Optimality conditions for multidimensional control problems with polyconvex gradient restrictions

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1. Introduction.

The present investigation continues a series of papers concerned with possible extensions of Pontryagin's principle to multidimensional control problems where the usual convexity of the data is replaced by generalized convexity notions. We are concerned with so-called Dieudonné-Rashevsky type problems, which are obtained from the basic problem of multidimensional calculus of variations

$$\Phi(x) \,=\, \int_\Omega f(s, x(s), Jx(s))\, ds \longrightarrow \inf!\,; \quad x \in W_0^{1,p}(\Omega, \mathbb{R}^n)\,; \quad \Omega \subset \mathbb{R}^m \tag{1.1}$$

in the vectorial case $n \geqslant 2$, $m \geqslant 2$ with $m < p < \infty$ by addition of constraints for the partial derivatives of x. More precisely, we impose the gradient restriction

$$Jx(s) \,=\, \begin{pmatrix} \partial x_1(s)/\partial s_1 & \dots & \partial x_1(s)/\partial s_m \\ \vdots & & \vdots \\ \partial x_n(s)/\partial s_1 & \dots & \partial x_n(s)/\partial s_m \end{pmatrix} \in \mathrm{A} \subset \mathbb{R}^{nm} \quad \text{for almost all } s \in \Omega \tag{1.2}$$

where $\mathrm{A} \subset \mathbb{R}^{nm}$ is a compact set with nonempty interior. We thus arrive at a multidimensional control problem of the shape

$$(\mathrm{P}) \quad F(x,u) \,=\, \int_\Omega f(s, x(s), u(s))\, ds \longrightarrow \inf!\,; \quad x \in W_0^{1,p}(\Omega, \mathbb{R}^n) \,\times\, L^p(\Omega, \mathbb{R}^{nm})\,; \tag{1.3}$$

$$Jx(s) \,=\, \begin{pmatrix} \partial x_1(s)/\partial s_1 & \dots & \partial x_1(s)/\partial s_m \\ \vdots & & \vdots \\ \partial x_n(s)/\partial s_1 & \dots & \partial x_n(s)/\partial s_m \end{pmatrix} \,=\, u(s) \quad \text{for almost all } s \in \Omega\,; \tag{1.4}$$

$$u(s) \in \mathrm{A} \subset \mathbb{R}^{nm} \quad \text{for almost all } s \in \Omega\,. \tag{1.5}$$

The motivation for a closer investigation of Dieudonné-Rashevsky type problems (P) is two-fold. First, due to its close affinity to the basic problem of multidimensional calculus of variations, the problem (1.3) − (1.5) is well-suited as a model problem in order to ascertain how the proof of optimality conditions is influenced through the weakening of the convexity properties of the data. Since the classical proof of Pontryagin's principle is based on an implicit convexification of the integrand as well as of the set of feasible controls, [01] an answer to this question is of conceptual interest. On the other hand, Dieudonné-Rashevsky type problems find applications in such different areas as convex geometry, [02] material sciences, [03] population dynamics [04] and mathematical image processing, [05] thus proving considerable practical importance.

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[01] Compare [Ginsburg/Ioffe 96] , p. 92, Definition 3.2., and p. 96, Theorem 3.6., where a "local relaxability" property of the problem is required, as well as [Ioffe/Tichomirov 79] , pp. 201 ff.

[02] [Andrejewa/Klötzler 84a] and [Andrejewa/Klötzler 84b] , p. 149 f.

[03] See, for instance, [Lur'e 75] , pp. 240 ff., [Ting 69a] , p. 531 f., [Ting 69b] and [Wagner 96] , pp. 76 ff.

[04] [Brokate 85] , [Feichtinger/Tragler/Veliov 03] .

[05] [Angelov/Wagner 12] , [Brune/Maurer/Wagner 09] , [Franek/Franek/Maurer/Wagner 12] , [Wagner 10] and [Wagner 12] .

As yet, first-order necessary optimality conditions for the global minimizers of (1.3) − (1.5) have been established in the case that the restriction set A is convex and the integrand $f(s, \xi, \cdot\,)$ is either convex or polyconvex with respect to v. [06] In the present paper, we will extend these results to problems (P) involving a polyconvex integrand as well as *a polyconvex gradient restriction*.
Within the hierarchy of semiconvexity notions, polyconvexity is the closest one to usual convexity. [07] Polyconvex integrands, which arise as a composition of the vector of all minors of a matricial argument with a convex function, are well-introduced in optimization problems in material sciences, hydrodynamics and image processing. [08] Polyconvex gradient restrictions frequently originate from volumetric constraints. A nice illustration is given if the function $x \in W^{1,\infty}(\Omega, \Omega)$ within the transformation formula for multiple integrals [09]

$$\int_\Omega I(s)\, ds \,=\, \int_\Omega I\big(\,x(s)\,\big) \cdot \big|\det Jx(s)\,\big|\, ds \tag{1.6}$$

is considered as an unknown. In order to keep the formula applicable, we must ensure that $\det Jx(s) \neq 0$ a. e. Consequently, we obtain a polyconvex gradient restriction for x, e. g. $|\det Jx(s)\,| > 0$ or $\det Jx(s) > 0$. In the literature, an explicit statement of polyconvex restrictions is often avoided. Instead, the objectives are augmented with corresponding penalty terms. [10]
A similar idea is employed in the proof of Pontryagin's principle for the problem (1.3) − (1.5) presented here. Assuming that the restriction set A = K ∩ P is the intersection of a convex body K with nonempty interior and a polyconvex set P, we introduce an *exact penalty term corresponding to* P. Then the proof technique developed in [Wagner 13], which makes explicit use of the polyconvex structure of the integrand as well as of the restriction set, can be applied with specific modifications.
The outline of the present paper is running parallel to [Wagner 13]. This introductory section is closed with some remarks about notation. Then in *Section 2*, we describe the notions of polyconvexity for functions as well as for subsets of $\mathbb{R}^{nm}$. In *Section 3*, we prove the equivalence of three different formulations of the control problem (P), the last one containing an exact penalty term for the polyconvex gradient restriction (Propositions 3.3. and 3.8.). Further, we prove the existence of global minimizers for the problems (Theorem 3.9.). In *Section 4*, we provide first the formulation of Pontryagin's principle in the special case of dimensions $n = m = 2$. Then we state and prove the first-order necessary optimality conditions in the general case as our main result (Theorem 4.3.). The occurence of the regular case and the a. e. pointwise reformulation of the maximum condition are discussed (Proposition 4.4. and Theorem 4.5.). The paper closes with an application of our theorems to a problem of three-dimensional hyperelastic image registration (*Section 5*).

Notations.

Let $\Omega \subset \mathbb{R}^m$ be the closure of a bounded Lipschitz domain (in strong sense). Then $C^k(\Omega, \mathbb{R}^r)$ denotes the space of r-dimensional vector functions $f\colon\, \Omega \to \mathbb{R}^r$, whose components are continuous ($k = 0$) or k-times

[06] [Wagner 09], p. 549 f., Theorems 2.2. and 2.3. (convex case), and [Wagner 13], p. 7, Theorems 4.3. and 4.4. (polyconvex case).

[07] Cf. [Ball 77], [Dacorogna 08], p. 156 f., and [Müller 99], pp. 126 ff.

[08] We refer e. g. to [Lubkoll/Schiela/Weiser 12], [Kunisch/Vexler 07], [Burger/Modersitzki/Ruthotto 13], [Droske/Rumpf 04], [Pöschl/Modersitzki/Scherzer 10] and [Wagner 10].

[09] [Elstrodt 96], p. 208, Corollary 4.9.

[10] See e. g. the discussion of the hyperelastic registration problem from [Burger/Modersitzki/Ruthotto 13] in Section 5 below.

continuously differentiable ($k = 1, \ldots, \infty$), respectively; $L^p(\Omega, \mathbb{R}^r)$ denotes the space of r-dimensional vector functions $f\colon \Omega \to \mathbb{R}^r$, whose components are integrable in the pth power ($1 \leqslant p < \infty$) or are measurable and esentially bounded ($p = \infty$). $W_0^{1,p}(\Omega, \mathbb{R}^r)$ denotes the Sobolev space of r-dimensional vector functions $f\colon \Omega \to \mathbb{R}^r$ with compactly supported components, which possess first-order weak partial derivatives and belong together with them to the space $L^p(\Omega, \mathbb{R})$ ($1 \leqslant p < \infty$). $W_0^{1,\infty}(\Omega, \mathbb{R}^r)$ is understood as the Sobolev space of all r-vector functions $f\colon \Omega \to \mathbb{R}^r$, whose components are Lipschitz with zero boundary values. [11] Jx denotes the Jacobi matrix of the vector function $x \in W_0^{1,p}(\Omega, \mathbb{R}^r)$. $\mathbb{Q}^r$ is the space of all r-vectors with rational coordinates. The abbreviation "$(\forall)\, s \in \mathrm{A}$" has to be read as "for almost all $s \in \mathrm{A}$" or "for all $s \in \mathrm{A}$ except a Lebesgue null set". Finally, the symbol $\mathfrak{o}$ denotes, depending on the context, the zero element or the zero function of the underlying space.

2. Polyconvex functions and polyconvex sets.

a) Polyconvex functions.

Throughout the paper, the following notation for the vector of the minors of a matricial argument will be used. [12]

Definition 2.1. (The operator T) *Let $n, m \geqslant 1$ and denote $\mathrm{Min}(n,m) = n \wedge m$.*

1) We consider elements $v \in \mathbb{R}^{nm}$ as (n,m)-matrices and define $T(v) = \big(v, T_2 v, T_3 v, \ldots, T_{(n\wedge m)} v\big) \in \mathbb{R}^{\tau(n,m)} = \mathbb{R}^{\sigma(1)} \times \mathbb{R}^{\sigma(2)} \times \mathbb{R}^{\sigma(3)} \times \ldots \times \mathbb{R}^{\sigma(n\wedge m)}$ as the row vector consisting of all minors of v: $T_2 v = \mathrm{adj}_2 v$, $T_3 v = \mathrm{adj}_3 v, \ldots, T_{(n\wedge m)} v = \mathrm{adj}_{(n\wedge m)} v$. Consequently, we have $\sigma(r) = \binom{n}{r} \cdot \binom{m}{r}$, $1 \leqslant r \leqslant n \wedge m$. The sum of the dimensions is denoted by $\tau(n,m) = \sigma(1) + \ldots + \sigma(n \wedge m)$.

2) Let $(m \wedge n) \leqslant p \leqslant \infty$. We consider elements $u \in L^p(\Omega, \mathbb{R}^{nm})$ as (n,m)-matrix functions and define the operator $T\colon L^p(\Omega, \mathbb{R}^{nm}) \to L^p(\Omega, \mathbb{R}^{\sigma(1)}) \times L^{p/2}(\Omega, \mathbb{R}^{\sigma(2)}) \times L^{p/3}(\Omega, \mathbb{R}^{\sigma(3)}) \times \ldots \times L^{p/(n\wedge m)}(\Omega, \mathbb{R}^{\sigma(n\wedge m)})$ by $u \longmapsto Tu = \big(u, T_2 u, T_3 u, \ldots, T_{(n\wedge m)} u\big)$ with $T_2 u = \mathrm{adj}_2 u$, $T_3 u = \mathrm{adj}_3 u, \ldots, T_{(n\wedge m)} u = \mathrm{adj}_{(n\wedge m)} u$.

As mentioned in the introduction, a polyconvex function is defined as a composition of the vector of all minors $T(v)$ of a matricial argument v with a convex function.

Definition 2.2. (Polyconvex function) [13] *We consider elements $v \in \mathbb{R}^{nm}$ as (n,m)-matrices and elements $V \in \mathbb{R}^{\tau(n,m)}$ as row vectors. A function $f(v)\colon \mathbb{R}^{nm} \to \mathbb{R} \cup \{(+\infty)\}$ is called polyconvex iff there exists a convex function $g(V)\colon \mathbb{R}^{\tau(n,m)} \to \mathbb{R} \cup \{(+\infty)\}$ such that*

$$f(v) = g\big(T(v)\big) \quad \forall\, v \in \mathbb{R}^{nm}. \tag{2.1}$$

The function g is called a convex representative for the polyconvex function f.

Note that a polyconvex function may possess more than one convex representative. To a given polyconvex function f, we may associate the special convex representative [14]

$$g(V) = \inf\Big\{ \sum_{r=1}^{\tau(n,m)+1} \lambda_r f(v_r) \;\Big|\; \sum_{r=1}^{\tau(n,m)+1} \lambda_r T(v_r) = V,\ \sum_{r=1}^{\tau(n,m)+1} \lambda_r = 1,\ \lambda_r \geqslant 0,\ v_r \in \mathbb{R}^{nm}, \tag{2.2}$$
$$1 \leqslant r \leqslant \tau(n,m) + 1 \Big\},$$

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11) [EVANS/GARIEPY 92], p. 131, Theorem 5.

12) Concerning the notations related to polyconvexity and matricial arguments, we adopt the conventions from [DACOROGNA 08].

13) [DACOROGNA 08], p. 156 f., Definition 5.1.(iii).

14) Ibid., p. 163, Theorem 5.6., Part 2.

which is called the Busemann representative of f. [15] Any polyconvex function is locally Lipschitz continuous on the interior of its effective domain [16] and, consequently, differentiable a. e. on its effective domain. Stronger smoothness properties as continuous differentiability are not automatically inherited by the convex representative. [17] Consequently, as mentioned in [WAGNER 13], within the framework of optimal control it is advisable to state the growth and smoothness assumptions about a polyconvex integrand rather in terms of a fixed convex representative than of the function itself.

In the special case $n = m = 2$, we get $\sigma(1) = 4$, $\sigma(2) = 1$, $\tau(2,2) = 5$ and $T(v) = \binom{v}{\det v}$. Consequently, any polyconvex function $f\colon \mathbb{R}^{2\times 2} \to \mathbb{R} \cup \{ (+\infty) \}$ must take the form $f(v) = g\big(v, \det v \big)$ with a convex function $g\colon \mathbb{R}^5 \to \mathbb{R} \cup \{ (+\infty) \}$. For $n = m = 3$, we have $\sigma(1) = 9$, $\sigma(2) = 9$, $\sigma(3) = 1$ and $\tau(3,3) = 19$. Here $\operatorname{adj}_2 v$ is the transpose of the cofactor matrix and $\operatorname{adj}_3 v = \det v$.

b) Polyconvex sets.

Definition 2.3. (Polyconvex set) [18] *We consider elements $v \in \mathbb{R}^{nm}$ as (n,m)-matrices and elements $V \in \mathbb{R}^{\tau(n,m)}$ as row vectors. A set $\mathrm{P} \subseteq \mathbb{R}^{nm}$ is called polyconvex iff there exists a convex set $\mathrm{Q} \subseteq \mathbb{R}^{\tau(n,m)}$ such that*

$$\mathrm{P} = \big\{ v \in \mathbb{R}^{nm} \ \big|\ T(v) \in \mathrm{Q} \big\}. \tag{2.3}$$

The set Q is called a convex representative for the polyconvex set P.

Equivalently, a set $\mathrm{P} \subseteq \mathbb{R}^{nm}$ can be defined as polyconvex iff its indicator function $\chi_{\mathrm{P}}\colon \mathbb{R}^{nm} \to \mathbb{R} \cup \{ (+\infty) \}$ is a polyconvex function. [19] Elementary examples of polyconvex sets are quasiaffine hyperplanes $\mathrm{H} = \{ v \in \mathbb{R}^{n\times m} \ |\ \langle V_0\,, T(v) \rangle = \alpha_0 \}$ for $V_0 \in \mathbb{R}^{\tau(n,m)}$, $\alpha_0 \in \mathbb{R}$ (e. g. the group $SO(n)$), open quasiaffine half-spaces $\mathrm{H}^+ = \{ v \in \mathbb{R}^{n\times n} \ |\ \langle V_0\,, T(v) \rangle > \alpha_0 \}$ (e. g. the group $GL^+(n)$), polyconvex polytopes obtained as the polyconvex hull of finitely many points [20] and polyconvex polyhedral sets obtained as the intersection of finitely many affine and quasiaffine half-spaces. Any convex set is polyconvex as well. [21]

Analogously to polyconvex functions, the convex representative of a polyconvex set is not necessarily uniquely determined. By the following lemma, the smallest possible convex representative is singled out, which will be called the precise representative $\widetilde{\mathrm{Q}}$ of P.

Lemma 2.4. (Precise representative of a polyconvex set) *1)* [22] *If $\mathrm{P} \subseteq \mathbb{R}^{nm}$ is a polyconvex set then $\widetilde{\mathrm{Q}} = \operatorname{co} \big\{ T(v) \in \mathbb{R}^{\tau(n,m)} \ \big|\ v \in \mathrm{P} \big\}$ forms a convex representative of P.*

2) For any convex representative $Q \subseteq \mathbb{R}^{\tau(n,m)}$ of P, it holds that $\widetilde{\mathrm{Q}} \subseteq \mathrm{Q}$.

Proof. The proof of Part 2) is obvious. ■

Lemma 2.5. (Compactness of the precise representative) *If $\mathrm{P} \subset \mathbb{R}^{nm}$ is a compact polyconvex set then its precise convex representative $\widetilde{\mathrm{Q}} \subset \mathbb{R}^{\tau(n,m)}$ of P is compact as well.*

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15) [BEVAN 06], p. 24, Definition 2.1. An effective numerical procedure for the evaluation of $g(V)$ has been provided in [ENEYA/BOSSE/GRIEWANK 13].

16) [DACOROGNA 08], p. 47, Theorem 2.31.

17) Cf. [BEVAN 03] and [BEVAN 06], pp. 44 ff., Section 5.

18) [DACOROGNA 08], p. 316, Definition 7.2. (ii). The definition goes back to [DACOROGNA/RIBEIRO 06], p. 108, Definition 3.1. (ii).

19) [DACOROGNA 08], p. 318, Proposition 7.5.

20) Cf. ibid., pp. 323 ff.

21) Ibid., p. 318, Theorem 7.7.

22) Ibid., p. 317, Theorem 7.4. (iii).

Proof. Consider the precise representative of P, which is given by Lemma 2.4. through $\widetilde{\mathrm{Q}} = \mathrm{co}\,\big\{\, T(v) \;\big|\; v \in \mathrm{P}\,\big\} \subseteq \mathbb{R}^{\tau(n,m)}$. First, the continuous function $T\colon \mathbb{R}^{nm} \to \mathbb{R}^{\tau(n,m)}$ maps the compact set P onto a compact image. Secondly, the convex hull of a compact set is compact again, cf. [Schneider 93], p. 6, Theorem 1.1.10. ∎

3. Existence of optimal solutions.

a) Statement of the control problem; basic assumptions.

We consider the following multidimensional control problem of Dieudonné-Rashevsky type:

$$(\mathrm{P})_0 \quad F(x,u) \;=\; \int_\Omega f(s,x(s),u(s))\,ds \;\longrightarrow\; \inf!\,; \tag{3.1}$$

$$(x,u) \in W_0^{1,p}(\Omega,\mathbb{R}^n) \times L^p(\Omega,\mathbb{R}^{nm})\,; \tag{3.2}$$

$$E(x,u) \;=\; Jx(s) - u(s) \;=\; 0 \quad (\forall)\, s \in \Omega\,; \tag{3.3}$$

$$u(s) \;\in\; \mathrm{K} \cap \mathrm{P} \subset \mathbb{R}^{nm} \quad (\forall)\, s \in \Omega\,. \tag{3.4}$$

The basic assumptions about problem $(\mathrm{P})_0$ involve the presence of a polyconvex integrand as well as of a polyconvex gradient restriction. In particular, the following properties of the data will be imposed.

