Pontryagin's principle for Dieudonné-Rashevsky type problems with polyconvex integrands

Marcus Wagner

1. Introduction.

The present paper is concerned with the proof of first-order necessary optimality conditions for multidimensional control problems of Dieudonné-Rashevsky type:

$$F(x,u) \;=\; \int_\Omega f(s,x(s),u(s))\,ds \longrightarrow \inf!\,; \quad (x,u) \in W_0^{1,p}(\Omega,\mathbb{R}^n) \,\times\, L^p(\Omega,\mathbb{R}^{nm})\,; \tag{1.1}$$

$$Jx(s) \;=\; \begin{pmatrix} \partial x_1(s)/\partial s_1 & \ldots & \partial x_1(s)/\partial s_m \\ \vdots & & \vdots \\ \partial x_n(s)/\partial s_1 & \ldots & \partial x_n(s)/\partial s_m \end{pmatrix} = u(s) \quad \text{for almost all } s \in \Omega\,; \tag{1.2}$$

$$u(s) \in \mathrm{K} \subset \mathbb{R}^{nm} \quad \text{for almost all } s \in \Omega \tag{1.3}$$

with $n,\, m \geqslant 2$, $\Omega \subset \mathbb{R}^m$, $m < p < \infty$ and a compact set $\mathrm{K} \subset \mathbb{R}^{nm}$ with nonempty interior. In the case of a convex integrand $f(s,\xi,\,\cdot\,)$ and a convex restriction set K, the global minimizers of (1.1) – (1.3) satisfy optimality conditions in the form of Pontryagin's principle [01)] even though the usual regularity condition for the equality operator (1.2) fails. [02)] The question arises whether the Pontryagin principle and its proof can be extended to situations where the usual convexity of the data is replaced by generalized convexity notions. An answer to this question is of conceptual interest since the classical proof of the Pontryagin principle is based on an implicit convexification of the integrand as well as of the set of feasible controls. [03)]
Within the hierarchy of the generalized convexity notions, [04)] *polyconvexity* is the closest one to usual convexity. In short, a polyconvex function arises as a composition of the vector of all minors of a matricial argument with a convex function. Appearing e. g. in problems from material science, [05)] hydrodynamics [06)] and mathematical image processing, [07)] objectives with polyconvex integrands are of considerable practical importance. In the present paper, it will be shown that the proof of Pontryagin's principle for the problem (1.1) – (1.3) can be maintained if the integrand $f(s,\xi,v)$ is polyconvex with respect to v while the control restriction set K is still convex (Theorems 4.3., 4.4. and 4.11.). To the best of the author's knowledge, a proof of optimality conditions, which makes explicit use of the polyconvex structure of the integrand, is still missing in the literature. The incorporation of polyconvex control constraints into the proof scheme, which turns out to be possible as well, will be achieved in a subsequent publication.

---

01) [Wagner 09] , p. 549 f., Theorems 2.2. and 2.3.

02) Cf. [Ioffe/Tichomirov 79] , p. 73 f., Theorem 3, Assumption c), and [Ito/Kunisch 08] , p. 5 f.

03) See [Ginsburg/Ioffe 96] , p. 92, Definition 3.2., and p. 96, Theorem 3.6. ("local relaxability" of the problem), as well as [Ioffe/Tichomirov 79] , pp. 201 ff.

04) [Dacorogna 08] , p. 156 f.

05) [Lubkoll/Schiela/Weiser 12] , p. 12 f. (deformation of a compressible Ogden-type material).

06) [Kunisch/Vexler 07] , p. 1371, (1.9), and p. 1376 f., (2.8) (vortex reduction for instationary flows).

07) [Burger/Modersitzki/Ruthotto 13] , [Droske/Rumpf 04] and [Wagner 10] , p. 5, (2.16) (hyperelastic image registration).

