x_1}(s, x^*(s), u^*(s), \det u^*(s)) \quad \text{for almost all } s \in \Omega; \\
(\mathcal{X}_2) & \quad \text{div} \, y_{(1)}^{(1)}(s) = \lambda_0 \frac{\partial g}{\partial x_2}(s, x^*(s), u^*(s), \det u^*(s)) \quad \text{for almost all } s \in \Omega.
\end{align*}
\]

2. Proof of Theorem 1.1.

Let \( q_1 = p/(p-1) \) and \( q_2 = p/(p-2) \). The proof of the maximum principle is based on the separation of two convex variational sets \( C \) and \( D \) within the space \( \mathbb{R} \times L^p(\Omega, \mathbb{R}^4) \times L^{p/2}(\Omega, \mathbb{R}) \), cf. [WAGNER 14], pp. 10 – 16. The desired gain of regularity for the multiplier \( y^{(1)} \) will be obtained by replacing \( L^p(\Omega, \mathbb{R}^4) \), the original target space of the state equation (1.2), by the dual space \( (W^{1,q_1}(\Omega, \mathbb{R}^4))^* \leftarrow (L^p(\Omega, \mathbb{R}^4))^* \cong L^p(\Omega, \mathbb{R}^4) \).

It turns out that the separation argument still works in this extended framework. Recall that any functional \( Z \in (W^{1,q_1}(\Omega, \mathbb{R}))^* \) may be represented as

\[
(Z, \psi)_{W^{1,q_1} \rightarrow W^{1,q_1}} = \int_\Omega (Z_0 \psi + Z_1 \frac{\partial \psi}{\partial s_1} + Z_2 \frac{\partial \psi}{\partial s_2}) \, ds \quad \text{with} \quad Z_0, Z_1, Z_2 \in (L^{q_1}(\Omega, \mathbb{R}))^* \cong L^p(\Omega, \mathbb{R}),
\]

cf. [ADAMS/FOURNIER 07], p. 62 f., Theorem 3.9.. Consequently, \( L^p(\Omega, \mathbb{R}^4) \) may be continuously imbedded into \( (W^{1,q_1}(\Omega, \mathbb{R}^4))^* \leftarrow (L^{q_1}(\Omega, \mathbb{R}^4))^* \leftarrow (L^p(\Omega, \mathbb{R}^4))^* \) by \( z \mapsto (z, \psi, \sigma) \). Since \( 1 < q_1 < \infty \), \( (W^{1,q_1}(\Omega, \mathbb{R}^4))^* \) is a reflexive Banach space. In relating to a global minimizer \((x^*, u^*)\) of (1.1) – (1.2), we define the sets

\[
C = \{ (\varrho, z_1, z_2) \in \mathbb{R} \times (L^p(\Omega, \mathbb{R}^4) \cap (W^{1,q_1}(\Omega, \mathbb{R}^4))^*) \times L^{p/2}(\Omega, \mathbb{R}) \quad \text{with} \]

\[
\varrho = \varrho + D_xG(x^*, u^*, w^*)(x-x^*) + D_uG(x^*, u^*, w^*)(u-u^*) + D_wG(x^*, u^*, w^*)(w-w^*) ;
\]

\[
z_1 = Jx - Jx^* - (u-u^*) ; \quad z_2 = (w_2 - w_2^*) - D_u \det(u^*)(u-u^*) ;
\]

\[
\varepsilon \geq 0, \quad x \in W_0^{1,p}(\Omega, \mathbb{R}^n), \quad u \in U, \quad w_2 \in L^{p/2}(\Omega, \mathbb{R}) ;
\]

\[
\tilde{C}_0 = \{ (\varrho, z_1, z_2) \in \mathbb{R} \times (W^{1,q_1}(\Omega, \mathbb{R}^4))^* \times (W^{1,q_2}(\Omega, \mathbb{R}))^* \quad \text{with} \]

\[
z_1 = Jx - Jx^* - (u-u^*), \quad \| z_1 \|_{(W^{1,q_1})^*} \leq \eta ;
\]

\[
z_2 = (w_2 - w_2^*) - \tilde{D}_u \det(u^*)(u-u^*), \quad \| z_2 \|_{(W^{1,q_2})^*} \leq \eta ;
\]

\[
x \in W_0^{1,p}(\Omega, \mathbb{R}^n), \quad u \in U_0 + K(\psi, \eta) \subset (W^{1,q_1}(\Omega, \mathbb{R}^4))^*, \quad w_2 \in (W^{1,q_1}(\Omega, \mathbb{R}))^*, \quad \eta \geq 0 ;
\]

\[
\tilde{D} = \{ (\varrho, z_1, z_2) \in \mathbb{R} \times (W^{1,q_1}(\Omega, \mathbb{R}^4))^* \times L^{p/2}(\Omega, \mathbb{R}) \quad \text{with} \]