Assumptions 3.1. (Basic assumptions about the data within $(\mathrm{P})_0$**)**

1) We assume that $n,\, m \geqslant 2$ *and* $m < p < \infty$ *(thus* $n \wedge m < p$*).*

2) $\Omega \subset \mathbb{R}^m$ *is the closure of a bounded strongly Lipschitz domain,* $\mathrm{K} \subset \mathbb{R}^{nm}$ *is a convex body with* $\mathfrak{o} \in \mathrm{int}\,(\mathrm{K})$ *and* $\mathrm{P} \subset \mathbb{R}^{nm}$ *is a nonempty compact, polyconvex set (cf. Definition 2.3. above).*

3) The integrand $f(s,\xi,v)\colon\, \Omega \times \mathbb{R}^n \times \mathbb{R}^{nm} \to \mathbb{R}$ *is measurable with respect to* s*, continuous with respect to* ξ *and* v *and polyconvex as a function of* v *for all fixed* $(\hat{s},\hat{\xi}\,) \in \Omega \times \mathbb{R}^n$*.*

4) The polyconvex integrand $f(s,\xi,v)$ *admits a convex representative* $g(s,\xi,v,\omega)\colon\, \Omega \times \mathbb{R}^n \times \mathbb{R}^{nm} \times \big(\mathbb{R}^{\sigma(2)} \times \mathbb{R}^{\sigma(3)} \times \ldots \times \mathbb{R}^{\sigma(n\wedge m)}\big) \to \mathbb{R}$*, which is measurable with respect to* s *and continuously differentiable with respect to* ξ*,* v *and* ω*. Moreover,* g *satisfies the following growth condition:*

$$\big|\, g(\,s,\xi,v,\omega_2,\omega_3,\,\ldots\,,\omega_{(n\wedge m)}\,)\,\big| \;\leqslant\; A_0(s) \,+\, B_0(\xi) \,+\, C_0\,\Big(\,1 + \big|\,v\,\big|^p + \sum_{r=2}^{(n\wedge m)} \big|\,\omega_r\,\big|^{p/r}\,\Big) \tag{3.5}$$

$$(\forall)\, s \in \Omega \quad \forall\,(\xi,v,\omega) \in \mathbb{R}^n \times \mathbb{R}^{nm} \times \big(\mathbb{R}^{\sigma(2)} \times \mathbb{R}^{\sigma(3)} \times \ldots \times \mathbb{R}^{\sigma(n\wedge m)}\big)$$

where $A_0 \in L^1(\Omega,\mathbb{R})$*,* $A_0\,\big|\,\mathrm{int}\,(\Omega)$ *is continuous,* B_0 *is measurable and bounded on every bounded subset of* $\mathbb{R}^n$*, and* $C_0 > 0$*.*

b) Equivalent formulations of the problem.

By Lemma 2.5., the compact polyconvex set P admits a convex, compact representative $\mathrm{Q} \subset \mathbb{R}^{nm} \times \mathbb{R}^{\sigma(2)} \times \mathbb{R}^{\sigma(3)} \times \ldots \times \mathbb{R}^{\sigma(n\wedge m)}$. Further, we choose for the polyconvex integrand $f(s,\xi,v)$ a convex representative $g(s,\xi,v,\omega)\colon\, \Omega \times \mathbb{R}^n \times \mathbb{R}^{nm} \times \big(\mathbb{R}^{\sigma(2)} \times \mathbb{R}^{\sigma(3)} \times \ldots \times \mathbb{R}^{\sigma(n\wedge m)}\big) \to \mathbb{R}$ according to Assumption 3.1., 4). Then the problem $(\mathrm{P})_0$ may be reformulated in the following way:

$$(\mathrm{P})_1 \quad G(x,u,w) \;=\; \int_\Omega g(s,x(s),u(s),w(s))\,ds \;\longrightarrow\; \inf!\,; \tag{3.6}$$

$$\begin{aligned}(x,u,w) \in W_0^{1,p}(\Omega,\mathbb{R}^n) &\times L^p(\Omega,\mathbb{R}^{nm}) \\ &\times \Big(\,L^{p/2}(\Omega,\mathbb{R}^{\sigma(2)}) \times L^{p/3}(\Omega,\mathbb{R}^{\sigma(3)}) \times \ldots \times L^{p/(n\wedge m)}(\Omega,\mathbb{R}^{\sigma(n\wedge m)})\Big);\end{aligned} \tag{3.7}$$

$$E_1(x,u) \;=\; Jx(s) - u(s) \;=\; 0 \quad (\forall)\, s \in \Omega\,; \tag{3.8}$$

$$E_2(u,w) \;=\; w_2(s) - \mathrm{adj}_2\, u(s) \;=\; 0 \quad (\forall)\, s \in \Omega\,; \tag{3.9}$$

$$E_3(u,w) \;=\; w_3(s) - \operatorname{adj}_3 u(s) \;=\; 0 \quad (\forall)\, s \in \Omega\,; \tag{3.10}$$
$$\vdots$$
$$E_{(n\wedge m)}(u,w) \;=\; w_{(n\wedge m)}(s) - \operatorname{adj}_{(n\wedge m)} u(s) \;=\; 0 \quad (\forall)\, s \in \Omega\,; \tag{3.11}$$
$$u \in \mathrm{U} \;=\; \big\{\, z_1 \in L^p(\Omega, \mathbb{R}^{nm}) \;\big|\; z_1(s) \in \mathrm{K} \;\; (\forall)\, s \in \Omega \,\big\}\,; \tag{3.12}$$
$$(u,w) \in \mathrm{W} \;=\; \big\{\, \big(\, z_1, z_2, z_3, \,\dots\,, z_{(n\wedge m)} \,\big) \in L^p(\Omega, \mathbb{R}^{nm}) \,\times\, L^{p/2}(\Omega, \mathbb{R}^{\sigma(2)}) \,\times\, L^{p/3}(\Omega, \mathbb{R}^{\sigma(3)}) \tag{3.13}$$
$$\times \dots \times\, L^{p/(n\wedge m)}(\Omega, \mathbb{R}^{\sigma(n\wedge m)}) \;\big|\; \big(\, z_1(s)\,,\, z_2(s)\,,\, z_3(s)\,,\,\dots\,,\, z_{(n\wedge m)}(s) \,\big) \;\in\; \mathrm{Q} \;\; (\forall)\, s \in \Omega \,\big\}\,.$$

We establish the following properties of the data.

Lemma 3.2. *Let Assumptions 3.1. hold. Then the sets* U *and* W *are nonempty, convex, bounded and closed with respect to the norm topologies in* $L^p(\Omega, \mathbb{R}^{nm})$ *and* $L^{p/2}(\Omega, \mathbb{R}^{\sigma(2)}) \times L^{p/3}(\Omega, \mathbb{R}^{\sigma(3)}) \times \dots \times L^{p/(n\wedge m)}(\Omega, \mathbb{R}^{\sigma(n\wedge m)})$*, respectively. Consequently,* U *and* W *are weakly sequentially compact as subsets of the mentioned spaces.*

Proof. U and W are nonempty and convex together with the convex sets K and Q. Obviously, the restrictions $z_1(s) \in \mathrm{K}$ and $\big(\, z_1(s)\,,\, z_2(s)\,,\, z_3(s)\,,\,\dots\,,\, z_{(n\wedge m)}(s) \,\big) \in \mathrm{Q}$ will be conserved under norm convergence in the mentioned spaces due to the existence of a. e. pointwise convergent subsequences. Now the weak sequential compactness of the sets follows from [Rolewicz 76] , p. 157, Theorem IV.5.6′, and its proof. ∎

Proposition 3.3. (Equivalent formulations of the basic problem, I) *Let Assumptions 3.1. hold. If* (x^*, u^*) *is a global minimizer of* $(\mathrm{P})_0$ *then* $\big(\, x^*,\, u^*,\, T_2(u^*),\, T_3(u^*),\, \dots\,,\, T_{(n\wedge m)}(u^*) \,\big)$ *is a global minimizer of* $(\mathrm{P})_1$*. Conversely, if* (x^*, u^*, w^*) *is a global minimizer of* $(\mathrm{P})_1$ *then* (x^*, u^*) *is a global minimizer of* $(\mathrm{P})_0$*.*

Proof. Assume that (x^*, u^*) is a global minimizer of $(\mathrm{P})_0$ and let (x, u, w) be a feasible triple within $(\mathrm{P})_1$. Then, by definition of G, $G(x,u,w) = F(x,u) \geqslant F(x^*, u^*) = G(x^*, u^*, w^*)$ with $w^* = \big(T_2(u^*),\, T_3(u^*),\, \dots\,,\, T_{(n\wedge m)}(u^*) \big)$, and (x^*, u^*, w^*) is a global minimizer of $(\mathrm{P})_1$ as well. Conversely, if (x^*, u^*, w^*) is a global minimizer of $(\mathrm{P})_1$ then, again by definition of G, we have $F(x,u) = G(x,u,w) \geqslant G(x^*, u^*, w^*) = F(x^*, u^*)$ for every feasible pair (x, u) within $(\mathrm{P})_0$ where $w = \big(T_2(u),\, T_3(u),\, \dots\,,\, T_{(n\wedge m)}(u) \big)$. Consequently, (x^*, u^*) forms a global minimizer of $(\mathrm{P})_0$. ∎

In the proof of the necessary optimality conditions below, a further equivalent formulation of problem $(\mathrm{P})_0$ will be used. Namely, we will introduce an exact penalty for the control restriction (3.13), thus obtaining the problem

$$(\mathrm{P})_2 \quad \widetilde{G}(x,u,w) \;=\; \int_\Omega g(s, x(s), u(s), w(s))\, ds + K_1 \cdot \mathrm{Dist}\,\big(\, (x,u,w)\,,\, L^p(\Omega, \mathbb{R}^n) \times \mathrm{W} \,\big) \;\longrightarrow\; \inf!\,; \tag{3.14}$$
$$(x,u,w) \in W^{1,p}_0(\Omega, \mathbb{R}^n) \times L^p(\Omega, \mathbb{R}^{nm}) \tag{3.15}$$
$$\times \Big(\, L^{p/2}(\Omega, \mathbb{R}^{\sigma(2)}) \times L^{p/3}(\Omega, \mathbb{R}^{\sigma(3)}) \times \dots \times L^{p/(n\wedge m)}(\Omega, \mathbb{R}^{\sigma(n\wedge m)}) \Big);$$
$$E_1(x,u) \;=\; Jx(s) - u(s) \;=\; 0 \quad (\forall)\, s \in \Omega\,; \tag{3.16}$$
$$E_2(u,w) \;=\; w_2(s) - \operatorname{adj}_2 u(s) \;=\; 0 \quad (\forall)\, s \in \Omega\,; \tag{3.17}$$
$$E_3(u,w) \;=\; w_3(s) - \operatorname{adj}_3 u(s) \;=\; 0 \quad (\forall)\, s \in \Omega\,; \tag{3.18}$$
$$\vdots$$
$$E_{(n\wedge m)}(u,w) \;=\; w_{(n\wedge m)}(s) - \operatorname{adj}_{(n\wedge m)} u(s) \;=\; 0 \quad (\forall)\, s \in \Omega\,; \tag{3.19}$$
$$u \in \mathrm{U} \;=\; \big\{\, z_1 \in L^p(\Omega, \mathbb{R}^{nm}) \;\big|\; z_1(s) \in \mathrm{K} \;\; (\forall)\, s \in \Omega \,\big\}\,, \tag{3.20}$$

which turns out to be equivalent to $(\mathrm{P})_0$ and $(\mathrm{P})_1$ provided that a sufficiently large constant $K_1 > 0$ will be chosen (see Proposition 3.8. below) and the partial derivatives of g satisfy additional growth conditions.

These will be collected in the following assumptions:

Assumptions 3.4. (Additional assumptions about the data within $(\mathrm{P})_0$**)** *Assume that* $g(s,\xi,v,\omega)$: $\Omega\times\mathbb{R}^n\times\mathbb{R}^{nm}\times\big(\mathbb{R}^{\sigma(2)}\times\mathbb{R}^{\sigma(3)}\times\ldots\times\mathbb{R}^{\sigma(n\wedge m)}\big)\to\mathbb{R}$ *is a convex representative of the polyconvex integrand* $f(s,\xi,v)$, *which is measurable with respect to* s *and continuously differentiable with respect to* ξ, v *and* ω. *Let the partial derivatives of* g *satisfy the following growth conditions:*

$$\Big|\,\frac{\partial g}{\partial\xi_i}(\,s,\xi,v,\omega_2,\omega_3,\,\ldots,\omega_{(n\wedge m)}\,)\,\Big|\;\leqslant\;A_i(s)\,+\,B_i(\xi)\,+\,C_i\,\Big(1+\big|\,v\,\big|^{p-1}+\sum_{r=2}^{(n\wedge m)}\big|\,\omega_r\,\big|^{(p-1)/r}\Big)\tag{3.21}$$
$$(\forall)\,s\in\Omega\quad\forall\,(\xi,v,\omega)\in\mathbb{R}^n\,\times\,\mathbb{R}^{nm}\times\big(\mathbb{R}^{\sigma(2)}\times\mathbb{R}^{\sigma(3)}\times\ldots\,\times\mathbb{R}^{\sigma(n\wedge m)}\big)$$

where $A_i\in L^{p/(p-1)}(\Omega,\mathbb{R})$, B_i *is measurable and bounded on every bounded subset of* $\mathbb{R}^n$, *and* $C_i>0$, $1\leqslant i\leqslant n$;

$$\Big|\,\frac{\partial g}{\partial v_l}(\,s,\xi,v,\omega_2,\omega_3,\,\ldots,\omega_{(n\wedge m)}\,)\,\Big|\;\leqslant\;A_l^{(1)}(s)\,+\,B_l^{(1)}(\xi)\,+\,C_l^{(1)}\,\Big(1+\big|\,v\,\big|^{p-1}+\sum_{r=2}^{(n\wedge m)}\big|\,\omega_r\,\big|^{(p-1)/r}\Big)\tag{3.22}$$
$$(\forall)\,s\in\Omega\quad\forall\,(\xi,v,\omega)\in\mathbb{R}^n\,\times\,\mathbb{R}^{nm}\times\big(\mathbb{R}^{\sigma(2)}\times\mathbb{R}^{\sigma(3)}\times\ldots\,\times\mathbb{R}^{\sigma(n\wedge m)}\big)$$

where $A_l^{(1)}\in L^{p/(p-1)}(\Omega,\mathbb{R})$, $B_l^{(1)}$ *is measurable and bounded on every bounded subset of* $\mathbb{R}^n$, *and* $C_l^{(1)}>0$, $1\leqslant l\leqslant\sigma(1)=nm$;

$$\Big|\,\frac{\partial g}{\partial\omega_{2,l}}(\,s,\xi,v,\omega_2,\omega_3,\,\ldots,\omega_{(n\wedge m)}\,)\,\Big|\;\leqslant\;A_l^{(2)}(s)\,+\,B_l^{(2)}(\xi)\,+\,C_l^{(2)}\,\Big(1+\big|\,v\,\big|^{p-2}+\sum_{r=2}^{(n\wedge m)}\big|\,\omega_r\,\big|^{(p-2)/r}\Big)\tag{3.23}$$
$$(\forall)\,s\in\Omega\quad\forall\,(\xi,v,\omega)\in\mathbb{R}^n\,\times\,\mathbb{R}^{nm}\times\big(\mathbb{R}^{\sigma(2)}\times\mathbb{R}^{\sigma(3)}\times\ldots\,\times\mathbb{R}^{\sigma(n\wedge m)}\big)$$

where $A_l^{(2)}\in L^{p/(p-2)}(\Omega,\mathbb{R})$, $B_l^{(2)}$ *is measurable and bounded on every bounded subset of* $\mathbb{R}^n$, *and* $C_l^{(2)}>0$, $1\leqslant l\leqslant\sigma(2)$;

$$\vdots$$

$$\Big|\,\frac{\partial g}{\partial\omega_{(n\wedge m),l}}(\,s,\xi,v,\omega_2,\omega_3,\,\ldots,\omega_{(n\wedge m)}\,)\,\Big|\;\leqslant\;A_l^{(n\wedge m)}(s)\,+\,B_l^{(n\wedge m)}(\xi)\,+\,C_l^{(n\wedge m)}\,\Big(1+\big|\,v\,\big|^{p-(n\wedge m)}\tag{3.24}$$
$$+\sum_{r=2}^{(n\wedge m)}\big|\,\omega_r\,\big|^{(p-(n\wedge m))/r}\Big)\quad(\forall)\,s\in\Omega\quad\forall\,(\xi,v,\omega)\in\mathbb{R}^n\,\times\,\mathbb{R}^{nm}\times\big(\mathbb{R}^{\sigma(2)}\times\mathbb{R}^{\sigma(3)}\times\ldots\,\times\mathbb{R}^{\sigma(n\wedge m)}\big)$$

where $A_l^{(n\wedge m)}\in L^{p/(p-(n\wedge m))}(\Omega,\mathbb{R})$, $B_l^{(n\wedge m)}$ *is measurable and bounded on every bounded subset of* $\mathbb{R}^n$, *and* $C_l^{(n\wedge m)}>0$, $1\leqslant l\leqslant\sigma(n\wedge m)$.

Before stating the next lemma, let us define the closed balls

$$\mathrm{K}(\mathfrak{o},R_0)\subset W_0^{1,p}(\Omega,\mathbb{R}^n)\hookrightarrow C_0^0(\Omega,\mathbb{R}^n)\,;\tag{3.25}$$
$$\mathrm{K}'(\mathfrak{o},R')\;=\;\mathrm{K}(\mathfrak{o},R')\,\times\,\mathrm{K}(\mathfrak{o},R')\,\times\,\ldots\,\times\,\mathrm{K}(\mathfrak{o},R')\tag{3.26}$$
$$\subset L^{p/2}(\Omega,\mathbb{R}^{\sigma(2)})\times L^{p/3}(\Omega,\mathbb{R}^{\sigma(3)})\times\,\ldots\,\times L^{p/(n\wedge m)}(\Omega,\mathbb{R}^{\sigma(n\wedge m)})$$

with the radii

$$R_0\;=\;\sup\big\{\,\|\,x\,\|_{C_0^0(\Omega,\mathbb{R}^n)}\;\big|\;Jx\in\mathrm{U}\,\big\}\,;\tag{3.27}$$
$$R'\;=\;\operatorname*{Max}_{2\,\leqslant\,r\,\leqslant\,(n\wedge m)}\;C_r\cdot\sup\big\{\,|\,\big(\operatorname{adj}_r(v)\big)_l\,|\;\big|\;1\leqslant l\leqslant\sigma(r)\,,\;v\in\mathrm{K}\,\big\}\tag{3.28}$$

where the constants $C_r > 0$ are taken from the imbedding inequalities

$$\| z_r \|_{L^{p/r}} \leqslant C_r \| z_r \|_{L^\infty}\,,\ 2 \leqslant r \leqslant (n \wedge m)\,. \tag{3.29}$$

Lemma 3.5. *Let Assumptions 3.1. together with Assumptions 3.4. hold. Then the functional G within* (P)$_1$ *satisfies the Lipschitz condition*

$$\big|\, G(x', u', w') - G(x'', u'', w'')\,\big| \leqslant K_0 \Big(\| x' - x'' \|_{L^p} + \| u' - u'' \|_{L^p} + \sum_{r=2}^{(n\wedge m)} \| w_r' - w_r'' \|_{L^{p/r}} \Big) \tag{3.33}$$

for all triples (x', u', w'), $(x'', u'', w'') \in \mathrm{K}(\mathfrak{o}, R_0) \times \mathrm{U} \times \mathrm{K}'(\mathfrak{o}, R') \subset \big(W_0^{1,p}(\Omega, \mathbb{R}^n) \cap C_0^0(\Omega, \mathbb{R}^n) \big) \times L^p(\Omega, \mathbb{R}^{nm}) \times \big(L^{p/2}(\Omega, \mathbb{R}^{\sigma(2)}) \times L^{p/3}(\Omega, \mathbb{R}^{\sigma(3)}) \times \ldots \times L^{p/(n\wedge m)}(\Omega, \mathbb{R}^{\sigma(n\wedge m)}) \big)$.