The outline of the paper is as follows: After closing this section with some remarks about notation, we turn in *Section 2* to the description of polyconvexity. In *Section 3*, we state the control problem to be investigated, provide two equivalent reformulations of the problem and ensure first the existence of global minimizers (Theorem 3.3.). In *Section 4*, we start with the formulation of Pontryagin's principle in the special case of dimensions $n = m = 2$. Then we state and prove the first-order necessary optimality conditions in full generality as our main result (Theorem 4.3.) and provide an a. e. pointwise reformulation of the maximum condition (Theorem 4.4.). In the final *Section 5*, we apply our theorems to a problem of hyperelastic image registration.

Notations.

Let $\Omega \subset \mathbb{R}^m$ be the closure of a bounded Lipschitz domain (in strong sense). Then $C^k(\Omega, \mathbb{R}^r)$ denotes the space of r-dimensional vector functions $f\colon \Omega \to \mathbb{R}^r$, whose components are continuous ($k = 0$) or k-times continuously differentiable ($k = 1, \dots, \infty$), respectively; $L^p(\Omega, \mathbb{R}^r)$ denotes the space of r-dimensional vector functions $f\colon \Omega \to \mathbb{R}^r$, whose components are integrable in the pth power ($1 \leqslant p < \infty$) or are measurable and esentially bounded ($p = \infty$). $W_0^{1,p}(\Omega, \mathbb{R}^r)$ denotes the Sobolev space of r-dimensional vector functions $f\colon \Omega \to \mathbb{R}^r$ with compactly supported components, possessing first-order weak partial derivatives and belonging together with them to the space $L^p(\Omega, \mathbb{R})$ ($1 \leqslant p < \infty$). $W_0^{1,\infty}(\Omega, \mathbb{R}^r)$ is understood as the Sobolev space of all r-vector functions $f\colon \Omega \to \mathbb{R}^r$ with Lipschitz continuous components and boundary values zero.[08] Jx denotes the Jacobi matrix of the vector function $x \in W_0^{1,p}(\Omega, \mathbb{R}^r)$. The abbreviation "$(\forall)\, s \in \mathrm{A}$" has to be read as "for almost all $s \in \mathrm{A}$" or "for all $s \in \mathrm{A}$ except a Lebesgue null set". Finally, the symbol $\mathfrak{o}$ denotes, depending on the context, the zero element or the zero function of the underlying space. The notion of a polyconvex function will be precisely stated in the following section.

2. Polyconvex functions.

In order to describe polyconvexity, we introduce first the following notation for the vector of the minors of a matricial argument.[09]

Definition 2.1. (The operator T) *Let $n,\, m \geqslant 1$ and denote* $\mathrm{Min}(n, m) = n \wedge m$.

1) We consider elements $v \in \mathbb{R}^{nm}$ as (n, m)-matrices and define $T(v) = \big(v, T_2 v, T_3 v, \dots, T_{(n\wedge m)} v\big) \in \mathbb{R}^{\tau(n,m)} = \mathbb{R}^{\sigma(1)} \times \mathbb{R}^{\sigma(2)} \times \mathbb{R}^{\sigma(3)} \times \dots \times \mathbb{R}^{\sigma(n\wedge m)}$ as the row vector consisting of all minors of v: $T_2 v = \mathrm{adj}_2 v$, $T_3 v = \mathrm{adj}_3 v, \dots, T_{(n\wedge m)} v = \mathrm{adj}_{(n\wedge m)} v$. Consequently, we have $\sigma(k) = \binom{n}{k} \cdot \binom{m}{k}$, $1 \leqslant k \leqslant n \wedge m$. The sum of the dimensions is denoted by $\tau(n, m) = \sigma(1) + \dots + \sigma(n \wedge m)$.

2) Let $(m \wedge n) \leqslant p \leqslant \infty$. We consider elements $u \in L^p(\Omega, \mathbb{R}^{nm})$ as (n, m)-matrix functions and define the operator $T\colon L^p(\Omega, \mathbb{R}^{nm}) \to L^p(\Omega, \mathbb{R}^{\sigma(1)}) \times L^{p/2}(\Omega, \mathbb{R}^{\sigma(2)}) \times L^{p/3}(\Omega, \mathbb{R}^{\sigma(3)}) \times \dots \times L^{p/(n\wedge m)}(\Omega, \mathbb{R}^{\sigma(n\wedge m)})$ by $u \longmapsto Tu = \big(u, T_2 u, T_3 u, \dots, T_{(n\wedge m)} u\big)$ with $T_2 u = \mathrm{adj}_2 u$, $T_3 u = \mathrm{adj}_3 u, \dots, T_{(n\wedge m)} u = \mathrm{adj}_{(n\wedge m)} u$.