\[
\varrho < 0, \quad \varrho \in K(\psi, \frac{1}{2} | \varrho | / K) \subset (W^{1,q_1}(\Omega, \mathbb{R}^4))^*, \quad z_2 \in K(\psi, \frac{1}{2} | \varrho | / K_0) \subset L^{p/2}(\Omega, \mathbb{R}) \}
\]

where

\[
U = \{ z_1 \in L^p(\Omega, \mathbb{R}^4) \mid z_1(s) \in A = K \text{ for almost all } s \in \Omega \},
\]

\[
U_0 = U \cap \{ z_1 \in L^p(\Omega, \mathbb{R}^4) \mid \exists x \in W_0^{1,p}(\Omega, \mathbb{R}^2) \text{ such that } z_1 = Jx \}
\]

and \( \tilde{D}_u \det(u^*) : (W^{1,q_1}(\Omega, \mathbb{R}^4))^* \rightarrow (W^{1,q_2}(\Omega, \mathbb{R}^4))^* \) is the natural extension of the linear, continuous Gâteaux derivative operator \( D_u \det(u^*) : L^p(\Omega, \mathbb{R}^4) \rightarrow L^{p/2}(\Omega, \mathbb{R}) \). For given \( Z \in (W^{1,q_1}(\Omega, \mathbb{R}^4))^* \), \( Z \cong (Z_0, Z_1, Z_2) \in (L^p(\Omega, \mathbb{R}^4))^3 \), the output of \( \tilde{D}_u \det(u^*) (Z) \) acts as a linear, continuous functional on \( \psi \in W^{1,q_2}(\Omega, \mathbb{R}^4) \) in the following way:
\[
\langle \tilde{D}_u T_2 (u^*) (Z), \psi \rangle_{(W^{1,q}_2)'} \sim_{W^{1,q}_2} \langle \tilde{D}_u T_2 (u^*) (Z_0, Z_2), \psi \rangle_{(W^{1,q}_2)'} \sim_{W^{1,q}_2} \quad (2.14)
\]

\[
= \int_{\Omega} \left( u_1^* Z_{0,1} \psi_1 - u_3^* Z_{0,2} \psi_2 - u_2^* Z_{0,3} \psi_3 + u_4^* Z_{0,4} \psi_4 \right) ds + \int_{\Omega} \left( u_1^* Z_{1,1} \frac{\partial \psi_1}{\partial s_1} - u_3^* Z_{1,2} \frac{\partial \psi_2}{\partial s_1} \right) ds + \int_{\Omega} \left( u_4^* Z_{2,1} \frac{\partial \psi_3}{\partial s_2} - u_2^* Z_{2,2} \frac{\partial \psi_4}{\partial s_2} \right) ds + \int_{\Omega} \left( u_1^* Z_{2,4} \frac{\partial \psi_1}{\partial s_2} - u_3^* Z_{2,4} \frac{\partial \psi_2}{\partial s_2} + u_1^* Z_{2,4} \frac{\partial \psi_3}{\partial s_2} \right) ds.
\]

\(K_0 > 0\) is chosen according to the following assertion (ii). It holds still true that (i) \(C\) and \(\tilde{D}\) are nonempty, convex variational sets with \(\text{int}(D) \neq \emptyset\), cf. [Wagner 14], p. 10, Proposition 4.5.; (ii) \((g, z_1, z_2) \in C \cap \tilde{C}_g\) implies that \(g \geq -K_0 \eta\) where \(K_0 > 0\) is a constant independent of \(\eta\), cf. [Wagner 14], p. 11, Proposition 4.7.; and (iii) \(\tilde{D}\) is a subset of \(\tilde{C}_g\) with \(\eta = \frac{1}{2} |g|/K_0\), cf. [Wagner 14], p. 11, Proposition 4.6.

Consequently, \(C \cap \tilde{D} = 0\), and \(C\) and \(\tilde{D}\) may be weakly separated within the space \(\mathbb{R} \times (W^{1,p} (\Omega, \mathbb{R}^4))^* \times L^{p/2} (\Omega, \mathbb{R})\) by a nontrivial, continuous functional \((\lambda_0, y^{(1)}, y^{(2)})\). Thus \(y^{(1)}\) gains the claimed Sobolev regularity. \(\blacksquare\)

Remark 2.1. The nonexistence case from [Bourdaud/Wojciechowski 00] cannot occur since the assumed growth condition [Wagner 14], p. 17, (4.96), guarantees that \(\partial g(s, x^*, u^*, \det \xi) / \partial s_i \in L^{p/(p-1)} (\Omega, \mathbb{R}), 1 \leq i \leq 2\), with \(1 < p/(p-1) < \infty\).

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References.


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