Proof. Fix a number $\varepsilon > 0$ and consider an arbitrary pair of elements (x', u', w'), $(x'', u'', w'') \in \mathrm{K}(\mathfrak{o}, R_0) \times \mathrm{U} \times \mathrm{K}'(\mathfrak{o}, R')$. By convexity, this set contains the whole segment $\mathrm{S} = \big[\, (x', u', w')\,,\, (x'', u'', w'')\,\big]$. Assumptions 3.1. and 3.4. guarantee the Gâteaux differentiability of the functional G with respect to x, u, w_2, w_3, ... , $w_{(n\wedge m)}$ even on the larger set $\mathrm{K}(\mathfrak{o}, R_0 + \varepsilon) \times \big(\mathrm{U} + \mathrm{K}(\mathfrak{o}, \varepsilon) \big) \times \mathrm{K}'(\mathfrak{o}, R' + \varepsilon)$ and, consequently, along S. Now the mean value theorem [23] yields the estimate

$$\begin{aligned} &\big|\, G(x', u', w') - G(x'', u'', w'')\,\big| \\ &\leqslant \sup_{(\hat{x}, \hat{u}, \hat{w}) \in \mathrm{S}} \big\| DG(\hat{x}, \hat{u}, \hat{w}) \big\| \cdot \Big(\| x' - x'' \|_{L^p} + \| u' - u'' \|_{L^p} + \sum_{r=2}^{(n\wedge m)} \| w_r' - w_r'' \|_{L^{p/r}} \Big) \end{aligned} \tag{3.34}$$

where

$$\sup_{(\hat{x}, \hat{u}, \hat{w}) \in \mathrm{S}} \big\| DG(\hat{x}, \hat{u}, \hat{w}) \big\| \leqslant \sup_{\hat{x} \in \mathrm{K}(\mathfrak{o}, R_0+\varepsilon)}\ \sup_{\hat{u} \in \mathrm{U}+\mathrm{K}(\mathfrak{o},\varepsilon)}\ \sup_{\hat{w} \in \mathrm{K}'(\mathfrak{o}, R'+\varepsilon)} \big\| DG(\hat{x}, \hat{u}, \hat{w}) \big\| \tag{3.35}$$

$$\begin{aligned} &\leqslant \sup_{\hat{x} \in \mathrm{K}(\mathfrak{o}, R_0+\varepsilon)}\ \sup_{\hat{u} \in \mathrm{U}+\mathrm{K}(\mathfrak{o},\varepsilon)}\ \sup_{\hat{w} \in \mathrm{K}'(\mathfrak{o}, R'+\varepsilon)} C \Big(\sum_{i=1}^{n} \Big\| \frac{\partial g}{\partial \xi_i}(\hat{x}, \hat{u}, \hat{w}) \Big\|_{L^{p/(p-1)}} + \sum_{i=1}^{n} \sum_{j=1}^{m} \Big\| \frac{\partial g}{\partial v_{ij}}(\hat{x}, \hat{u}, \hat{w}) \Big\|_{L^{p/(p-1)}} \\ &\quad + \sum_{l=1}^{\sigma(2)} \Big\| \frac{\partial g}{\partial \omega_{2,l}}(\hat{x}, \hat{u}, \hat{w}) \Big\|_{L^{p/(p-2)}} + \ldots + \sum_{l=1}^{\sigma(n\wedge m)} \Big\| \frac{\partial g}{\partial \omega_{(n\wedge m),l}}(\hat{x}, \hat{u}, \hat{w}) \Big\|_{L^{p/(p-(n\wedge m))}} \Big). \end{aligned} \tag{3.36}$$

The suprema in (3.36) are formed over bounded function sets. Consequently, the expression in (3.36) remains finite as far as the boundedness of the Nemytskij operators $(\hat{x}, \hat{u}, \hat{w}) \longmapsto \partial g(\hat{x}, \hat{u}, \hat{w})/\partial \xi_i \in L^{p/(p-1)}$, $(\hat{x}, \hat{u}, \hat{w}) \longmapsto \partial g(\hat{x}, \hat{u}, \hat{w})/\partial v_{ij} \in L^{p/(p-1)}$, $(\hat{x}, \hat{u}, \hat{w}) \longmapsto \partial g(\hat{x}, \hat{u}, \hat{w})/\partial \omega_{2,l} \in L^{p/(p-2)}$, $(\hat{x}, \hat{u}, \hat{w}) \longmapsto \partial g(\hat{x}, \hat{u}, \hat{w})/\partial \omega_{3,l} \in L^{p/(p-3)}$, ... , $(\hat{x}, \hat{u}, \hat{w}) \longmapsto \partial g(\hat{x}, \hat{u}, \hat{w})/\ \partial \omega_{(n\wedge m),l} \in L^{p/(p-(n\wedge m))}$ can be guaranteed. However, this is implied by the growth conditions (3.21) – (3.24). For example, from (3.21) it follows that

$$\Big\| \frac{\partial g}{\partial \xi_i}(\hat{x}, \hat{u}, \hat{w}) \Big\|_{L^{p/(p-1)}} = \int_\Omega \Big| \frac{\partial}{\partial \xi_i} g(s, \hat{x}(s), \hat{u}(s), \hat{w}(s)) \Big|^{p/(p-1)} ds \tag{3.37}$$

$$\leqslant \int_\Omega \Big| A_i(s) + B_i\big(\hat{x}(s) \big) + C_i \Big(1 + \big| \hat{u}(s) \big|^{p-1} + \sum_{r=2}^{(n\wedge m)} \big| \hat{w}_r(s) \big|^{(p-1)/r} \Big) \Big|^{p/(p-1)} ds \tag{3.38}$$

$$\leqslant C \int_\Omega \Big(A_i(s)^{p/(p-1)} + B_i\big(\hat{x}(s) \big)^{p/(p-1)} + C_i \Big(1 + \big| \hat{u}(s) \big|^p + \sum_{r=2}^{(n\wedge m)} \big| \hat{w}_r(s) \big|^{p/r} \Big) \Big) ds \tag{3.39}$$

$$\leqslant C \Big(\| A_i \|_{L^{p/(p-1)}}^{p/(p-1)} + \big(\tilde{B}_i(R_0 + \varepsilon) \big)^{p/(p-1)} + C_i \Big(1 + \| \hat{u} \|_{L^p}^p + \sum_{r=2}^{(n\wedge m)} \| \hat{w}_r \|_{L^{p/r}}^{p/r} \Big) \Big) \tag{3.40}$$

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[23] [Ioffe/Tichomirow 79], p. 40.

with an appropriate constant $\tilde{B}_i(R_0+\varepsilon) > 0$ such that $\|\,\hat{x}\,\|_{C^0} \leqslant R_0+\varepsilon \Longrightarrow |\,B_i(\,\hat{x}(s)\,)\,| \leqslant \tilde{B}_i(R_0+\varepsilon)$, and (3.40) remains uniformly bounded for all $(\hat{x},\hat{u},\hat{w}) \in \mathrm{K}(\mathfrak{o},R_0+\varepsilon) \times \big(\mathrm{U}+\mathrm{K}(\mathfrak{o},\varepsilon)\big) \times \mathrm{K}'(\mathfrak{o},R'+\varepsilon)$. For the other partial derivatives occuring in (3.36), we may reason analogously. Consequently, condition (3.33) holds true with a constant $K_0 \geqslant \sup_{(\hat{x},\hat{u},\hat{w}) \in \mathrm{K}(\mathfrak{o},R_0+\varepsilon) \times (\mathrm{U}+\mathrm{K}(\mathfrak{o},\varepsilon)) \times \mathrm{K}'(\mathfrak{o},R'+\varepsilon)} \|\,DG(\hat{x},\hat{u},\hat{w})\,\|$. ∎

Remark 3.7. For the application of the mean value theorem in this proof, the Gâteaux differentiability of the functional G is required not only on the set $\mathrm{K}(\mathfrak{o},R_0) \times \mathrm{U} \times \big\{\,\big(\,z_2, z_3, \,...\,, z_{(n\wedge m)}\,\big) \;\big|\; \big(\,z_1, z_2, z_3, \,...\,, z_{(n\wedge m)}\,\big) \in \mathrm{W}\,\big\}$, which belongs in fact to the subspace $W_0^{1,\infty}(\Omega,\mathbb{R}^n) \times L^\infty(\Omega,\mathbb{R}^{nm}) \times \big(\,L^\infty(\Omega,\mathbb{R}^{\sigma(2)}) \times L^\infty(\Omega,\mathbb{R}^{\sigma(3)}) \times \,...\, \times L^\infty(\Omega,\mathbb{R}^{\sigma(n\wedge m)})\,\big)$, but on an open neighbourhood of it. In order to ensure this, the growth conditions (3.21) – (3.24) must be imposed.

Now we are in position to prove the equivalence of the control problems $(\mathrm{P})_0$, $(\mathrm{P})_1$ and $(\mathrm{P})_2$. The feasible domains of the three problems will be denoted by $\mathcal{B}_0$, $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively. Obviously, we have $\mathcal{B}_1 \subset \mathcal{B}_2 \subset \mathrm{K}(\mathfrak{o},R_0) \times \mathrm{U} \times \mathrm{K}'(\mathfrak{o},R')$.

Proposition 3.8. (Equivalent formulations of the basic problem, II) *Let Assumptions 3.1. and 3.4. hold and fix in* (3.14) *a sufficiently large constant* $K_1 > K_0 > 0$.[24] *Then every global minimizer* (x^*,u^*,w^*) *of* $(\mathrm{P})_1$ *is a global minimizer of* $(\mathrm{P})_2$ *as well. Conversely, every global minimizer of* $(\mathrm{P})_2$ *is feasible in* $(\mathrm{P})_1$ *and forms a global minimizer of* $(\mathrm{P})_1$.

Proof. Assume that (x^*,u^*,w^*) is a global minimizer of $(\mathrm{P})_1$. Let us apply [Clarke 90], p. 51 f., Proposition 2.4.3., to the following data: $\mathrm{S} = \mathcal{B}_2 \subset L^p(\Omega,\mathbb{R}^n) \times \mathrm{U} \times L^{p/2}(\Omega,\mathbb{R}^{\sigma(2)}) \times L^{p/3}(\Omega,\mathbb{R}^{\sigma(3)}) \times \,...\, \times L^{p/(n\wedge m)}(\Omega,\mathbb{R}^{\sigma(n\wedge m)})$, $\mathrm{C} = \mathcal{B}_1 = \mathcal{B}_2 \cap \big(\,L^p(\Omega,\mathbb{R}^n) \times \mathrm{W}\,\big)$, and $f\colon\, \mathrm{S} \to \mathbb{R}$ is the functional $G\colon\, \mathcal{B}_2 \to \mathbb{R}$. By Lemma 3.5., G is Lipschitz on S with constant K_0. Moreover, since W is closed by Lemma 3.2., the assertion follows from the cited result. Conversely, let a global minimizer (x^*,u^*,w^*) of $(\mathrm{P})_2$ be given. Then the cited theorem ensures that (x^*,u^*,w^*) is feasible in $(\mathrm{P})_1$ and forms even a global minimizer there. ∎

c) Existence of global minimizers.

We will see that the assumptions stated above guarantee the existence of a global minimizer for problem $(\mathrm{P})_1$ and, consequently, for problems $(\mathrm{P})_0$ and $(\mathrm{P})_2$ as well. No structural assumptions about the polyconvex restriction set P must be added.

Theorem 3.9. (Existence of global minimizers for $(\mathrm{P})_0 - (\mathrm{P})_2$) *Consider problem* $(\mathrm{P})_1$ *under Assumptions 3.1. Then there exists a global minimizer* (x^*,u^*,w^*) *of* $(\mathrm{P})_1$ *and, consequently, a global minimizer* (x^*,u^*) *of* $(\mathrm{P})_0$. *If, additionally, Assumptions 3.4. are imposed then* $(\mathrm{P})_2$ *admits a global minimizer as well.*

Proof. Due to the control restrictions (3.12) and (3.13), the feasible domain $\mathcal{B}_1$ of $(\mathrm{P})_1$ forms a bounded subset of $W_0^{1,p}(\Omega,\mathbb{R}^n) \times L^p(\Omega,\mathbb{R}^{nm}) \times L^{p/2}(\Omega,\mathbb{R}^{\sigma(2)}) \times L^{p/3}(\Omega,\mathbb{R}^{\sigma(3)}) \times \,...\, \times L^{p/(n\wedge m)}(\Omega,\mathbb{R}^{\sigma(n\wedge m)})$. By Assumption 3.1., 4), the objective (3.6) remains bounded on $\mathcal{B}_1$, and $(\mathrm{P})_1$ admits a minimizing sequence $\{\,(x^N,u^N,w^N)\,\}$. First, we must confirm ourselves that this sequence contains a subsequence, which converges with respect to the product of the norm topology of $W_0^{1,p}(\Omega,\mathbb{R}^n)$ and the weak topologies of $L^p(\Omega,\mathbb{R}^{nm})$ and $L^{p/2}(\Omega,\mathbb{R}^{\sigma(2)}) \times L^{p/3}(\Omega,\mathbb{R}^{\sigma(3)}) \times \,...\, \times L^{p/(n\wedge m)}(\Omega,\mathbb{R}^{\sigma(n\wedge m)})$ to a feasible element $(\hat{x},\hat{u},\hat{w})$. It is clear that we may pass over to subsequences, which satisfy $x^N \rightharpoonup \hat{x}$, $u^N \rightharpoonup \hat{u}$ and $w^N \rightharpoonup \hat{w}$ (we keep the index N). By the Rellich-Kondrachev theorem,[25] we may ensure further that x^N converges uniformly

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[24] The constant K_0 is taken from Lemma 3.5.

[25] [Adams/Fournier 07], p. 168, Theorem 6.3.

to $\hat{x}$, and $\hat{x}$ satisfies the zero boundary condition. Moreover, the weak continuity of the generalized derivative yields

$$Jx^N - u^N \;=\; E_1(x^N, u^N) \rightharpoonup E_1(\hat{x}, \hat{u}) \;=\; J\hat{x} - \hat{u} = \mathfrak{o}\,. \tag{3.41}$$

From [DACOROGNA 08], p. 395 f., Theorem 8.20, Parts 3 and 4, we infer that $u^N = Jx^N \rightharpoonup J\hat{x} = \hat{u}$ implies

$$\operatorname{adj}_2 u^N \;=\; \operatorname{adj}_2 Jx^N \rightharpoonup \operatorname{adj}_2 J\hat{x} \;=\; \operatorname{adj}_2 \hat{u} \quad\Longrightarrow\quad E_2(u^N, w^N) \rightharpoonup E_2(\hat{u}, \hat{w}) \;=\; \mathfrak{o}\,; \tag{3.42}$$

$$\operatorname{adj}_3 u^N \;=\; \operatorname{adj}_3 Jx^N \rightharpoonup \operatorname{adj}_3 J\hat{x} \;=\; \operatorname{adj}_3 \hat{u} \quad\Longrightarrow\quad E_3(u^N, w^N) \rightharpoonup E_3(\hat{u}, \hat{w}) \;=\; \mathfrak{o}\,; \tag{3.43}$$

$$\vdots$$

$$\begin{aligned} \operatorname{adj}_{(n\wedge m)} u^N \;=\; \operatorname{adj}_{(n\wedge m)} Jx^N \rightharpoonup \operatorname{adj}_{(n\wedge m)} J\hat{x} \;=\; \operatorname{adj}_{(n\wedge m)}\hat{u} \\ \Longrightarrow\quad E_{(n\wedge m)}(u^N, w^N) \rightharpoonup E_{(n\wedge m)}(\hat{u}, \hat{w}) \;=\; \mathfrak{o}\,. \end{aligned} \tag{3.44}$$

By Lemma 3.2., $\hat{u}$ and $(\hat{u}, \hat{w})$ belong to U and W, respectively, and $(\hat{x}, \hat{u}, \hat{w})$ is feasible in $(\mathrm{P})_1$. In order to confirm the lower semicontinuity of the objective (3.6) with respect to the mentioned topology, we observe that the growth condition (3.5) guarantees that the function $\widetilde{f}(s, \xi, v)\colon\, \Omega \times \mathbb{R}^n \times \mathbb{R}^{nm} \to \mathbb{R} \cup \{\, +(\infty)\,\}$ defined through

$$\widetilde{f}(s, \xi, v) \;=\; f(s, \xi, v) + \begin{cases} 0 & |\ (s, \xi, v) \in \Omega \times \mathbb{R}^n \times \mathrm{K}\,; \\ (+\infty) & |\ (s, \xi, v) \in \Omega \times \mathbb{R}^n \times (\mathbb{R}^{nm} \setminus \mathrm{K}) \end{cases} \tag{3.45}$$

belongs to the function class $\widetilde{\mathfrak{F}}_{\mathrm{K}}$ mentioned in [WAGNER 11], p. 191, Definition 1.1., 2). Consequently, the existence theorems [WAGNER 11], p. 193, Theorems 1.4. and 1.5., imply the weak lower semicontinuity relation

$$\liminf_{n\to\infty} G(x^N, u^N, w^N) \;=\; \liminf_{n\to\infty} \int_\Omega g(s, x^N(s), u^N(s), w^N(s))\, ds \tag{3.46}$$

$$= \liminf_{n\to\infty} \int_\Omega g\big(\, s, x^N(s), u^N(s), T_2\big(\, u^N(s)\,\big), T_3\big(\, u^N(s)\,\big),\, \ldots\,, T_{(n\wedge m)}\big(\, u^N(s)\,\big)\,\big)\, ds \tag{3.47}$$

$$= \liminf_{n\to\infty} \int_\Omega f(s, x^N(s), u^N(s))\, ds \;=\; \liminf_{n\to\infty} \int_\Omega \widetilde{f}(s, x^N(s), u^N(s)) \;\geqslant\; \int_\Omega \widetilde{f}(s, \hat{x}(s), \hat{u}(s))\, ds \tag{3.47}$$

$$= \int_\Omega f(s, \hat{x}(s), \hat{u}(s))\, ds \;=\; \int_\Omega g(s, \hat{x}(s), \hat{u}(s), \hat{w}(s))\, ds \;=\; G(\hat{x}, \hat{u}, \hat{w})\,, \tag{3.48}$$

and $(\hat{x}, \hat{u}, \hat{w})$ is a global minimizer for $(\mathrm{P})_1$. Now Proposition 3.3. implies that $(\hat{x}, \hat{u})$ is a global minimizer of $(\mathrm{P})_0$. If, additionally, Assumptions 3.4. are satisfied then $(\hat{x}, \hat{u}, \hat{w})$ is a global minimizer of $(\mathrm{P})_2$ by Proposition 3.8. ∎

4. The first-order necessary optimality conditions.

a) The conditions in the special case $n = m = 2$.

In order to illustrate the structure of the optimality conditions, we state them first in the special case of dimensions $n = m = 2$. For instance, this case appears in the two-dimensional image registration problems discussed in [WAGNER 10], p. 5 f.

Theorem 4.1. (Pontryagin's principle for $(\mathrm{P})_0$ **with** $n = m = 2$**)** [26] *Consider the problem* $(\mathrm{P})_0$ *with* $n = m = 2$ *under Assumptions 3.1. and 3.4. mentioned above. Choose for the polyconvex set* P *a compact, convex representative* $\mathrm{Q} \subset \mathbb{R}^5$ *and for the polyconvex integrand* $f(s, \xi, v)$ *in* $(\mathrm{P})_0$ *a convex representative* $g(s, \xi, v, \omega_2)$ *in accordance with Assumptions 3.1., 4) and 3.4. If* (x^*, u^*) *is a global minimizer of* $(\mathrm{P})_0$ *then there exist multipliers* $\lambda_0 \geqslant 0$, $y^{(1)} \in L^{p/(p-1)}(\Omega, \mathbb{R}^4)$ *and* $y^{(2)} \in L^{p/(p-2)}(\Omega, \mathbb{R})$ *such that the following conditions are satisfied:*

$$(\mathcal{M}) \quad \lambda_0 \int_\Omega \Big(g(\, s, x^*(s), u(s), w_2(s)\,) - g(\, s, x^*(s), u^*(s), \det u^*(s)\,)\Big)\, ds - \int_\Omega \big(u(s) - u^*(s)\,\big)^{\mathrm{T}} y^{(1)}(s)\, ds \tag{4.1}$$

$$+ \int_\Omega \big(w_2(s) - \det u^*(s)\,\big)\, y^{(2)}(s)\, ds \;-\; \int_\Omega \nabla_v \det\big(u^*(s)\,\big)^{\mathrm{T}}\, \big(u(s) - u^*(s)\,\big)\, y^{(2)}(s)\, ds \;\geqslant\; 0$$

$$\forall\, (u, w_2) \,\in\, \Big(\mathrm{U} \,\times\, L^{p/2}(\Omega, \mathbb{R}) \Big) \,\cap\, \mathrm{W}\,;$$

$$(\mathcal{K}) \quad \lambda_0 \sum_{i=1}^{2} \int_\Omega \frac{\partial g}{\partial \xi_i}(\, s, x^*(s), u^*(s), \det u^*(s)\,)\, \big(x_i(s) - x_i^*(s)\,\big)\, ds \tag{4.2}$$

$$+ \sum_{i=1}^{2} \sum_{j=1}^{2} \int_\Omega \Big(\frac{\partial x_i}{\partial s_j}(s) - \frac{\partial x_i^*}{\partial s_j}(s)\,\Big)\, y_{ij}^{(1)}(s)\, ds \;=\; 0 \quad \forall\, x \in W_0^{1,p}(\Omega, \mathbb{R}^2)\,.$$

The function sets U *and* W *are defined by means of* K *and* Q *through* (3.12) *and* (3.13).