Now we may state the definition of a polyconvex function.

Definition 2.2. (Polyconvex function)[10] *We consider elements $v \in \mathbb{R}^{nm}$ as (n, m)-matrices and elements $V \in \mathbb{R}^{\tau(n,m)}$ as row vectors. A function $f(v)\colon \mathbb{R}^{nm} \to \mathbb{R} \cup \{(+\infty)\}$ is called polyconvex iff there*

---

[08] [EVANS/GARIEPY 92], p. 131, Theorem 5.

[09] For all notations related to matricial arguments and polyconvexity, we adopt the conventions from [DACOROGNA 08].

[10] [DACOROGNA 08], p. 156 f., Definition 5.1.(iii).

exists a convex function $g(V)\colon\ \mathbb{R}^{\tau(n,m)} \to \mathbb{R} \cup \{\,(+\infty)\,\}$ *such that*

$$f(v) \;=\; g\big(T(v)\big) \quad \forall\, v \in \mathbb{R}^{nm}\ . \tag{2.1}$$

The function g is called a convex representative for the polyconvex function f.

Note that, in general, the convex representative for a given polyconvex function is not uniquely determined. Given a polyconvex function f, a convex representative may be obtained through [11)]

$$g(V) \;=\; \inf\Big\{ \sum_{r=1}^{\tau(n,m)+1} \lambda_r\, f(v_r) \;\Big|\; \sum_{r=1}^{\tau(n,m)+1} \lambda_r\, T(v_r) \;=\; V\,,\ \sum_{r=1}^{\tau(n,m)+1} \lambda_r = 1\,,\ \lambda_r \geqslant 0\,,\ v_r \in \mathbb{R}^{nm}\,, \quad 1 \leqslant r \leqslant \tau(n,m)+1 \Big\}\,. \tag{2.2}$$

(2.2) is called the Busemann representative of f. [12)] Any polyconvex function is locally Lipschitz continuous on the interior of its effective domain [13)] and, consequently, differentiable a. e. on $\mathrm{dom}\,(f)$. Surprisingly, smoothness properties as continuous differentiability of a polyconvex function f are not automatically inherited by its convex representatives. [14)] For the purposes of optimal control, it is therefore advisable to state the smoothness and growth assumptions about the integrand in terms of a fixed convex representative g rather than of the original function f.
In the special case $n = m = 2$, we get $\sigma(1) = 4$, $\sigma(2) = 1$, $\tau(2,2) = 5$ and $T(v) = \binom{v}{\det v}$. Consequently, any polyconvex function $f\colon\ \mathbb{R}^{2\times 2} \to \mathbb{R} \cup \{\,(+\infty)\,\}$ must take the form $f(v) = g\big(v,\, \det v\big)$ with a convex function $g\colon\ \mathbb{R}^5 \to \mathbb{R} \cup \{\,(+\infty)\,\}$. For $n = m = 3$, we have $\sigma(1) = 9$, $\sigma(2) = 9$, $\sigma(3) = 1$ and $\tau(3,3) = 19$. Here $\mathrm{adj}_2\, v$ is the transpose of the cofactor matrix and $\mathrm{adj}_3\, v = \det v$.

3. Existence of optimal solutions.

a) Statement of the control problem and basic assumptions.