Theorem 4.2. (Pointwise maximum condition for $(\mathrm{P})_0$ **with** $n = m = 2$**)** [27] *Consider the problem* $(\mathrm{P})_0$ *with* $n = m = 2$ *under the assumptions of Theorem 4.1. If* (x^*, u^*) *is a global minimizer of* $(\mathrm{P})_0$ *then the maximum condition* $(\mathcal{M})$ *from Theorem 4.1. implies the following pointwise maximum condition:*

$$(\mathcal{MP}) \quad \lambda_0 \Big(g(\, s, x^*(s), v, \omega_2\,) - g(\, s, x^*(s), u^*(s), \det u^*(s)\,)\Big) \;-\; \sum_{i=1}^{2} \sum_{j=1}^{2} \big(v_{ij} - u_{ij}^*(s)\,\big)\, y_{ij}^{(1)}(s) \tag{4.3}$$

$$+ \;\big(\omega_2 - \det u^*(s)\,\big)\, y^{(2)}(s) \;-\; \sum_{i=1}^{2} \sum_{j=1}^{2} \frac{\partial}{\partial v_{ij}} \det(\, u^*(s)\,)\, \big(v_{ij} - u_{ij}^*(s)\,\big)\, y^{(2)}(s) \;\geqslant\; 0$$

$$(\forall)\, s \in \Omega \quad \forall\, \big(\, v,\, \omega_2\,\big) \in \big(\, \mathrm{K} \times \mathbb{R}\,\big) \,\cap\, \mathrm{Q}\,. \quad \blacksquare$$

b) Statement of the theorems in the general case $n \geqslant 2$, $m \geqslant 2$.

In the following main theorem, the first-order necessary optimality conditions for a global minimizer of the multidimensional control problem $(\mathrm{P})_0$ will be stated for general dimensions $n \geqslant 2$, $m \geqslant 2$.

Theorem 4.3. (Pontryagin's principle for $(\mathrm{P})_0$**)** *Consider the problem* $(\mathrm{P})_0$ *under Assumptions 3.1. and 3.4. and choose for the polyconvex set* P *a compact, convex representative* $\mathrm{Q} \subset \mathbb{R}^{nm} \times \mathbb{R}^{\sigma(2)} \times \mathbb{R}^{\sigma(3)} \times \ldots \times \mathbb{R}^{\sigma(n \wedge m)}$. *Further, choose for the integrand* $f(s, \xi, v)$ *in* $(\mathrm{P})_0$ *a convex representative* $g(s, \xi, v, \omega)$ *in accordance with Assumption 3.1., 4) and Assumptions 3.4. If* (x^*, u^*) *is a global minimizer of* $(\mathrm{P})_0$ *then there*

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[26] Special case of Theorem 4.3. below.

[27] Special case of Theorem 4.5. below.

exist multipliers $\lambda_0 \geqslant 0$, $y^{(1)} \in L^{p/(p-1)}(\Omega, \mathbb{R}^{nm})$, $y^{(2)} \in L^{p/(p-2)}(\Omega, \mathbb{R}^{\sigma(2)})$, $y^{(3)} \in L^{p/(p-3)}(\Omega, \mathbb{R}^{\sigma(3)})$, ... , $y^{(n\wedge m)} \in L^{p/(p-(n\wedge m))}(\Omega,\ \mathbb{R}^{\sigma(n\wedge m)})$ *such that the following conditions are satisfied:*

$$(\mathcal{M}) \quad \lambda_0 \int_\Omega \Big(g(\,s, x^*(s), u(s), w(s)\,) - g(\,s, x^*(s), u^*(s), w^*(s)\,)\Big)\, ds - \int_\Omega \big(u(s) - u^*(s)\big)^{\mathrm{T}} y^{(1)}(s)\, ds \tag{4.4}$$
$$+ \sum_{r=2}^{(n\wedge m)} \int_\Omega \big(w_r(s) - w_r^*(s)\big)^{\mathrm{T}} y^{(r)}(s)\, ds - \sum_{r=2}^{(n\wedge m)} \int_\Omega \nabla_v \operatorname{adj}_r(\,u^*(s)\,)\big(u(s) - u^*(s)\big)^{\mathrm{T}} y^{(r)}(s)\, ds \geqslant 0$$
$$\forall\,(u,w) \in \Big(\mathrm{U} \times L^{p/2}(\Omega, \mathbb{R}^{\sigma(2)}) \times L^{p/3}(\Omega, \mathbb{R}^{\sigma(3)}) \times \dots \times L^{p/(n\wedge m)}(\Omega, \mathbb{R}^{\sigma(n\wedge m)})\Big) \cap \mathrm{W};$$

$$(\mathcal{K}) \quad \lambda_0 \sum_{i=1}^{n} \int_\Omega \frac{\partial g}{\partial \xi_i}(\,s, x^*(s), u^*(s), w^*(s)\,)\big(x_i(s) - x_i^*(s)\big)\, ds \tag{4.5}$$
$$+ \sum_{i=1}^{n}\sum_{j=1}^{m} \int_\Omega \Big(\frac{\partial x_i}{\partial s_j}(s) - \frac{\partial x_i^*}{\partial s_j}(s)\Big)\, y_{ij}^{(1)}(s)\, ds = 0 \quad \forall\, x \in W_0^{1,p}(\Omega, \mathbb{R}^n)\,.$$

The function sets U *and* W *are defined by means of* K *and* Q *through* (3.12) *and* (3.13)*.*

Let us define

$$\mathrm{Q}' = \big\{\, \big(\omega_2, \omega_3, \dots, \omega_{(n\wedge m)}\big) \in \mathbb{R}^{\sigma(2)} \times \mathbb{R}^{\sigma(3)} \times \dots \times \mathbb{R}^{\sigma(n\wedge m)} \;\big|\; \big(v, \omega_2, \omega_3, \dots, \omega_{(n\wedge m)}\big) \in \mathrm{Q} \,\big\}\,. \tag{4.6}$$

Proposition 4.4. (Occurence of the regular case) *Consider the problem* $(\mathrm{P})_0$ *under the assumptions of Theorem 4.3. and let* (x^*, u^*) *be a global minimizer of* $(\mathrm{P})_0$*. If there exists a number* $\gamma > 0$ *such that* $\big(T_2(\,u^*(s)\,), T_3(\,u^*(s)\,), \dots, T_{(n\wedge m)}(\,u^*(s)\,)\big) + \mathrm{K}(\mathfrak{o}, \gamma) \in \operatorname{int}(\mathrm{Q}')$ *for almost all* $s \in \Omega$ *then in the necessary optimality conditions* $(\mathcal{M})$ *and* $(\mathcal{K})$ *the regular case occurs, i. e.* $\lambda_0 > 0$*.*

The maximum condition $(\mathcal{M})$ from Theorem 4.3. implies the following condition $(\mathcal{MP})$, which holds a. e. pointwise:

Theorem 4.5. (Pointwise maximum condition for $(\mathrm{P})_0$**)** *Consider the problem* $(\mathrm{P})_0$ *under the assumptions of Theorem 4.3. If* (x^*, u^*) *is a global minimizer of* $(\mathrm{P})_0$ *then the maximum condition* $(\mathcal{M})$ *from Theorem 4.3. implies the following pointwise maximum condition:*

$$(\mathcal{MP}) \quad \lambda_0 \Big(g(\,s, x^*(s), v, \omega\,) - g(\,s, x^*(s), u^*(s), w^*(s)\,)\Big) - \big(v - u^*(s)\big)^{\mathrm{T}} y^{(1)}(s) \tag{4.7}$$
$$+ \sum_{r=2}^{(n\wedge m)} \big(\omega_r - w_r^*(s)\big)^{\mathrm{T}} y^{(r)}(s) - \sum_{r=2}^{(n\wedge m)} \nabla_v \operatorname{adj}_r(\,u^*(s)\,)\big(v - u^*(s)\big)^{\mathrm{T}} y^{(r)}(s) \geqslant 0$$
$$(\forall)\, s \in \Omega \quad \forall\,\big(v, \omega_2, \omega_3, \dots, \omega_{(n\wedge m)}\big) \in \big(\mathrm{K} \times \mathbb{R}^{\sigma(2)} \times \mathbb{R}^{\sigma(3)} \times \dots \times \mathbb{R}^{\sigma(n\wedge m)}\big) \cap \mathrm{Q}\,.$$

c) Proof of Pontryagin's principle.

Proof of Theorem 4.3. • Sketch of the proof. The proof of Theorem 4.3. is based on the equivalence of the problems $(\mathrm{P})_0$ and $(\mathrm{P})_2$. Thus to the given global minimizer (x^*, u^*) of $(\mathrm{P})_0$, a global minimizer (x^*, u^*, w^*) of $(\mathrm{P})_2$ corresponds, which will be used in order to define a pair of convex variational sets C and D as subsets of the space $\mathbb{R} \times L^p(\Omega, \mathbb{R}^{nm}) \times L^{p/2}(\Omega, \mathbb{R}^{\sigma(2)}) \times L^{p/3}(\Omega, \mathbb{R}^{\sigma(3)}) \times \dots \times L^{p/(n\wedge m)}(\Omega, \mathbb{R}^{\sigma(n\wedge m)})$. We establish first that C is closed, and the interior of D is nonempty (Step 1). Although the usual regularity condition for the equality operator E_1 fails, [28] we are able to show that $\mathrm{C} \cap \mathrm{D} = \emptyset$ by applying Lyusternik's theorem to the operators $E_2, E_3, \dots, E_{(n\wedge m)}$ and exploiting the Lipschitz property of the penalty term in the objective of $(\mathrm{P})_2$ (Steps 2 – 4). This fact allows for the application of the weak separation theorem

[28] See [Ioffe/Tichomirov 79], p. 73 f., Theorem 3, Assumption c), and [Ito/Kunisch 08], p. 5 f.

and a subsequent derivation of the first-order necessary optimality conditions from the resulting variational inequality (Steps 5 and 6).

• **Step 1.** *The variational sets* C *and* D. Let a global minimizer (x^*, u^*) of $(\mathrm{P})_0$ be given. Then, by Proposition 3.8., $(x^*, u^*, w^*) = \big(\, x^*, u^*, T_2(u^*), T_3(u^*), \,...\, , T_{(n\wedge m)}(u^*)\,\big)$ is a global minimizer of $(\mathrm{P})_2$ provided that K_1 has been chosen in accordance with Proposition 3.8. We fix a number $\alpha > 0$ and define the variational sets

$$\mathrm{C} \,=\, \Big\{\, \big(\, \varrho,\, z_1,\, z_2,\, z_3,\, ...\, ,\, z_{(n\wedge m)}\,\big) \tag{4.8}$$

$$\in \mathbb{R} \times L^p(\Omega, \mathbb{R}^{nm}) \times L^{p/2}(\Omega, \mathbb{R}^{\sigma(2)}) \times L^{p/3}(\Omega, \mathbb{R}^{\sigma(3)}) \times \,...\, \times L^{p/(n\wedge m)}(\Omega, \mathbb{R}^{\sigma(n\wedge m)}) \quad \text{with}$$

$$\varrho = \varepsilon + D_x G(x^*, u^*, w^*)(x - x^*) + D_u G(x^*, u^*, w^*)(u - u^*) + D_w G(x^*, u^*, w^*)(w - w^*)\,; \tag{4.9}$$

$$z_1 = Jx - Jx^* - (u - u^*)\,; \tag{4.10}$$

$$z_2 = (w_2 - w_2^*) - D_u T_2(u^*)(u - u^*)\,; \tag{4.11}$$

$$z_3 = (w_3 - w_3^*) - D_u T_3(u^*)(u - u^*)\,; \tag{4.12}$$

$$\vdots$$

$$z_{(n\wedge m)} = (w_{(n\wedge m)} - w_{(n\wedge m)}^*) - D_u T_{(n\wedge m)}(u^*)(u - u^*)\,; \tag{4.13}$$

$$\varepsilon \geqslant 0\,,\ x \in W_0^{1,p}(\Omega, \mathbb{R}^n)\,,\ u \in \mathrm{U}\,,\ (u, w) \in \mathrm{W}\,\Big\}\,; \tag{4.14}$$

$$\mathrm{D} \,=\, \Big\{\, \big(\, \varrho,\, z_1,\, z_2,\, z_3,\, ...\, ,\, z_{(n\wedge m)}\,\big) \tag{4.15}$$

$$\in \mathbb{R} \times L^p(\Omega, \mathbb{R}^{nm}) \times L^{p/2}(\Omega, \mathbb{R}^{\sigma(2)}) \times L^{p/3}(\Omega, \mathbb{R}^{\sigma(3)}) \times \,...\, \times L^{p/(n\wedge m)}(\Omega, \mathbb{R}^{\sigma(n\wedge m)}) \quad \text{with}$$

$$\varrho \leqslant -2\,K_2\,\Big(\,\|\, z_1 \,\|_{L^p} + \|\, z_2 \,\|_{L^{p/2}} + \|\, z_3 \,\|_{L^{p/3}} + \,...\, + \|\, z_{(n\wedge m)} \,\|_{L^{p/(n\wedge m)}}\,\Big)\,; \tag{4.16}$$

$$z_1 \in \mathrm{K}(\mathfrak{o}, \alpha) \subset L^p(\Omega, \mathbb{R}^{nm})\,; \tag{4.17}$$

$$z_2 \in \mathrm{K}(\mathfrak{o}, \alpha) \subset L^{p/2}(\Omega, \mathbb{R}^{\sigma(2)})\,; \tag{4.18}$$

$$z_3 \in \mathrm{K}(\mathfrak{o}, \alpha) \subset L^{p/3}(\Omega, \mathbb{R}^{\sigma(3)})\,; \tag{4.19}$$

$$\vdots$$

$$z_{(n\wedge m)} \in \mathrm{K}(\mathfrak{o}, \alpha) \subset L^{p/(n\wedge m)}(\Omega, \mathbb{R}^{\sigma(n\wedge m)})\,\Big\} \tag{4.20}$$

The value of the constant $K_2 > 0$ will be specified later, cf. inequality (4.88) below.

Proposition 4.6. *The variational sets* C *and* D *are nonempty and convex. Moreover,* D *admits a nonempty interior.*

Proof. The set C contains the origin and is convex together with U and K. The set D is described as the subgraph of a concave function over a convex range of definition in the space $L^p(\Omega, \mathbb{R}^{nm}) \times L^{p/2}(\Omega, \mathbb{R}^{\sigma(2)}) \times L^{p/3}(\Omega, \mathbb{R}^{\sigma(3)}) \times \,...\, \times L^{p/(n\wedge m)}(\Omega, \mathbb{R}^{\sigma(n\wedge m)})$. Consequently, D is convex as well. Obviously, the point $\big(-2\,K_1,\, \mathfrak{o},\, \mathfrak{o},\, \mathfrak{o},\, ...\, ,\, \mathfrak{o}\big)$ belongs to int (D) ∎

• **Step 2.** *Definition of the sets* C_η. We denote by $\mathrm{G} = \big\{\, z_1 \in L^p(\Omega, \mathbb{R}^{nm}) \,\big|\, \exists\, x \in W_0^{1,p}(\Omega, \mathbb{R}^n)$ such that $z_1 = Jx \,\big\}$ the subspace of the "gradients" within $L^p(\Omega, \mathbb{R}^{nm})$ and by $\mathrm{U}_0 = \mathrm{U} \cap \mathrm{G}$ the subset of those admissible controls of $(\mathrm{P})_0$, which may be completed to feasible pairs for $(\mathrm{P})_0$. For every $\eta \geqslant 0$, we define a set

$$\mathrm{C}_\eta \,=\, \Big\{\, \big(\, \varrho,\, z_1,\, z_2,\, z_3,\, ...\, ,\, z_{(n\wedge m)}\,\big) \tag{4.21}$$

$$\in \mathbb{R} \times L^p(\Omega, \mathbb{R}^{nm}) \times L^{p/2}(\Omega, \mathbb{R}^{\sigma(2)}) \times L^{p/3}(\Omega, \mathbb{R}^{\sigma(3)}) \times \,...\, \times L^{p/(n\wedge m)}(\Omega, \mathbb{R}^{\sigma(n\wedge m)}) \quad \text{with}$$

$$z_1 = Jx - Jx^* - (u - u^*)\,,\ \|\, z_1 \,\|_{L^p} \leqslant \eta\,; \tag{4.22}$$

$$z_2 = (w_2 - w_2^*) - D_u T_2(u^*)(u - u^*)\,,\ \|\, z_2 \,\|_{L^{p/2}} \leqslant \eta\,; \tag{4.23}$$

$$z_3 = (w_3 - w_3^*) - D_u T_3(u^*)(u - u^*)\,,\ \| z_3 \|_{L^{p/3}} \leqslant \eta\,; \tag{4.24}$$

$$\vdots$$

$$z_{(n\wedge m)} = (w_{(n\wedge m)} - w_{(n\wedge m)}^*) - D_u T_{(n\wedge m)}(u^*)(u - u^*)\,,\ \| z_{(n\wedge m)} \|_{L^{p/(n\wedge m)}} \leqslant \eta\,; \tag{4.25}$$

$$x \in W_0^{1,p}(\Omega, \mathbb{R}^n)\,,\ u \in \mathrm{U}_0 + \mathrm{K}(\mathfrak{o}, \eta) \subset L^p(\Omega, \mathbb{R}^{nm})\,, \tag{4.26}$$

$$w_2 \in L^{p/2}(\Omega, \mathbb{R}^{\sigma(2)})\,,\ w_3 \in L^{p/3}(\Omega, \mathbb{R}^{\sigma(3)})\,,\ \dots,\ w_{(n\wedge m)} \in L^{p/(n\wedge m)}(\Omega, \mathbb{R}^{\sigma(n\wedge m)})\,\big\}\,. \tag{4.27}$$

• **Step 3. Proposition 4.7.** *The variational set* D *is a subset of* C_α.

Proof. We must confirm that the components of a given element $\big(\varrho,\, z_1,\, z_2,\, z_3,\, \dots,\, z_{(n\wedge m)}\big) \in \mathrm{D}$ can be represented through

$$z_1 = Jx - Jx^* - (u - u^*)\,; \tag{4.28}$$

$$z_2 = (w_2 - w_2^*) - D_u T_2(u^*)(u - u^*)\,; \tag{4.29}$$

$$z_3 = (w_3 - w_3^*) - D_u T_3(u^*)(u - u^*)\,; \tag{4.30}$$

$$\vdots$$

$$z_{(n\wedge m)} = (w_{(n\wedge m)} - w_{(n\wedge m)}^*) - D_u T_{(n\wedge m)}(u^*)(u - u^*) \tag{4.31}$$

with functions $x \in W_0^{1,p}(\Omega, \mathbb{R}^n)$, $u \in \mathrm{U}_0 + \mathrm{K}(\mathfrak{o}, \alpha) \subset L^p(\Omega, \mathbb{R}^{nm})$, $w_2 \in L^{p/2}(\Omega, \mathbb{R}^{\sigma(2)})$, $w_3 \in L^{p/3}(\Omega, \mathbb{R}^{\sigma(3)})$, $\dots$, $w_{(n\wedge m)} \in L^{p/(n\wedge m)}(\Omega, \mathbb{R}^{\sigma(n\wedge m)})$. Indeed, since $\mathfrak{o} \in \mathrm{U}_0 \subset L^p(\Omega, \mathbb{R}^{nm})$ and $\| z_1 \|_{L^p} \leqslant \alpha$, we may choose $x = \mathfrak{o} \in W_0^{1,p}(\Omega, \mathbb{R}^n)$, $u = \mathfrak{o} + z_1 \in \mathrm{U}_0 + \mathrm{K}(\mathfrak{o}, \alpha) \subset L^p(\Omega, \mathbb{R}^{nm})$, $w_2 = z_2 + D_u T_2(u^*) z_1 \in L^{p/2}(\Omega, \mathbb{R}^{\sigma(2)})$, $w_3^0 = z_3 + D_u T_3(u^*) z_1 \in L^{p/3}(\Omega, \mathbb{R}^{\sigma(3)})$, $\dots$, $w_{(n\wedge m)}^0 = z_{(n\wedge m)} + D_u T_{(n\wedge m)}(u^*) z_1 \in L^{p/(n\wedge m)}(\Omega, \mathbb{R}^{\sigma(n\wedge m)})$, thus obtaining the claimed representation. ∎

• **Step 4. Proposition 4.8.** *Let* $\eta \geqslant 0$ *be given. If* $\big(\varrho,\, z_1,\, z_2,\, z_3,\, \dots,\, z_{(n\wedge m)}\big)$ *belongs to* $\mathrm{C}_\eta \cap \mathrm{C}$ *then it follows that* $\varrho \geqslant -K_2\, \eta$ *where* $K_2 > 0$ *is a constant independent on* η.

Proof. Step 4.1. Assume that an element $\big(\varrho,\, z_1,\, z_2,\, z_3,\, \dots,\, z_{(n\wedge m)}\big)$ belongs to the intersection $\mathrm{C}_\eta \cap \mathrm{C}$. Consequently, there exist a number $\tilde\varepsilon \geqslant 0$ and functions $\tilde x \in W_0^{1,p}(\Omega, \mathbb{R}^n)$, $\tilde u \in \mathrm{U} \cap \big(\mathrm{U}_0 + \mathrm{K}(\mathfrak{o}, \eta)\big) \subset L^p(\Omega, \mathbb{R}^{nm})$, $\tilde w_2 \in L^{p/2}(\Omega, \mathbb{R}^{\sigma(2)})$, $\tilde w_3 \in L^{p/3}(\Omega, \mathbb{R}^{\sigma(3)})$, $\dots$, $\tilde w_{(n\wedge m)} \in L^{p/(n\wedge m)}(\Omega, \mathbb{R}^{\sigma(n\wedge m)})$ such that $(\tilde u, \tilde w) \in \mathrm{W}$ and

$$\varrho \ = \tilde\varepsilon + D_x G(x^*, u^*, w^*)(\tilde x - x^*) + D_u G(x^*, u^*, w^*)(\tilde u - u^*) + D_w G(x^*, u^*, w^*)(\tilde w - w^*)\,; \tag{4.32}$$

$$z_1 \ = \ J\tilde x - Jx^* - (\tilde u - u^*)\,; \tag{4.33}$$

$$z_2 \ = \ (\tilde w_2 - w_2^*) - D_u T_2(u^*)(\tilde u - u^*)\,; \tag{4.34}$$

$$z_3 \ = \ (\tilde w_3 - w_3^*) - D_u T_3(u^*)(\tilde u - u^*)\,; \tag{4.35}$$

$$\vdots$$

$$z_{(n\wedge m)} \ = \ (\tilde w_{(n\wedge m)} - w_{(n\wedge m)}^*) - D_u T_{(n\wedge m)}(u^*)(\tilde u - u^*) \quad \text{as well as} \tag{4.36}$$

$$\| z_1 \|_{L^p} \leqslant \eta\,,\ \| z_2 \|_{L^{p/2}} \leqslant \eta\,,\ \| z_3 \|_{L^{p/3}} \leqslant \eta\,,\ \dots,\ \| z_{(n\wedge m)} \|_{L^{p/(n\wedge m)}} \leqslant \eta\,. \tag{4.37}$$

First, in relation to $\tilde u \in \mathrm{U}_0 + \mathrm{K}(\mathfrak{o}, \eta)$, we find $u^0 \in \mathrm{U}_0$ with $u^0 = Jx^0$, $x^0 \in W_0^{1,p}(\Omega, \mathbb{R}^n)$, and $\| \tilde u - u^0 \| \leqslant \eta$.[29] Thus we obtain

$$\mathfrak{o} \ = \ Jx^0 - Jx^* - (u^0 - u^*) \quad \text{and} \tag{4.38}$$

$$\| J\tilde x - Jx^0 \|_{L^p} \ = \ \| J\tilde x - u^0 \|_{L^p} \ \leqslant\ \| J\tilde x - \tilde u \|_{L^p} + \| \tilde u - u^0 \|_{L^p} \ \leqslant\ 2\,\eta \quad \Longrightarrow \tag{4.39}$$

$$\| \tilde x - x^0 \|_{W_0^{1,p}} \ \leqslant\ C_1\, \| J\tilde x - Jx^0 \|_{L^p} \ \leqslant\ 2\, C_1\, \eta \tag{4.40}$$

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[29] If $\eta = 0$ then we may employ $u^0 = \tilde u$, $x^0 = \tilde x$ and $w^0 = \tilde w$ throughout the proof.

by application of the Poincaré inequality with constant $C_1 > 0$.[30] Next, we find that

$$\tilde{w}_2 - w_2^* \;=\; D_u T_2(u^*)(\tilde{u} - u^*) + z_2 \;=\; D_u T_2(u^*)(\tilde{u} - u^0) + D_u T_2(u^*)(u^0 - u^*) + z_2 \quad \Longrightarrow \tag{4.41}$$

$$\big(\,\tilde{w}_2 - D_u T_2(u^*)(\tilde{u} - u^0) - z_2\,\big) - w_2^* \;=\; D_u T_2(u^*)(u^0 - u^*)\,. \tag{4.42}$$

Using the abbreviation $\tilde{w}_2 - D_u T_2(u^*)(\tilde{u} - u^0) - z_2 = w_2^0 \in L^{p/2}(\Omega, \mathbb{R}^{\sigma(2)})$, we obtain

$$\mathfrak{o} \;=\; (w_2^0 - w_2^*) - D_u T_2(u^*)(u^0 - u^*) \quad \text{and} \tag{4.43}$$

$$\|\, w_2^0 - \tilde{w}_2 \,\|_{L^{p/2}} \;\leqslant\; \|\, D_u T_2(u^*) \,\|_{\mathcal{L}(L^p, L^{p/2})} \cdot \|\, \tilde{u} - u^0 \,\|_{L^p} + \|\, z_2 \,\|_{L^{p/2}} \;\leqslant\; (1 + C_2)\, \eta\,. \tag{4.44}$$

Analogously, we find elements $w_3^0 \in L^{p/3}(\Omega, \mathbb{R}^{\sigma(3)}),\ \ldots\ ,\ w_{(n\wedge m)}^0 \in L^{p/(n\wedge m)}(\Omega, \mathbb{R}^{\sigma(n\wedge m)})$ such that

$$\mathfrak{o} \;=\; (w_3^0 - w_3^*) - D_u T_3(u^*)(u^0 - u^*) \quad \text{and} \tag{4.45}$$

$$\|\, w_3^0 - \tilde{w}_3 \,\|_{L^{p/3}} \;\leqslant\; \|\, D_u T_3(u^*) \,\|_{\mathcal{L}(L^p, L^{p/3})} \cdot \|\, \tilde{u} - u^0 \,\|_{L^p} + \|\, z_3 \,\|_{L^{p/3}} \;\leqslant\; (1 + C_3)\, \eta\,; \tag{4.46}$$

$$\vdots$$

$$\mathfrak{o} \;=\; (w_{(n\wedge m)}^0 - w_{(n\wedge m)}^*) - D_u T_{(n\wedge m)}(u^*)(u^0 - u^*) \quad \text{and} \tag{4.47}$$

$$\begin{aligned} \|\, w_{(n\wedge m)}^0 - \tilde{w}_{(n\wedge m)} \,\|_{L^{p/(n\wedge m)}} \;\leqslant\; \|\, D_u T_{(n\wedge m)}(u^*) \,\|_{\mathcal{L}(L^p, L^{p/(n\wedge m)})} \cdot \|\, \tilde{u} - u^0 \,\|_{L^p} & \\ + \|\, z_{(n\wedge m)} \,\|_{L^{p/(n\wedge m)}} \;\leqslant\; (1 + C_{(n\wedge m)})\, \eta\,. & \end{aligned} \tag{4.48}$$

The constants $C_2 > 0$, $C_3 > 0$, ... , $C_{(n\wedge m)} > 0$ depend only on (x^*, u^*) and the data of $(\mathrm{P})_0 \;-\; (\mathrm{P})_2$.

• **Step 4.2.** As a next step, we will employ Lyusternik's theorem, which reads as follows:

Theorem 4.9. (Ljusternik's theorem)[31] *Consider Banach spaces* X, Y, *the (possibly nonlinear) operator* $M\colon \mathrm{X} \to \mathrm{Y}$ *and its kernel* $\mathcal{M} = \{\, r \in \mathrm{X} \;\big|\; M(r) = \mathfrak{o} \,\}$. *If* $r^* \in \mathcal{M}$, M *is continuously Fréchet differentiable in a neighbourhood of* r^* *and* $DM(r^*)$ *maps onto* Y *then the set of the tangential vectors for* $\mathcal{M}$ *at the point* r^* *coincides with the kernel* $\{\, r \in \mathrm{X} \;\big|\; DM(r^*)(r) = \mathfrak{o} \,\}$.

Let us apply Theorem 4.9. to the data

$$\mathrm{X} \;=\; L^p(\Omega, \mathbb{R}^{nm}) \,\times\, L^{p/2}(\Omega, \mathbb{R}^{\sigma(2)}) \,\times\, L^{p/3}(\Omega, \mathbb{R}^{\sigma(3)}) \,\times\, \ldots \,\times\, L^{p/(n\wedge m)}(\Omega, \mathbb{R}^{\sigma(n\wedge m)})\,; \tag{4.49}$$

$$\mathrm{Y} \;=\; L^{p/2}(\Omega, \mathbb{R}^{\sigma(2)}) \,\times\, L^{p/3}(\Omega, \mathbb{R}^{\sigma(3)}) \,\times\, \ldots \,\times\, L^{p/(n\wedge m)}(\Omega, \mathbb{R}^{\sigma(n\wedge m)})\,; \tag{4.50}$$

$$M \;=\; \big(\, E_2\,,\, \ldots\,,\, E_{(n\wedge m)} \,\big)\,; \tag{4.51}$$

$$r^* \;=\; (u^*, w^*)\,. \tag{4.52}$$

Then we may observe that the Fréchet derivative $DM(u^*, w^*)\colon\ \mathrm{X} \times \mathrm{Y} \to \mathrm{Y}$, which is given through

$$DM(u^*, w^*)(\, u - u^*, w - w^*) \;=\; \begin{pmatrix} w_2 - w_2^* - D_u T_2(u^*)(u - u^*) \\ w_3 - w_3^* - D_u T_3(u^*)(u - u^*) \\ \vdots \\ w_{(n\wedge m)} - w_{(n\wedge m)}^* - D_u T_{(n\wedge m)}(u^*)(u - u^*) \end{pmatrix}, \tag{4.53}$$

is a mapping onto the target space Y. The continuity of DM with respect to the reference point is obvious. Consequently, equations (4.43), (4.45) and (4.47) imply that $(u^0 - u^*, w^0 - w^*)$ is a tangential vector for the

[30] [Adams/Fournier 07] , p. 184, Corollary 6.31.

[31] [Ioffe/Tichomirow 79] , p. 42.

set $\mathcal{M} = \{ (u,w) \in \mathrm{X} \mid M(u,w) = \mathfrak{o} \}$ at (u^*, w^*), and we find elements $\big(Q(u^0,\lambda)\,,\ R(w^0,\lambda) \big) \in \mathrm{X}$ satisfying

$$\big(u^* + \lambda\,(u^0 - u^*) + Q(u^0,\lambda)\,,\ w^* + \lambda\,(w^0 - w^*) + R(w^0,\lambda) \big) \in \mathcal{M} \quad \Longleftrightarrow \tag{4.54}$$

$$w_2^*(s) + \lambda\,(w_2^0 - w_2^*) + R_2(w^0,\lambda) - \mathrm{adj}_2\big(u^*(s) + \lambda\,(u^0 - u^*) + Q(u^0,\lambda) \big) = 0 \quad (\forall)\, s \in \Omega\,; \tag{4.55}$$

$$w_3^*(s) + \lambda\,(w_3^0 - w_3^*) + R_3(w^0,\lambda) - \mathrm{adj}_3\big(u^*(s) + \lambda\,(u^0 - u^*) + Q(u^0,\lambda) \big) = 0 \quad (\forall)\, s \in \Omega\,; \tag{4.56}$$

$$\vdots$$

$$w_{(n\wedge m)}^*(s) + \lambda\,(w_{(n\wedge m)}^0 - w_{(n\wedge m)}^*) + R_{(n\wedge m)}(w^0,\lambda) - \mathrm{adj}_{(n\wedge m)}\big(u^*(s) + \lambda\,(u^0 - u^*) + Q(u^0,\lambda) \big) = 0 \quad (\forall)\, s \in \Omega \tag{4.57}$$

for all sufficiently small $0 \leqslant \lambda < 1$ where

$$\lim_{\lambda\to 0+0} \lambda^{-1}\,\|\, Q(u^0,\lambda) \,\|_{L^p} = 0\,; \tag{4.58}$$

$$\lim_{\lambda\to 0+0} \lambda^{-1}\,\|\, R_2(w^0,\lambda) \,\|_{L^{p/2}} = 0\,; \tag{4.59}$$

$$\lim_{\lambda\to 0+0} \lambda^{-1}\,\|\, R_3(w^0,\lambda) \,\|_{L^{p/3}} = 0\,; \tag{4.60}$$

$$\vdots$$

$$\lim_{\lambda\to 0+0} \lambda^{-1}\,\|\, R_{(n\wedge m)}(w^0,\lambda) \,\|_{L^{p/(n\wedge m)}} = 0\,. \tag{4.61}$$

• **Step 4.3.** We perform the decomposition

$$\mathrm{adj}_2\big(u^*(s) + \lambda\,(u^0 - u^*) + Q(u^0,\lambda) \big) = \mathrm{adj}_2\big(u^*(s) + \lambda\,(u^0 - u^*) \big) + S_2(u^*,u^0,\lambda)\,; \tag{4.62}$$

$$\mathrm{adj}_3\big(u^*(s) + \lambda\,(u^0 - u^*) + Q(u^0,\lambda) \big) = \mathrm{adj}_3\big(u^*(s) + \lambda\,(u^0 - u^*) \big) + S_3(u^*,u^0,\lambda)\,; \tag{4.63}$$

$$\vdots$$

$$\mathrm{adj}_{(n\wedge m)}\big(u^*(s) + \lambda\,(u^0 - u^*) + Q(u^0,\lambda) \big) = \mathrm{adj}_{(n\wedge m)}\big(u^*(s) + \lambda\,(u^0 - u^*) \big) + S_{(n\wedge m)}(u^*,u^0,\lambda)\,, \tag{4.64}$$

thus getting

$$w_2^*(s) + \lambda\,(w_2^0 - w_2^*) + R_2(w^0,\lambda) - S_2(u^*,u^0,\lambda) - \mathrm{adj}_2\big(u^*(s) + \lambda\,(u^0 - u^*) \big) = 0\,; \tag{4.65}$$

$$w_3^*(s) + \lambda\,(w_3^0 - w_3^*) + R_3(w^0,\lambda) - S_3(u^*,u^0,\lambda) - \mathrm{adj}_3\big(u^*(s) + \lambda\,(u^0 - u^*) \big) = 0\,; \tag{4.66}$$

$$\vdots$$

$$w_{(n\wedge m)}^*(s) + \lambda\,(w_{(n\wedge m)}^0 - w_{(n\wedge m)}^*) + R_{(n\wedge m)}(w^0,\lambda) - S_{(n\wedge m)}(u^*,u^0,\lambda) - \mathrm{adj}_{(n\wedge m)}\big(u^*(s) + \lambda\,(u^0 - u^*) \big) = 0\,, \tag{4.67}$$

and the triples

$$\Big(x^* + \lambda\,(x^0 - x^*)\,,\ u^* + \lambda\,(u^0 - u^*)\,,\ w^* + \lambda\,(w^0 - w^*) + R(w^0,\lambda) - S(u^*,u^0,\lambda) \Big) \tag{4.68}$$

are feasible in $(\mathrm{P})_2$ for all sufficiently small $0 < \lambda < 1$. We will convince ourselves that the expressions $S(u^*,u^0,\lambda)$ satisfy limit relations analogous to $R(w^0,\lambda)$.

• **Step 4.4. Lemma 4.10.** *It holds that* $\lim\limits_{\lambda\to 0+0} \lambda^{-1}\,\|\, S_2(u^*,u^0,\lambda) \,\|_{L^{p/2}} = 0$, $\lim\limits_{\lambda\to 0+0} \lambda^{-1}\,\|\, S_3(u^*,u^0,\lambda) \,\|_{L^{p/3}} = 0$, … , $\lim\limits_{\lambda\to 0+0} \lambda^{-1}\,\|\, S_{(n\wedge m)}(u^*,u^0,\lambda) \,\|_{L^{p/(n\wedge m)}} = 0$.

Proof. [32] We start with expanding (4.62). Then to every index $1 \leqslant l \leqslant \sigma(2)$ correspond indices $1 \leqslant i < k \leqslant n$, $1 \leqslant j < r \leqslant m$ such that

$$S_{2,l}(u^*, u^0, \lambda) \;=\; \big(u^*_{ij} + \lambda\,(u^0_{ij} - u^*_{ij}) \big)\, Q_{kr}(u^0, \lambda) - \big(u^*_{kj} + \lambda\,(u^0_{kj} - u^*_{kj}) \big)\, Q_{ir}(u^0, \lambda) \tag{4.69}$$

$$+\; Q_{ij}(u^0, \lambda)\, \big(u^*_{kr} + \lambda\,(u^0_{kr} - u^*_{kr}) \big) - Q_{kj}(u^0, \lambda)\, \big(u^*_{ir} + \lambda\,(u^0_{ir} - u^*_{ir}) \big)$$

$$+\, Q_{ij}(u^0, \lambda)\, Q_{kr}(u^0, \lambda) - Q_{kj}(u^0, \lambda)\, Q_{ir}(u^0, \lambda) \quad \Longrightarrow$$

$$\int_\Omega \big| \, S_{2,l}(u^*, u^0, \lambda) \, \big|^{p/2} \, ds \;\leqslant\; C \Big(\int_\Omega \big| \, Q_{kr}(u^0, \lambda) \big|^{p/2} \, ds + \int_\Omega \big| \, Q_{ir}(u^0, \lambda) \, \big|^{p/2} \, ds + \int_\Omega \big| \, Q_{ij}(u^0, \lambda) \big|^{p/2} \, ds \tag{4.70}$$

$$+\; \int_\Omega \big| \, Q_{kj}(u^0, \lambda) \big|^{p/2} \, ds + \int_\Omega \big| \, Q_{ij}(u^0, \lambda)\, Q_{kr}(u^0, \lambda) \, \big|^{p/2} \, ds + \int_\Omega \big| \, Q_{kj}(u^0, \lambda)\, Q_{ir}(u^0, \lambda) \, \big|^{p/2} \, ds \Big)$$

since $\big(u^*_{ij}(s) + \lambda\,(u^0_{ij}(s) - u^*_{ij}(s)) \big)$, $\big(u^*_{kj}(s) + \lambda\,(u^0_{kj}(s) - u^*_{kj}(s)) \big)$, $\big(u^*_{kr}(s) + \lambda\,(u^0_{kr}(s) - u^*_{kr}(s)) \big)$ and $\big(u^*_{ir}(s) + \lambda\,(u^0_{ir}(s) - u^*_{ir}(s)) \big)$ belong to the compact set K for almost all $s \in \Omega$. This implies the estimate

$$\| \, S_{2,l}(u^*, u^0, \lambda) \, \|_{L^{p/2}(\Omega)} \tag{4.71}$$

$$\leqslant\; C \Big(\| \, Q_{kr}(u^0, \lambda) \, \|_{L^{p/2}(\Omega)} + \| \, Q_{ir}(u^0, \lambda) \, \|_{L^{p/2}(\Omega)} + \| \, Q_{ij}(u^0, \lambda) \, \|_{L^{p/2}(\Omega)} + \| \, Q_{kj}(u^0, \lambda) \, \|_{L^{p/2}(\Omega)}$$

$$+ \| \, Q_{ij}(u^0, \lambda) \, \|_{L^p(\Omega)} \cdot \| \, Q_{kr}(u^0, \lambda) \, \|_{L^p(\Omega)} + \| \, Q_{kj}(u^0, \lambda) \, \|_{L^p(\Omega)} \cdot \| \, Q_{ir}(u^0, \lambda) \, \|_{L^p(\Omega)} \Big)$$

$$\leqslant\; \widetilde{C} \Big(\| \, Q_{kr}(u^0, \lambda) \, \|_{L^p(\Omega)} + \| \, Q_r(u^0, \lambda) \, \|_{L^p(\Omega)} + \| \, Q_{ij}(u^0, \lambda) \, \|_{L^p(\Omega)} + \| \, Q_{kj}(u^0, \lambda) \, \|_{L^p(\Omega)} \tag{4.72}$$

$$+ \| \, Q_{ij}(u^0, \lambda) \, \|_{L^p(\Omega)} \cdot \| \, Q_{kr}(u^0, \lambda) \, \|_{L^p(\Omega)} + \| \, Q_{kj}(u^0, \lambda) \, \|_{L^p(\Omega)} \cdot \| \, Q_{ir}(u^0, \lambda) \, \|_{L^p(\Omega)} \Big) \quad \Longrightarrow$$

$$\lim_{\lambda \to 0+0} \; \lambda^{-1} \, \| \, S_{2,l}(u^*, u^0, \lambda) \, \|_{L^{p/2}} \tag{4.73}$$

$$\leqslant\; \widetilde{C} \Big(\lim_{\lambda \to 0+0} \; \lambda^{-1} \, \| \, Q_{kr}(u^0, \lambda) \, \|_{L^p(\Omega)} + \lim_{\lambda \to 0+0} \; \lambda^{-1} \, \| \, Q_{ir}(u^0, \lambda) \, \|_{L^p(\Omega)}$$

$$+\; \lim_{\lambda \to 0+0} \; \lambda^{-1} \, \| \, Q_{ij}(u^0, \lambda) \, \|_{L^p(\Omega)} + \lim_{\lambda \to 0+0} \; \lambda^{-1} \| \, Q_{kj}(u^0, \lambda) \, \|_{L^p(\Omega)}$$

$$+\; \lim_{\lambda \to 0+0} \; \lambda^{-1} \, \| \, Q_{ij}(u^0, \lambda) \, \|_{L^p(\Omega)} \cdot \lim_{\lambda \to 0+0} \; \| \, Q_{kr}(u^0, \lambda) \, \|_{L^p(\Omega)}$$

$$+\; \lim_{\lambda \to 0+0} \; \lambda^{-1} \, \| \, Q_{kj}(u^0, \lambda) \, \|_{L^p(\Omega)} \cdot \lim_{\lambda \to 0+0} \; \| \, Q_{ir}(u^0, \lambda) \, \|_{L^p(\Omega)} \Big) \;=\; 0 \tag{4.74}$$

by assumption about $Q(u^0, \lambda)$. Analogously, the limit relations $\lambda^{-1} \, \| \, S_{3,l}(u^*, u^0, \lambda) \, \|_{L^{p/3}} \to 0$ for all $1 \leqslant l \leqslant \sigma(3)$, ... , $\lambda^{-1} \, \| \, S_{(n \wedge m),l}(u^*, u^0, \lambda) \, \|_{L^{p/(n \wedge m)}} \to 0$ for all $1 \leqslant l \leqslant \sigma(n \wedge m)$ may be confirmed. ∎

• **Step 4.5.** We are ready to compute the limit

$$\lim_{\lambda \to 0+0} \; \frac{1}{\lambda} \, \Big(\widetilde{G}\big(\, x^* + \lambda\,(x^0 - x^*) \, , \, u^* + \lambda\,(u^0 - u^*) \, , \tag{4.75}$$

$$w^* + \lambda\,(w^0 - w^*) + R(w^0, \lambda) - S(u^*, u^0, \lambda) \big) \;-\; \widetilde{G}(x^*, u^*, w^*) \Big)$$

$$=\; \lim_{\lambda \to 0+0} \; \frac{1}{\lambda} \, \Big(G\big(\, x^* + \lambda\,(x^0 - x^*) \, , \, u^* + \lambda\,(u^0 - u^*) \, , \tag{4.76}$$

$$w^* + \lambda\,(w^0 - w^*) + R(w^0, \lambda) - S(u^*, u^0, \lambda) \big) \;-\; G(x^*, u^*, w^*) \Big)$$

$$+\; \lim_{\lambda \to 0+0} \; \frac{K_1}{\lambda} \, \Big(\mathrm{Dist} \, \Big(\, \big(\, x^* + \lambda\,(x^0 - x^*) \, , \, u^* + \lambda\,(u^0 - u^*) \, , \, w^* + \lambda\,(w^0 - w^*) + R(w^0, \lambda) - S(u^*, u^0, \lambda) \, \big) \, ,$$

$$L^p(\Omega, \mathbb{R}^n) \times \mathrm{W} \Big) \;-\; \mathrm{Dist} \, \Big(\, \big(\, x^*, \, u^*, \, w^* \, \big) \, , \, L^p(\Omega, \mathbb{R}^n) \times \mathrm{W} \, \Big) \Big) \;\geqslant\; 0 \, .$$

---

[32] The proof, which is identical with [WAGNER 13], p. 12, Proof of Lemma 4.8., will be repeated here for sake of completeness.

Since (x^*, u^*, w^*) and $(\tilde{x}, \tilde{u}, \tilde{w})$ belong to the convex set $L^p(\Omega, \mathbb{R}^n) \times \mathrm{W}$, we have

$$\mathrm{Dist}\Big(\,(\,x^*,\,u^*,\,w^*\,)\,,\,L^p(\Omega,\mathbb{R}^n)\times\mathrm{W}\Big)\Big) \;=\; 0\,; \tag{4.77}$$

$$\mathrm{Dist}\Big(\,\big(x^*+\lambda\,(\tilde{x}-x^*)\,,\,u^*+\lambda\,(\tilde{u}-u^*)\,,\,w^*+\lambda\,(\tilde{w}-w^*)\,\big)\,,\,L^p(\Omega,\mathbb{R}^n)\times\mathrm{W}\Big)\Big) \;=\; 0\,, \tag{4.78}$$

and in the last term of (4.76), the expression (4.77) may be replaced by (4.78). Thus we get

$$\begin{aligned} &\ldots + \lim_{\lambda\to 0+0} \frac{K_1}{\lambda}\,\Big(\mathrm{Dist}\Big(\,\big(x^*+\lambda\,(x^0-x^*)\,,\,u^*+\lambda\,(u^0-u^*)\,,\,w^*+\lambda\,(w^0-w^*)+R(w^0,\lambda)-S(u^*,u^0,\lambda)\big)\,, \\ &\quad L^p(\Omega,\mathbb{R}^n)\times\mathrm{W}\Big) - \mathrm{Dist}\Big(\,\big(\,x^*+\lambda\,(\tilde{x}-x^*)\,,\,u^*+\lambda\,(\tilde{u}-u^*)\,,\,w^*+\lambda\,(\tilde{w}-w^*)\big)\,,\,L^p(\Omega,\mathbb{R}^n)\times\mathrm{W}\Big)\Big) \geqslant 0 \end{aligned} \tag{4.79}$$

$$\Longrightarrow \lim_{\lambda\to 0+0} \frac{1}{\lambda}\,\Big(G\big(x^*+\lambda\,(x^0-x^*)\,,\,u^*+\lambda\,(u^0-u^*)\,,\,w^*+\lambda\,(w^0-w^*)+R(w^0,\lambda)-S(u^*,u^0,\lambda)\big) \tag{4.80}$$

$$- G(x^*,u^*,w^*)\Big) \geqslant -\lim_{\lambda\to 0+0} \frac{K_1}{\lambda}\cdot\big|\,\mathrm{Dist}\,(\ldots) - \mathrm{Dist}\,(\ldots)\,\big|$$

$$\geqslant -\lim_{\lambda\to 0+0} \frac{K_1}{\lambda}\,\Big(\lambda\,\|\,\tilde{x}-x^0\,\|_{L^p}+\lambda\,\|\,\tilde{u}-u^0\,\|_{L^p} \tag{4.81}$$

$$+\sum_{r=2}^{(n\wedge m)}\Big(\lambda\,\|\,\tilde{w}_r-w_r^0\,\|_{L^{p/r}}+\|\,R_r(w^0,\lambda)\,\|_{L^{p/r}}+\|\,S_r(u^*,u^0,\lambda)\,\|_{L^{p/r}}\Big)\Big)$$

since the distance function to a closed set of a normed space satisfies a Lipschitz condition with constant 1.[33] The left-hand side of (4.80) may be expanded as follows:

$$\lim_{\lambda\to 0+0} \frac{1}{\lambda}\,\Big(G\big(\,x^*+\lambda\,(x^0-x^*)\,,\,u^*+\lambda\,(u^0-u^*)\,, \tag{4.82}$$

$$w^*+\lambda\,(w^0-w^*)+R(w^0,\lambda)-S(u^*,u^0,\lambda)\big)\;-\;G(x^*,u^*,w^*)\Big)$$

$$= \lim_{\lambda\to 0+0} \frac{1}{\lambda}\,\Big(G\big(\,x^*+\lambda\,(x^0-x^*)\,,\,u^*+\lambda\,(u^0-u^*)\,,\,w^*+\lambda\,(w^0-w^*)+R(w^0,\lambda)-S(u^*,u^0,\lambda)\big) \tag{4.83}$$

$$-G\big(\,x^*\,,\,u^*+\lambda\,(u^0-u^*)\,,\,w^*+\lambda\,(w^0-w^*)+R(w^0,\lambda)-S(u^*,u^0,\lambda)\big)\Big)$$

$$+\lim_{\lambda\to 0+0} \frac{1}{\lambda}\,\Big(G\big(\,x^*\,,\,u^*+\lambda\,(u^0-u^*)\,,\,w^*+\lambda\,(w^0-w^*)+R(w^0,\lambda)-S(u^*,u^0,\lambda)\big)$$

$$-G\big(\,x^*\,,\,u^*\,,\,w^*+\lambda\,(w^0-w^*)+R(w^0,\lambda)-S(u^*,u^0,\lambda)\,\big)\Big)$$

$$+\lim_{\lambda\to 0+0} \frac{1}{\lambda}\,\Big(G\big(\,x^*\,,\,u^*\,,\,w^*+\lambda\,(w^0-w^*)+R(w^0,\lambda)-S(u^*,u^0,\lambda)\big)-G(x^*,u^*,w^*)\Big)$$

$$= D_xG(x^*,u^*,w^*)\,(x^0-x^*)+D_uG(x^*,u^*,w^*)\,(u^0-u^*)+D_wG(x^*,u^*,w^*)\,(w^0-w^*)\,. \tag{4.84}$$

On the other hand, using (4.40), (4.44), (4.46) and (4.48), we may continue (4.80) and (4.81) through

$$D_xG(x^*,u^*,w^*)\,(\tilde{x}-x^*)+D_uG(x^*,u^*,w^*)\,(\tilde{u}-u^*)+D_wG(x^*,u^*,w^*)\,(\tilde{w}-w^*) \tag{4.85}$$

$$+D_xG(x^*,u^*,w^*)\,(x^0-\tilde{x})+D_uG(x^*,u^*,w^*)\,(u^0-\tilde{u})+D_wG(x^*,u^*,w^*)\,(w^0-\tilde{w})$$

$$\geqslant -K_1\,\eta\,\Big(2\,C_1+1+\sum_{r=2}^{(n\wedge m)}(1+C_r)\Big)$$

$$-\lim_{\lambda\to 0+0} K_1\sum_{r=2}^{(n\wedge m)}\lambda^{-1}\Big(\|\,R_r(w^0,\lambda)\,\|_{L^{p/r}}+\|\,S_r(u^*,u^0,\lambda)\,\|_{L^{p/r}}\Big)\,.$$

---

[33] [Clarke 90], p. 50, Proposition 2.4.1.

Consequently, the first component ϱ of our element from Step 4.1. satisfies

$$\begin{aligned}
\varrho \,=\, & \tilde{\varepsilon} + D_x G(x^*,u^*,w^*)\,(\tilde{x}-x^*) + D_u G(x^*,u^*,w^*)\,(\tilde{u}-u^*) + D_w G(x^*,u^*,w^*)\,(\tilde{w}-w^*) && (4.86)\\
\geqslant\, & -\Big(\, \|\, D_x G(x^*,u^*,w^*)\,\| \cdot \|\, x^0-\tilde{x}\,\|_{L^p} + \|\, D_u G(x^*,u^*,w^*)\,\| \cdot \|\, u^0-\tilde{u}\,\|_{L^p} && (4.87)\\
& + \|\, D_w G(x^*,u^*,w^*)\,\| \cdot \sum_{r=2}^{(n\wedge m)} \|\, w_r^0 - \tilde{w}_r\,\|_{L^{p/r}} \Big) - K_1\,\eta\,\Big(\, 2\,C_1 + 1 + \sum_{r=2}^{(n\wedge m)} (1+C_r) \Big) \geqslant -K_2\,\eta && (4.88)
\end{aligned}$$

with a sufficiently large number $K_2 > 0$. The proof of Proposition 4.8. is complete. ■

In particular, Proposition 4.8. implies that the origin $\big(0,\, \mathfrak{o},\, \mathfrak{o},\, \mathfrak{o},\, \dots,\, \mathfrak{o}\big)$, which belongs to $\mathrm{C}_0 \cap \mathrm{C}$, must be a boundary point of C.

• **Step 5.** *Separation of* C *and* D*.* Propositions 4.7. and 4.8. imply together that the convex sets C and D are disjoint while int (D) $\neq \emptyset$. Consequently, we may apply the weak separation theorem [34] in order to find a nontrivial linear, continuous functional $(\,\lambda_0,\, y^{(1)},\, y^{(2)},\, y^{(3)},\, \dots,\, y^{(n\wedge m)}) \in \mathbb{R} \times L^{p/(p-1)}(\Omega,\mathbb{R}^{nm}) \times L^{p/(p-2)}(\Omega,\mathbb{R}^{\sigma(2)}) \times L^{p/(p-3)}(\Omega,\mathbb{R}^{\sigma(3)}) \times \dots \times L^{p/(p-(n\wedge m))}(\Omega,\mathbb{R}^{\sigma(n\wedge m)})$, which separates C and D properly. Consequently, we arrive at the variational inequality

$$\begin{aligned}
& \lambda_0\,\varrho' + \langle\, y^{(1)},\, z_1'\,\rangle + \langle\, y^{(2)},\, z_2'\,\rangle + \langle\, y^{(3)},\, z_3'\,\rangle + \dots + \langle\, y^{(n\wedge m)},\, z'_{(n\wedge m)}\,\rangle && (4.89)\\
& \qquad \geqslant \lambda_0\,\varrho'' + \langle\, y^{(1)},\, z_1''\,\rangle + \langle\, y^{(2)},\, z_2''\,\rangle + \langle\, y^{(3)},\, z_3''\,\rangle + \dots + \langle\, y^{(n\wedge m)},\, z''_{(n\wedge m)}\,\rangle\\
& \forall\,\big(\,\varrho',\, z_1',\, z_2',\, z_3',\, \dots,\, z'_{(n\wedge m)}\big) \in \mathrm{C} \quad \forall\,\big(\,\varrho'',\, z_1'',\, z_2'',\, z_3'',\, \dots,\, z''_{(n\wedge m)}\big) \quad \text{with}\\
& \|\, z_1''\,\|_{L^p} \leqslant \alpha\,,\ \|\, z_2''\,\|_{L^{p/2}} \leqslant \alpha\,,\ \|\, z_3''\,\|_{L^{p/3}} \leqslant \alpha\,,\ \dots,\ \|\, z''_{(n\wedge m)}\,\|_{L^{p/(n\wedge m)}} \leqslant \alpha \quad \text{and} && (4.90)\\
& \varrho'' \leqslant -2\,K\,\big(\,\|\, z_1''\,\|_{L^p} + \|\, z_2''\,\|_{L^{p/2}} + \|\, z_3\,\|_{L^{p/3}} + \dots + \|\, z_{(n\wedge m)}\,\|_{L^{p/(n\wedge m)}}\,\big)\,. && (4.91)
\end{aligned}$$

• **Step 6.** *Derivation of the optimality conditions from the variational inequality* (4.89).

a) Nonnegativity. Inserting $(\,1,\, \mathfrak{o},\, \mathfrak{o},\, \mathfrak{o},\, \dots,\, \mathfrak{o}\,) \in \mathrm{C}$ (generated with $\varepsilon = 1$, $x = x^*$, $u = u^*$ and $w = w^*$) and $(\,-1,\, \mathfrak{o},\, \mathfrak{o},\, \mathfrak{o},\, \dots,\, \mathfrak{o}\,) \in \mathrm{D}$ into the inequality, we find $\lambda_0 \geqslant 0$.

b) Derivation of $(\mathcal{M})$*.* Next we insert into the inequality (4.89) elements of C generated with $\varepsilon = 0$, $x = x^*$ and functions u and w such that $u \in \mathrm{U}$ and $(u,w) \in \mathrm{W}$ together with $(\,0,\, \mathfrak{o},\, \mathfrak{o},\, \mathfrak{o},\, \dots,\, \mathfrak{o}\,) \in \mathrm{cl}\,(\mathrm{D})$. This yields the maximum condition $(\mathcal{M})$, namely

$$\begin{aligned}
& \lambda_0\,\big(\, G(x^*,u,w) - G(x^*,u^*,w^*)\,\big) - \langle\, y^{(1)},\, u-u^*\,\rangle && (4.92)\\
& \qquad + \sum_{r=2}^{(n\wedge m)} \langle\, y^{(r)},\, w_r - w_r^*\,\rangle - \sum_{r=2}^{(n\wedge m)} \langle\, y^{(r)},\, D_u T_r(u^*)\,(u-u^*)\,\rangle \geqslant 0\,.
\end{aligned}$$

c) Derivation of $(\mathcal{K})$*.* Insert now into (4.89) elements of C generated with $\varepsilon = 0$, $u = u^*$, $w = w^*$ and arbitrary $x \in W_0^{1,p}(\Omega,\mathbb{R}^n)$ and $(\,0,\, \mathfrak{o},\, \mathfrak{o},\, \mathfrak{o},\, \dots,\, \mathfrak{o}\,) \in \mathrm{cl}\,(\mathrm{D})$. This yields

$$\lambda_0\, D_x G(x^*,u^*,w^*)\,(x-x^*) + \langle\, y^{(1)},\, Jx - Jx^*\,\rangle \,\geqslant\, 0\,. \tag{4.93}$$

Inserting further the element of C generated with $\varepsilon = 0$, $u = u^*$, $w = w^*$ and $(2x^* - x) \in W_0^{1,p}(\Omega,\mathbb{R}^n)$

[34] [Ioffe/Tichomirow 79], p. 152, Theorem 1.

instead of x, we obtain the reverse inequality

$$\lambda_0\, D_x G(x^*, u^*, w^*)\,(x - x^*) + \langle\, y^{(1)},\; Jx - Jx^*\,\rangle \;\leqslant\; 0\,, \tag{4.94}$$

and we arrive at the canonical equation $(\mathcal{K})$. The proof of Theorem 4.3. is complete. ■

Proof of Proposition 4.4. Let us assume, on the contrary, that $\lambda_0 = 0$. Then, inserting $u = u^*$ into the maximum condition $(\mathcal{M})$, we obtain the inequality

$$\sum_{r=2}^{(n\wedge m)} \langle\, y^{(r)},\; w_r - w_r^*\,\rangle \;=\; \sum_{r=2}^{(n\wedge m)} \langle\, y^{(r)},\; w_r - T_r(\,u^*\,)\,\rangle \;\geqslant\; 0\,, \tag{4.95}$$

which holds true for all functions w belonging to elements $(u, w) \in \mathrm{W} \cap \big(\, \mathrm{U} \times L^{p/2}(\Omega, \mathbb{R}^{\sigma(2)}) \times L^{p/3}(\Omega, \mathbb{R}^{\sigma(3)}) \times \ldots \times L^{p/(n\wedge m)}(\Omega, \mathbb{R}^{\sigma(n\wedge m)})\,\big)$. By assumption, we are allowed to insert into (4.95) arbitrary functions $w \in L^\infty(\Omega, \mathbb{R}^{\sigma(2)}) \times L^\infty(\Omega, \mathbb{R}^{\sigma(3)}) \times \ldots \times L^\infty(\Omega, \mathbb{R}^{\sigma(n\wedge m)})$ with

$$\big\|\, w_r - T_r(\,u^*\,)\,\big\|_{L^\infty(\Omega, \mathbb{R}^{\sigma(r)})} \;\leqslant\; \gamma\,,\;\; 2 \leqslant r \leqslant (n\wedge m)\,. \tag{4.96}$$

Consequently, for $2 \leqslant r \leqslant (n \wedge m)$, $y^{(r)}$ vanishes on all functions $z \in C_0^\infty(\Omega, \mathbb{R}^{\sigma(r)}) \cap L^p(\Omega, \mathbb{R}^{\sigma(r)})$ and thus on the whole space $L^p(\Omega, \mathbb{R}^{\sigma(r)})$, cf. [ADAMS/FOURNIER 07] , p. 38, Corollary 2.30., and we get $y^{(2)}$, $y^{(3)}$, ... , $y^{(n\wedge m)} = \mathfrak{o}$. Further, condition $(\mathcal{K})$ reduces to

$$\langle\, y^{(1)}\,,\; Jx\,\rangle \;=\; \langle\, y^{(1)}\,,\; Jx^*\,\rangle \quad \forall\, x \in W_0^{1,p}(\Omega, \mathbb{R}^n)\,, \tag{4.97}$$

and this implies $\langle\, y^{(1)}\,,\; Jx^*\,\rangle = \langle\, y^{(1)}\,,\; u^*\,\rangle = 0$. Thus the maximum condition reduces to

$$-\;\langle\, y^{(1)}\,,\; u - u^*\,\rangle \;=\; -\,\langle\, y^{(1)}\,,\; u\,\rangle \;\geqslant\; 0 \quad \forall\, u \in \mathrm{U}\,. \tag{4.98}$$

Since $\mathfrak{o} \in \mathrm{int}\,(\mathrm{K})$ by assumption, U contains some $L^\infty(\Omega, \mathbb{R}^{nm})$-norm ball V, and we see from (4.98) that $\langle\, y^{(1)}\,,\; u\,\rangle = 0$ for all $u \in \mathrm{U} \cap \mathrm{V}$. As above, this implies that $y^{(1)} = \mathfrak{o}$, and we get a contradiction since the sets C and D were separated by a nontrivial functional. Consequently, we arrive at $\lambda_0 > 0$. ■

Proof of Theorem 4.5. Consider the countable subsets

$$\mathrm{K}^0 \;=\; \big(\,\mathrm{K} \cap\; \mathbb{Q}^{nm}\,\big) \times \mathbb{Q}^{\sigma(2)} \times \mathbb{Q}^{\sigma(3)} \times \ldots \times \mathbb{Q}^{\sigma(n\wedge m)} \quad \text{and} \tag{4.99}$$

$$\mathrm{Q}^0 \;=\; \mathrm{Q} \cap \big(\,\mathbb{Q}^{nm} \times \mathbb{Q}^{\sigma(2)} \times \mathbb{Q}^{\sigma(3)} \times \ldots \times \mathbb{Q}^{\sigma(n\wedge m)}\,\big)\,, \tag{4.100}$$

which are dense in K and Q, respectively. Consider further the null sets of the non-Lebesgue points of the integrable functions $g(\,\cdot\,, x^*(\,\cdot\,), u^*(\,\cdot\,), w^*(\,\cdot\,))$, $g(\,\cdot\,, x^*(\,\cdot\,), v^0, \omega^0)$, $\big(\,v^0 - u^*(\,\cdot\,)\,\big)^{\mathrm{T}}\, y^{(1)}(\,\cdot\,)$, $\big(\,\omega_r^0 - w_r^*(\,\cdot\,)\,\big)^{\mathrm{T}}\, y^{(r)}(\,\cdot\,)$, $\nabla_v\, \mathrm{adj}_r(\,u^*(\,\cdot\,)\,)\,\big(\,v^0 - u^*(\,\cdot\,)\,\big)^{\mathrm{T}} \cdot y^{(r)}(\,\cdot\,)$, $2 \leqslant r \leqslant (n \wedge m)$, for $(v^0, \omega^0) \in \mathrm{K}^0 \cap \mathrm{Q}^0$. We form the countable union N of these null sets, which is still a null set. Since $\Omega \subset \mathbb{R}^m$ is the closure of a strongly Lipschitz domain, its boundary $\partial\Omega$ is a null set as well. [35] Let us fix now a point $s^0 \in \mathrm{int}\,(\Omega) \setminus \mathrm{N}$ as well as an element $(v^0, \omega^0) \in \mathrm{K}^0 \cap \mathrm{Q}^0$. Then a closed ball $\mathrm{B} = \mathrm{K}(s^0, \varepsilon)$ with sufficiently small radius $\varepsilon > 0$ is contained in $\mathrm{int}\,(\Omega)$, and the pair (u, w) of the functions

$$u(s) \;=\; \mathbb{1}_{\mathrm{B}}(s)\,\Big(\,\frac{\mathrm{Dist}\,(s, \partial\mathrm{B})}{\mathrm{Dist}\,(s^0, \partial B)} \cdot v^0 + \frac{\big(\,\mathrm{Dist}\,(s^0, \partial B) - \mathrm{Dist}\,(s, \partial B)\,\big)}{\mathrm{Dist}\,(s^0, \partial B)} \cdot u^*(s)\,\Big) + \mathbb{1}_{(\Omega \setminus \mathrm{B})}(s)\; u^*(s)\,; \tag{4.101}$$

$$w(s) \;=\; \mathbb{1}_{\mathrm{B}}(s)\,\Big(\,\frac{\mathrm{Dist}\,(s, \partial\mathrm{B})}{\mathrm{Dist}\,(s^0, \partial B)} \cdot \omega^0 + \frac{\big(\,\mathrm{Dist}\,(s^0, \partial B) - \mathrm{Dist}\,(s, \partial B)\,\big)}{\mathrm{Dist}\,(s^0, \partial B)} \cdot w^*(s)\,\Big) + \mathbb{1}_{(\Omega \setminus \mathrm{B})}(s)\; w^*(s) \tag{4.102}$$

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[35] [WAGNER 06] , p. 122, Lemma 9.2.

belongs to $\big(\mathrm{U}\times L^{p/2}(\Omega,\mathbb{R}^{\sigma(2)})\times L^{p/3}(\Omega,\mathbb{R}^{\sigma(3)})\times\ \dots\ \times L^{p/(n\wedge m)}(\Omega,\mathbb{R}^{\sigma(n\wedge m)})\big)\cap \mathrm{W}$. Observe now that all above mentioned functions are continuous with respect to v and ω and $(u(s^0),w(s^0)) = (v^0,\omega^0)$. Consequently, s^0 is a Lebesgue point of $g(\,\cdot\,,x^*(\,\cdot\,),u(\,\cdot\,),w(\,\cdot\,))$, $\big(u(\,\cdot\,)-u^*(\,\cdot\,)\big)^{\mathrm{T}}y^{(1)}(\,\cdot\,)$, $\big(w_r(\,\cdot\,)-w_r^*(\,\cdot\,)\big)^{\mathrm{T}}y^{(r)}(\,\cdot\,)$, $\nabla_v\,\mathrm{adj}_r(\,u^*(\,\cdot\,)\,)\,\big(u(\,\cdot\,)-u^*(\,\cdot\,)\big)^{\mathrm{T}}\cdot y^{(r)}(\,\cdot\,)$, $2\leqslant r\leqslant (n\wedge m)$, as well, and we may form the Lebesgue derivative of $(\mathcal{M})$ at the point s^0 after inserting (u,w) into the inequality.

In order to do so, let us consider a Vitali covering of Ω.[36] We specify therein some decreasing sequence $\{\,\Omega^N\,\}$ of closed subsets of $\Omega\cap\mathrm{B}$ with $\bigcap_N\,\Omega^N=\{\,s^0\,\}$. All function pairs (u^N,w^N) with

$$u^N(s) \;=\; \mathbb{1}_{\Omega^N}(s)\,u(s)+\mathbb{1}_{(\Omega\,\setminus\,\Omega^N)}(s)\,u^*(s)\,; \tag{4.103}$$

$$w^N(s) \;=\; \mathbb{1}_{\Omega^N}(s)\,w(s)+\mathbb{1}_{(\Omega\,\setminus\,\Omega^N)}(s)\,w^*(s) \tag{4.104}$$

form admissible controls together with (u,w), and we arrive at

$$\begin{aligned}
&\lim_{N\to\infty}\frac{1}{|\,\Omega^N\,|}\int_{\Omega^N}\lambda_0\Big(g(\,s,x^*(s),u^N(s),w^N(s)\,)-g(\,s,x^*(s),u^*(s),w^*(s)\,)\Big)\,ds \qquad (4.105)\\
&\quad-\lim_{N\to\infty}\frac{1}{|\,\Omega^N\,|}\int_{\Omega^N}\big(\,u^N(s)-u^*(s)\,\big)^{\mathrm{T}}y^{(1)}(s)\,ds+\sum_{r=2}^{(n\wedge m)}\lim_{N\to\infty}\frac{1}{|\,\Omega^N\,|}\int_{\Omega^N}\big(\,w_r^N(s)-w_r^*(s)\,\big)^{\mathrm{T}}y^{(r)}(s)\,ds\\
&\qquad-\sum_{r=2}^{(n\wedge m)}\lim_{N\to\infty}\frac{1}{|\,\Omega^N\,|}\int_{\Omega^N}\nabla_v\,\mathrm{adj}_r(\,u^*(s)\,)\,\big(\,u^N(s)-u^*(s)\,\big)^{\mathrm{T}}y^{(r)}(s)\,ds\\
&=\lambda_0\Big(g(\,s,x^*(s),v^0,\omega^0\,)-g(\,s,x^*(s),u^*(s),w^*(s)\,)\Big)-\big(\,v^0-u^*(s)\,\big)^{\mathrm{T}}y^{(1)}(s) \qquad (4.106)\\
&\qquad+\sum_{r=2}^{(n\wedge m)}\big(\,\omega_r^0-w_r^*(s)\,\big)^{\mathrm{T}}y^{(r)}(s)-\sum_{r=2}^{(n\wedge m)}\nabla_v\,\mathrm{adj}_r(\,u^*(s)\,)\,\big(\,v^0-u^*(s)\,\big)^{\mathrm{T}}y^{(r)}(s)\;\geqslant\;0\,.
\end{aligned}$$

This inequality holds for all fixed $s^0\in\mathrm{int}\,(\Omega)\setminus\mathrm{N}$ for arbitrary $(v^0,\omega^0)\in\mathrm{K}^0\cap\mathrm{Q}^0$. Since its left-hand side is a continuous function with respect to (v,ω), it may be extended to the whole set $\big(\mathrm{K}\times\mathbb{R}^{\sigma(2)}\times\mathbb{R}^{\sigma(3)}\times\dots\times\mathbb{R}^{\sigma(n\wedge m)}\big)\cap\mathrm{Q}$, and the proof is finished. ■

d) Remarks and generalizations.

Our first remark concerns the polyconvex set P. Here Assumption 3.1., 2) may be weakened as follows:

Corollary 4.11. (General polyconvex restriction set) *Propositions 3.3. and 3.8., Theorem 3.9. as well as Theorem 4.3., Proposition 4.4. and Theorem 4.5. remain true as far as the polyconvex set* $\mathrm{P}\subseteq\mathbb{R}^{nm}$ *in Assumption 3.1., 2) is closed but possibly unbounded.*

Proof. Since $\mathrm{K}\subset\mathbb{R}^{nm}$ is convex and compact, we may replace P by $\widetilde{\mathrm{P}}=\mathrm{K}\cap\mathrm{P}$ before starting the analysis of the problems. This causes no change in the feasible domains. However, the set $\widetilde{\mathrm{P}}$ is compact together with K and polyconvex as an intersection of a convex and a polyconvex set. ■

Remark 4.12. Compared with [Wagner 13], (4.123) – (4.126), the growth conditions (3.21) – (3.24) for the partial derivatives of g from Assumption 3.4. are slightly more restrictive. The conditions, however, are adapted to the L^p case and in this form required by the penalty technique used in our proofs.

Remark 4.13. As a consequence of the consideration of a polyconvex gradient constraint, the number of variables in $(\mathrm{P})_1$ and $(\mathrm{P})_2$ as well as in the conditions of Pontryagin's principle cannot be reduced even if the integrand does not depend explicitly on some of them. For the same reason, in contrast to [Wagner 13],

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[36] [Dunford/Schwartz 88], p. 212, Definition 2.

(4.9) – (4.12), the pointwise condition ($\mathcal{MP}$) from Theorem 4.5. allows for a decomposition into separate conditions in special cases only.

Remark 4.14. For the purposes of optimization, it would be desirable to know the *largest* possible convex representative of a given polyconvex set P, thus obtaining maximal significance of the conditions ($\mathcal{M}$) and ($\mathcal{MP}$).

5. Application to a problem from mathematical imaging.

a) The image registration problem.

In this final section, we provide an application of our theorems to an optimal control problem arising in mathematical image processing. Consider a given pair of greyscale images, which will be described through at least measurable, essentially bounded functions $I_0(s)$, $I_1(s)\colon\ \Omega \to [\,0\,,\,1\,]$ on a domain $\Omega \subset \mathbb{R}^k$, $k \geqslant 2$, and assign I_0 as the reference image. Based on the assumption that there is an overall correlation between the greyscale intensity distributions as well as the geometrical properties of the images, the unimodal registration problem for I_0, I_1 consists in the search for a deformation field $x\colon\ \Omega \to \mathbb{R}^k$, which satisfies the condition $I_1\big(\,x(s)\,\big) \approx I_0(s)$ for a. a. $s \in \Omega$. [37] Among the different approaches to determine a suitable vector field x, [38] the registration by means of an *elastic deformation* became particularly important. [39] In this approach, the changes in I_1 with respect to I_0 will be attributed to an elastic deformation of the pictured objects. (This is particulary reasonable in medical imaging since human tissue behaves according to hyperelastic material laws. [40]) Consequently, the deformation field will be obtained as a solution of a multidimensional variational problem, whose objective contains a stored-energy function connected with a linear-elastic or hyperelastic material law. Typically, the objective consists of a data fidelity term, e. g. $\big(\,I_1\big(\,x(s)\,\big) - I_0(s)\,\big)^2$, and a convex or polyconvex regularization term. For the numerical solution of these problems, a number of well-established methods is available.

There are at least two reasons for incorporating constraints for the gradient Jx into variational problems of this type. First, the validity of the underlying elasticity models is always bounded by a restriction for the maximal shear stress generated by the deformation. [41] This leads to the introduction of a convex gradient restriction of the type $Jx(s) \in \mathrm{K}$, thus altering the given variational problem into a multidimensional control problem of Dieudonné-Rashevsky type. [42] Secondly, it is often desirable to keep the deformation bijective,

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[37] For a detailed introduction to the registration problem, we refer to [Hintermüller/Keeling 09], [Modersitzki 04] and [Modersitzki 09].

[38] Depending on the shape of the pictured objects and their motion behaviour, methods as different as viscous fluid registration, Monge-Kantorovič transport optimization, rigid motion in a higher-dimensional space or level-set methods have been proposed in the literature, cf. [Christensen/Rabbitt/Miller 96], [Museyko/Stiglmayr/Klamroth/Leugering 09], [Breitenreicher/Schnörr 09] and [Vemuri/Ye/Chen/Leonard 00], respectively. The concepts of the optical displacement and the optical flow could be mentioned here as well, see e. g. [Alvarez/Weickert/Sánchez 00] and [Aubert/Kornprobst 06], pp. 250 ff.

[39] See e. g. [Burger/Modersitzki/Ruthotto 13], [Droske/Rumpf 04], [Droske/Rumpf 07], [Henn/Witsch 00], [Henn/Witsch 00], [Le Guyader/Vese 09], [Modersitzki 04], pp. 83 ff., and [Pöschl/Modersitzki/Scherzer 10]. The idea can be traced back to [Broit 81].

[40] See e. g. [Ogden 03]. [Balzani/Neff/Schröder/Holzapfel 06] provides examples for polyconvex stored-energy functions applicable in this context.

[41] This is true for a wide range of materials and even for living tissue. See e. g. [Chmelka/Melan 76], pp. 38 – 45, (material sciences,linear-elastic model) as well as [Gasser/Holzapfel 02], p. 340 f., and the literature cited there (human tissue, hyperelastic models).

[42] Cf. [Angelov/Wagner 12], [Wagner 10] and [Wagner 12] where this approach has been pursued.

which will be guaranteed by the restriction $\det Jx(s) > 0$. More generally, volumetric constraints lead to different types of polyconvex gradient restrictions.

b) Three-dimensional registration with mass-preserving data term and polyconvex regularizer.

As an example, we will reformulate a three-dimensional image registration problem provided in [Burger/ Modersitzki/Ruthotto 13] within the framework of optimal control. Within this problem, which originates from medical imaging, the authors consider the mass-preserving data fidelity term [43)]

$$\int_\Omega \big(I_1(x(s)) \cdot \det Jx(s) - I_0(s) \big)^2 \, ds \tag{5.1}$$

as well as a regularization with respect to Jx, its cofactor matrix and determinant. The resulting variational problem may be stated as follows: [44)]

$$\text{(V)} \quad F(x) = \int_\Omega \Big(I_1(x(s)) \, \det Jx(s) - I_0(s) \Big)^2 \, ds + \mu \cdot \int_\Omega \Big(c_1 \, \| Jx(s) - E_3 \|^2 + c_2 \, \| \mathrm{adj}_2 Jx(s) - \mathrm{adj}_2 E_3 \|^2 + c_3 \big(\det Jx(s) - 1 \big)^2 \Big) \, ds \longrightarrow \inf! ; \quad x \in W_0^{1,p}(\Omega, \mathbb{R}^3) . \tag{5.2}$$

For the reasons mentioned above, we incorporate the convex gradient restriction

$$Jx(s) \in \mathrm{K} = \big\{ v \in \mathbb{R}^{3\times 3} \,\big|\, |v_{ij}| \leqslant c_4 , \ 1 \leqslant i, j \leqslant 3 \big\} \quad (\forall)\, s \in \Omega \tag{5.3}$$

and, fixing a sufficiently small $\varepsilon > 0$, the polyconvex gradient restriction

$$Jx(s) \in \mathrm{P} = \mathrm{K} \cap \big\{ v \in \mathbb{R}^{3\times 3} \,\big|\, \det v \geqslant \varepsilon \big\} \quad (\forall)\, s \in \Omega \tag{5.4}$$

which allows to dispense with the growth condition $\lim_{\det Jx \to 0+0} F(x) = (+\infty)$ and the corresponding penalty term within the objective. Consequently, we obtain the multidimensional control problem

$$\text{(R)} \quad F(x) \longrightarrow \inf! ; \quad x \in W_0^{1,p}(\Omega, \mathbb{R}^3) ; \quad Jx(s) \in \mathrm{K} \cap \mathrm{P} \quad (\forall)\, s \in \Omega . \tag{5.5}$$

We assume that the region of imaging is a rectangular block $\Omega \subset \mathbb{R}^3$, $6 \leqslant p < \infty$, μ, c_1, c_2, c_3, $c_4 > 0$. The image data I_0 and I_1 belong to $L^\infty(\Omega, \mathbb{R})$ and $C_0^1(\Omega, \mathbb{R})$, respectively. We use the matrix norm $\| v \|^2 = | v_{11} |^2 + \ldots + | v_{33} |^2$.
For f, we choose the convex representative $g\colon \Omega \times \mathbb{R}^3 \times \mathbb{R}^9 \times \mathbb{R}^9 \times \mathbb{R} \to \mathbb{R}$ defined through

$$g(s, \xi, v, \omega_2, \omega_3) = \big(I_1(\xi) \cdot \omega_3 - I_0(s) \big)^2 + \mu \, c_1 \sum_{i=1}^{3} \sum_{j=1}^{3} \Big(\delta_{ij} \, (v_{ij} - 1)^2 + (1 - \delta_{ij}) \, (v_{ij})^2 \Big) + \mu \, c_2 \sum_{i=1}^{3} \sum_{j=1}^{3} \Big(\delta_{ij} \, (\omega_{2,ij} - 1)^2 + (1 - \delta_{ij}) \, (\omega_{2,ij})^2 \Big) + \mu \, c_3 \, (\omega_3 - 1)^2 \tag{5.6}$$

where δ_{ij} denotes the Kronecker symbol. Since $I_0(s), I_1(\xi) \in [0, 1]$ for almost all $s \in \Omega$ and all $\xi \in \mathbb{R}^3$, the function g is convex with respect to (v, ω_2, ω_3) for almost all $s \in \Omega$ and all $\xi \in \mathbb{R}^3$. The partial derivatives

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43) [Burger/Modersitzki/Ruthotto 13], p. B 134, (2.3).

44) Cf. [Burger/Modersitzki/Ruthotto 13], p. B 134 f., (2.3) – (2.7). For our purposes, the second and third part of the regularizer have been slightly modified.

of g read as

$$\frac{\partial g}{\partial \xi_i}(s,\xi,v,\omega_2,\omega_3) \;=\; 2\,\big(\,I_1(\xi)\,\omega_3^2 - I_0(s)\,\omega_3\,\big)\,\frac{\partial I_1}{\partial s_i}(\xi)\,,\ 1 \leqslant i \leqslant 3\ ; \tag{5.7}$$

$$\frac{\partial g}{\partial v_{ij}}(s,\xi,v,\omega_2,\omega_3) \;=\; 2\,\mu\,c_1 \sum_{i=1}^{3}\sum_{j=1}^{3}\Big(\,\delta_{ij}\,\big(\,v_{ij}-1\,\big) + \big(\,1-\delta_{ij}\,\big)\,v_{ij}\,\Big)\,,\ 1 \leqslant i \leqslant 3\,,\ 1 \leqslant j \leqslant 3\ ; \tag{5.8}$$

$$\frac{\partial g}{\partial \omega_{2,ij}}(s,\xi,v,\omega_2,\omega_3) \;=\; 2\,\mu\,c_2 \sum_{i=1}^{3}\sum_{j=1}^{3}\Big(\,\delta_{ij}\,\big(\,\omega_{2,ij}-1\,\big) + \big(\,1-\delta_{ij}\,\big)\,\omega_{2,ij}\,\Big)\,,\ 1 \leqslant i \leqslant 3\,,\ 1 \leqslant j \leqslant 3\ ; \tag{5.9}$$

$$\frac{\partial g}{\partial \omega_3}(s,\xi,v,\omega_2,\omega_3) \;=\; 2\,I_1(\xi)^2\,\omega_3 - 2\,I_1(\xi)\,I_0(s) + 2\,\mu\,c_3\,\big(\,\omega_3 - 1\,\big)\,. \tag{5.10}$$

As a convex compact representative of the polyconvex set $\mathrm{P} \in \mathbb{R}^{3\times 3}$, we will employ the set $\mathrm{Q} \in \mathbb{R}^{19}$ with

$$\mathrm{Q} \;=\; \big\{\,(v,\omega_2,\omega_3) \in \mathbb{R}^9 \times \mathbb{R}^9 \times \mathbb{R}\ \big|\ |\,v_{ij}\,| \;\leqslant\; c_4\,,\ |\,\omega_{2,ij}\,| \leqslant \kappa_{ij}^{(2)}\,,\ 1 \;\leqslant\; i,\,j \;\leqslant\; 3\,,\ \varepsilon \leqslant \omega_3 \leqslant \kappa^{(3)}\,\big\}\,; \tag{5.11}$$

$$\kappa_{ij}^{(2)} \;=\; \mathrm{Max}_{\,v\,\in\,\mathrm{K}}\ |\,\mathrm{adj}_{2,ij} v\,|\,,\ 1 \leqslant i \leqslant 3\,,\ 1 \leqslant j \leqslant 3\,; \tag{5.12}$$

$$\kappa^{(3)} \;=\; \mathrm{Max}_{\,v\,\in\,\mathrm{K}}\ |\,\det v\,|\,. \tag{5.13}$$

Note that we get still a convex, compact representative for P if the constants $\kappa_{ij}^{(2)}$ and $\kappa^{(3)}$ are replaced by $\kappa_{ij}^{(2)} + \gamma$ and $\kappa^{(3)} + \gamma$ with $\gamma > 0$, $1 \leqslant i,j \leqslant 3$.
In order to apply the theorems from Sections 3 and 4, we must verify Assumptions 3.1. and 3.4. for the problem (R). Obviously, Assumptions 3.1., 1) $-$ 3) are satisfied. In order to confirm Assumption 3.1., 4) as well as Assumption 3.4., we perform the following estimates wherein $C > 0$ denotes a generic constant, which may change from expression to expression. For almost all $s \in \Omega$ and for all $(\xi,v,\omega_2,\omega_3) \in \mathbb{R}^3 \times \mathbb{R}^9 \times \mathbb{R}^9 \times \mathbb{R}$, we obtain with $p \geqslant 6$

$$|\,g(s,\xi,v,\omega_2,\omega_3)\,| \;\leqslant\; C\,\Big(\,\big(\,|\,I_1(\xi)\,|^2\,|\,\omega_3\,|^2 + |I_0(s)\,|^2\,\big) + \big(\,1+|\,v\,|^2\,\big) + \big(\,1+|\,\omega_2\,|^2\,\big) + \big(\,1+|\,\omega_3\,|^2\,\big)\,\Big) \tag{5.14}$$

$$\leqslant\; C\,\Big(\,1+|\,v\,|^p + |\,\omega_2\,|^{p/2} + |\,\omega_3\,|^{p/3}\,\Big) \tag{5.15}$$

since $I_0(s)$, $I_1(\xi) \in [\,0\,,\,1\,]$ for almost all $s \in \Omega$ and all $\xi \in \mathbb{R}^3$. Consequently, (3.5) is satisfied with $A_0(s) \equiv 0$ and $B_0(\xi) \equiv 0$, and the problem (R) admits a global minimizer. For the partial derivatives of g, the following estimates hold:

$$|\,\frac{\partial g}{\partial \xi_i}(s,\xi,v,\omega_2,\omega_3)\,| \;\leqslant\; 2\,C\,\Big(\,|\,I_1(\xi)\,|\,|\,\omega_3\,|^2 + |\,I_0(s)\,|\,|\,\omega_3\,|\,\Big)\,\|\,I_1\,\|_{C^1} \;\leqslant\; C\,\Big(\,1+|\,\omega_3\,|^{p/3}\,\Big)\,,\ 1 \leqslant i \leqslant 3\,; \tag{5.16}$$

$$|\,\frac{\partial g}{\partial v_{ij}}(s,\xi,v,\omega_2,\omega_3)\,| \;\leqslant\; C\,\Big(\,1+|\,v\,|\,\Big) \;\leqslant\; C\,\Big(\,1+|\,v\,|^{p-1}\,\Big)\,,\ 1 \leqslant i \leqslant 3\,,\ 1 \leqslant j \leqslant 3\,; \tag{5.17}$$

$$|\,\frac{\partial g}{\partial \omega_{2,ij}}(s,\xi,v,\omega_2,\omega_3)\,| \;\leqslant\; C\,\Big(\,1+|\,\omega_2\,|\,\Big) \;\leqslant\; C\,\Big(\,1+|\,\omega_2\,|^{(p-2)/2}\,\Big)\,,\ 1 \leqslant i \leqslant 3\,,\ 1 \leqslant j \leqslant 3\,; \tag{5.18}$$

$$|\,\frac{\partial g}{\partial \omega_3}(s,\xi,v,\omega_2,\omega_3)\,| \;\leqslant\; C\,\Big(\,1+|\,\omega_3\,|\,\Big) \;=\; C\,\Big(\,1+|\,\omega_3\,|^{(p-3)/3}\,\Big)\,. \tag{5.19}$$

Consequently, the partial derivatives of g satisfy the growth conditions (3.21) $-$ (3.24) as well, and any global minimizer of (R) must satisfy the necessary optimality conditions from Theorems 4.3. and 4.5. Moreover, the box structure of Q allows for a decomposition of the maximum condition ($\mathcal{MP}$). We arrive at the following set of conditions:

Proposition 5.1. (Pontryagin's principle for (R)**)** *Consider the control problem* (R) *under the analytical assumptions mentioned above. In particular, we choose $p \geqslant 6$. If (x^*, u^*) is a global minimizer of* (R) *then* *there exist multipliers* $\lambda_0 \geqslant 0$, $y^{(1)} \in L^{p/(p-1)}(\Omega, \mathbb{R}^9)$, $y^{(2)} \in L^{p/(p-2)}(\Omega, \mathbb{R}^9)$ *and* $y^{(3)} \in L^{p/(p-3)}(\Omega, \mathbb{R})$ *such that the following conditions are satisfied:*

$$
\begin{aligned}
(\mathcal{MP})_1 \quad & \lambda_0\, \mu\, c_1 \sum_{i=1}^{3} \sum_{j=1}^{3} \Big(\delta_{ij} \Big(\big(v_{ij} - 1 \big)^2 - \big(u^*_{ij}(s) - 1 \big)^2 \Big) + \big(1 - \delta_{ij} \big) \Big(\big(v_{ij} \big)^2 - u^*_{ij}(s)^2 \Big) \Big) \qquad (5.20) \\
& - \sum_{i=1}^{3} \sum_{j=1}^{3} \Big(y^{(1)}_{ij}(s) + \sum_{l=1}^{9} \frac{\partial}{\partial v_{ij}} \mathrm{adj}_{2,l}\big(u^*(s) \big)\, y^{(2)}_l(s) + \frac{\partial}{\partial v_{ij}} \det\big(u^*(s) \big)\, y^{(3)}(s) \Big) \big(v_{ij} - u^*_{ij}(s) \big) \geqslant 0 \\
& (\forall)\, s \in \Omega \quad \forall\, v \in \mathrm{K}\,;
\end{aligned}
$$

$$
\begin{aligned}
(\mathcal{MP})_2 \quad & \lambda_0\, \mu\, c_2 \sum_{i=1}^{3} \sum_{j=1}^{3} \Big(\delta_{ij} \Big(\big(\omega_{2,ij} - 1 \big)^2 - \big(\mathrm{adj}_{2,ij} u^*(s) - 1 \big)^2 \Big) \qquad (5.21) \\
& + \big(1 - \delta_{ij} \big) \Big(\big(\omega_{2,ij} \big)^2 - \mathrm{adj}_{2,ij} u^*(s)^2 \Big) \Big) + \sum_{l=1}^{9} \big(\omega_{2,l} - \mathrm{adj}_{2,l} u^*(s) \big)\, y^{(2)}_l(s) \geqslant 0 \\
& (\forall)\, s \in \Omega \quad \forall\, \omega_2 \in \mathbb{R}^9 \ \ \textit{with}\ |\,\omega_{2,11}\,| \leqslant \kappa^{(2)}_{11}\,,\ \ldots,\ |\,\omega_{2,33}\,| \leqslant \kappa^{(2)}_{33}\,;
\end{aligned}
$$

$$
\begin{aligned}
(\mathcal{MP})_3 \quad & \lambda_0 \Big(I_1\big(x^*(s) \big)^2 \big(\big(\omega_3 \big)^2 - \big(\det u^*(s) \big)^2 \big) - 2\, I_0(s)\, I_1\big(x^*(s) \big) \big(\omega_3 - \det u^*(s) \big) \Big) \qquad (5.22) \\
& + \lambda_0\, \mu\, c_3 \Big(\big(\omega_3 - 1 \big)^2 - \big(\det u^*(s) - 1 \big)^2 \Big) + \big(\omega_3 - \det u^*(s) \big)\, y^{(3)}(s) \geqslant 0 \\
& (\forall)\, s \in \Omega \quad \forall\, \omega_3 \in \mathbb{R} \ \ \textit{with}\ \varepsilon \leqslant \omega_3 \leqslant \kappa^{(3)}\,;
\end{aligned}
$$

$$
\begin{aligned}
(\mathcal{K}) \quad & 2\, \lambda_0 \int_\Omega \Big(I_1\big(x^*(s) \big) \big(\det u^*(s) \big)^2 - I_0(s)\, \det u^*(s) \Big) \sum_{i=1}^{3} \frac{\partial}{\partial s_i} I_1\big(x^*(s) \big) \big(x_i(s) - x^*_i(s) \big)\, ds \qquad (5.23) \\
& + \sum_{i=1}^{3} \sum_{j=1}^{3} \int_\Omega \Big(\frac{\partial x_i}{\partial s_j}(s) - \frac{\partial x^*_i}{\partial s_j}(s) \Big)\, y^{(1)}_{ij}(s)\, ds = 0 \quad \forall\, x \in W^{1,p}_0(\Omega, \mathbb{R}^3)\,.
\end{aligned}
$$

Proof. Assume that (x^*, u^*) is a global minimizer of (R). As a consequence of (5.15) − (5.19), Theorems 4.3. and 4.5. may be applied, and we obtain the existence of multipliers $\lambda_0 \geqslant 0$, $y^{(1)} \in L^{p/(p-1)}(\Omega, \mathbb{R}^9)$, $y^{(2)} \in L^{p/(p-2)}(\Omega, \mathbb{R}^9)$ and $y^{(3)} \in L^{p/(p-3)}(\Omega, \mathbb{R})$, which satisfy together with (x^*, u^*) the conditions $(\mathcal{M})$, $(\mathcal{K})$ and $(\mathcal{MP})$. In particular, the pointwise maximum condition $(\mathcal{MP})$ reads as

$$
\begin{aligned}
& \lambda_0 \Big(I_1\big(x^*(s) \big)^2 \big(\big(\omega_3 \big)^2 - \big(\det u^*(s) \big)^2 \big) - 2\, I_0(s)\, I_1\big(x^*(s) \big) \big(\omega_3 - \det u^*(s) \big) \Big) \qquad (5.24) \\
& \quad + \lambda_0\, \mu\, c_1 \sum_{i=1}^{3} \sum_{j=1}^{3} \Big(\delta_{ij} \Big(\big(v_{ij} - 1 \big)^2 - \big(u^*_{ij}(s) - 1 \big)^2 \Big) + \big(1 - \delta_{ij} \big) \Big(\big(v_{ij} \big)^2 - u^*_{ij}(s)^2 \Big) \Big) \\
& \quad + \lambda_0\, \mu\, c_2 \sum_{i=1}^{3} \sum_{j=1}^{3} \Big(\delta_{ij} \Big(\big(\omega_{2,ij} - 1 \big)^2 - \big(\mathrm{adj}_{2,ij} u^*(s) - 1 \big)^2 \Big) + \big(1 - \delta_{ij} \big) \Big(\big(\omega_{2,ij} \big)^2 - \mathrm{adj}_{2,ij} u^*(s)^2 \Big) \Big) \\
& \quad + \lambda_0\, \mu\, c_3 \Big(\big(\omega_3 - 1 \big)^2 - \big(\det u^*(s) - 1 \big)^2 \Big) \\
& \quad - \sum_{i=1}^{3} \sum_{j=1}^{3} \Big(y^{(1)}_{ij}(s) + \sum_{l=1}^{9} \frac{\partial}{\partial v_{ij}} \mathrm{adj}_{2,l}\big(u^*(s) \big)\, y^{(2)}_l(s) + \frac{\partial}{\partial v_{ij}} \det\big(u^*(s) \big)\, y^{(3)}(s) \Big) \big(v_{ij} - u^*_{ij}(s) \big) \\
& \quad + \sum_{l=1}^{9} \big(\omega_{2,l} - \mathrm{adj}_{2,l} u^*(s) \big)\, y^{(2)}_l(s) + \big(\omega_3 - \det u^*(s) \big)\, y^{(3)}(s) \geqslant 0 \\
& (\forall)\, s \in \Omega \quad \forall\, (v,\, \omega_2,\, \omega_3) \in \big(\mathrm{K} \times \mathbb{R}^9 \times \mathbb{R} \big) \cap \mathrm{Q}\,.
\end{aligned}
$$

Observing that

$$
\mathrm{Q} = \big(\mathrm{K} \times \mathbb{R}^9 \times \mathbb{R} \big) \cap \mathrm{Q} = \mathrm{K} \times \prod_{i,j=1}^{3} [\, -\kappa^{(2)}_{ij}\,,\, \kappa^{(2)}_{ij}\,] \times [\, \varepsilon\,,\, \kappa^{(3)}\,]\,, \qquad (5.25)
$$

we may insert into (5.24) arbitrary vectors of the shape $\big(v, \mathrm{adj}_2 u^*(s), \det u^*(s)\big)$, $\big(u^*(s), \omega_2, \det u^*(s)\big)$ or $\big(u^*(s), \mathrm{adj}_2 u^*(s), \omega_3\big)$ with $v \in \mathrm{K}$, $|\,\omega_{2,ij}\,| \leqslant \kappa_{ij}^{(2)}$, $1 \leqslant i, j \leqslant 3$ and $\varepsilon \leqslant \omega_3 \leqslant \kappa^{(3)}$. Consequently, $(\mathcal{MP})$ implies the separate conditions $(\mathcal{MP})_1$, $(\mathcal{MP})_2$ and $(\mathcal{MP})_3$. ■

We close the study of (R) with the following observation:

Proposition 5.2. (Occurence of the regular case within the conditions) *Under the assumptions of Proposition 5.1., consider the control problem* (R) *together with its global minimizer* (x^*, u^*). *If* $\det u^*(s) \geqslant \varepsilon' > \varepsilon$ *for almost all* $s \in \Omega$ *then the conditions from Proposition 5.1. hold true with a multiplier* $\lambda_0 > 0$, *i. e. the regular case occurs.*

Proof. Together with u^*, the functions $\mathrm{adj}_2 u^*$ and $\det u^*$ are essentially bounded. Consequently, we may enlarge the constants $\kappa_{ij}^{(2)}$ and $\kappa^{(3)}$ within the definition (5.11) $-$ (5.13) of the convex representative Q until $\big(\mathrm{adj}_2 u^*(s)\,,\, \det u^*(s)\big) \in \mathrm{int}\,\Big(\prod_{i,j=1}^{3}[\,-\kappa_{ij}^{(2)} - (\varepsilon^p - \varepsilon)/2\,,\, \kappa_{ij}^{(2)} + (\varepsilon' - \varepsilon)/2\,] \times [\,\varepsilon\,,\, \kappa^{(3)} + (\varepsilon' - \varepsilon)/2\,]\Big)$ for almost all $s \in \Omega$. Proposition 4.4. implies now the occurence of the regular case. ■

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