We are concerned with the following multidimensional control problem of Dieudonné-Rashevsky type:

$$(\mathrm{P})_0 \quad F(x,u) \;=\; \int_\Omega f(s, x(s), u(s))\, ds \;\longrightarrow\; \inf!\,; \tag{3.1}$$

$$(x,u) \in W_0^{1,p}(\Omega, \mathbb{R}^n) \times L^p(\Omega, \mathbb{R}^{nm})\,; \tag{3.2}$$

$$E(x,u) \;=\; Jx(s) - u(s) \;=\; 0 \quad (\forall)\, s \in \Omega\,; \tag{3.3}$$

$$u(s) \;\in\; \mathrm{K} \subset \mathbb{R}^{nm} \quad (\forall)\, s \in \Omega\,. \tag{3.4}$$

About the data within the problem $(\mathrm{P})_0$, the following assumptions will be imposed:

Assumptions 3.1. (Basic assumptions about the data within $(\mathrm{P})_0$**)**

1) We assume that $n,\, m \geqslant 2$ *and* $m < p < \infty$ *(thus* $n \wedge m < p$*).*

2) $\Omega \subset \mathbb{R}^m$ *is the closure of a bounded strongly Lipschitz domain, and* $\mathrm{K} \subset \mathbb{R}^{nm}$ *is a convex body with* $\mathfrak{o} \in \mathrm{int}\,(\mathrm{K})$.

---

11) [Dacorogna 08] , p. 163, Theorem 5.6., Part 2.

12) [Bevan 06] , p. 24, Definition 2.1. Recently, [Eneya/Bosse/Griewank 13] provided an effective numerical procedure for the evaluation of $g(V)$.

13) [Dacorogna 08] , p. 47, Theorem 2.31.

14) Cf. [Bevan 03] and [Bevan 06] , pp. 44 ff., Section 5.

3) The integrand $f(s,\xi,v)\colon\ \Omega\times\mathbb{R}^n\times\mathbb{R}^{nm}\to\mathbb{R}$ *is continuous with respect to* s, ξ *and* v *and polyconvex as a function of* v *for all fixed* $(\hat{s},\hat{\xi})\in\Omega\times\mathbb{R}^n$.

4) The polyconvex integrand $f(s,\xi,v)$ *admits a convex representative* $g(s,\xi,v,\omega)\colon\ \Omega\times\mathbb{R}^n\times\mathbb{R}^{nm}\times\big(\mathbb{R}^{\sigma(2)}\times\mathbb{R}^{\sigma(3)}\times\ldots\times\mathbb{R}^{\sigma(n\wedge m)}\big)\to\mathbb{R}$*, which is continuous with respect to* s *and continuously differentiable with respect to* ξ, v *and* ω*. Moreover,* g *satisfies a growth condition*

$$\big|\,g(\,s,\xi,v,\omega_2,\omega_3,\,\ldots,\omega_{(n\wedge m)}\,)\,\big|\,\leqslant\,A_0(s)\,+\,B_0(\xi)\,+\,C_0\,\Big(\,1+\big|\,v\,\big|^p+\sum_{r=2}^{(n\wedge m)}\big|\,\omega_r\,\big|^{p/r}\,\Big)\tag{3.5}$$
$$(\forall)\,s\in\Omega\quad\forall\,(\xi,v,\omega)\in\mathbb{R}^n\times\mathbb{R}^{nm}\times\big(\,\mathbb{R}^{\sigma(2)}\times\mathbb{R}^{\sigma(3)}\times\ldots\times\mathbb{R}^{\sigma(n\wedge m)}\,\big)$$

where $A_0\in L^1(\Omega,\mathbb{R})$, $A_0\,\big|\,\mathrm{int}\,(\Omega)$ *is continuous,* B_0 *is measurable and bounded on every bounded subset of* $\mathbb{R}^n$*, and* $C_0>0$.

b) Equivalent formulations of the control problem.

Choosing for the polyconvex integrand $f(s,\xi,v)$ a convex representative $g(s,\xi,v,\omega)\colon\ \Omega\times\mathbb{R}^n\times\mathbb{R}^{nm}\times\big(\,\mathbb{R}^{\sigma(2)}\times\mathbb{R}^{\sigma(3)}\times\ldots\times\mathbb{R}^{\sigma(n\wedge m)}\,\big)\to\mathbb{R}$ according to Assumption 3.1., 4), the problem $(\mathrm{P})_0$ may be restated in the following way:

$$(\mathrm{P})_1\quad G(x,u,w)\;=\;\int_\Omega g(s,x(s),u(s),w(s))\,ds\;\longrightarrow\;\inf!\,;\tag{3.6}$$
$$(x,u,w)\in W_0^{1,p}(\Omega,\mathbb{R}^n)\times L^p(\Omega,\mathbb{R}^{nm})\times\Big(\,L^{p/2}(\Omega,\mathbb{R}^{\sigma(2)})\times L^{p/3}(\Omega,\mathbb{R}^{\sigma(3)})\times\ldots\times L^{p/(n\wedge m)}(\Omega,\mathbb{R}^{\sigma(n\wedge m)})\Big);\tag{3.7}$$
$$E_1(x,u)\;=\;Jx(s)-u(s)\;=\;0\quad(\forall)\,s\in\Omega\,;\tag{3.8}$$
$$E_2(u,w)\;=\;w_2(s)-\mathrm{adj}_2\,u(s)\;=\;0\quad(\forall)\,s\in\Omega\,;\tag{3.9}$$
$$E_3(u,w)\;=\;w_3(s)-\mathrm{adj}_3\,u(s)\;=\;0\quad(\forall)\,s\in\Omega\,;\tag{3.10}$$
$$\vdots$$
$$E_{(n\wedge m)}(u,w)\;=\;w_{(n\wedge m)}(s)-\mathrm{adj}_{(n\wedge m)}\,u(s)\;=\;0\quad(\forall)\,s\in\Omega\,;\tag{3.11}$$
$$u(s)\in\mathrm{K}\subset\mathbb{R}^{nm}\,;\tag{3.12}$$
$$w(s)\in\mathrm{K}'\subset\mathbb{R}^{\sigma(2)}\times\mathbb{R}^{\sigma(3)}\;\ldots\;\times\mathbb{R}^{\sigma(n\wedge m)}\quad(\forall)\,s\in\Omega\tag{3.13}$$

where

$$\mathrm{K}'=\mathrm{K}(\mathfrak{o}_{\sigma(2)},R_2)\times\mathrm{K}(\mathfrak{o}_{\sigma(3)},R_3)\times\;\ldots\;\times\mathrm{K}(\mathfrak{o}_{\sigma(n\wedge m)},R_{(n\wedge m)})\tag{3.14}$$

is a product of closed balls with the radii

$$R_2\;=\;\sup\big\{\,|\,\big(\,\mathrm{adj}_2(v)\,\big)_l\,|\;\big|\;1\leqslant l\leqslant\sigma(2)\,,\;v\in\mathrm{K}\,\big\}\,,\tag{3.15}$$
$$R_3\;=\;\sup\big\{\,|\,\big(\,\mathrm{adj}_3(v)\,\big)_l\,|\;\big|\;1\leqslant l\leqslant\sigma(3)\,,\;v\in\mathrm{K}\,\big\}\,,\tag{3.16}$$
$$\vdots$$
$$R_{(n\wedge m)}\;=\;\sup\big\{\,|\,\big(\,\mathrm{adj}_{(n\wedge m)}(v)\,\big)_l\,|\;\big|\;1\leqslant l\leqslant\sigma(n\wedge m)\,,\;v\in\mathrm{K}\,\big\}\,.\tag{3.17}$$

A further reformulation of $(\mathrm{P})_0$ is the problem $(\mathrm{P})_2$, which is identical with $(\mathrm{P})_1$ except the fact that the restriction (3.13) is replaced by $w(s)\in\mathbb{R}^{\sigma(2)}\times\mathbb{R}^{\sigma(3)}\;\ldots\;\times\mathbb{R}^{\sigma(n\wedge m)}$ $(\forall)\,s\in\Omega$, i. e. the additional variables w are governed by the equality restrictions (3.9) – (3.11) alone. In fact, the following proposition shows that the problems $(\mathrm{P})_0$ – $(\mathrm{P})_2$ are equivalent.

Proposition 3.2. (Equivalent formulations of the basic problem) *If* (x^*, u^*) *is a global minimizer of* $(\mathrm{P})_0$ *then* $\big(x^*,\, u^*,\, T_2(u^*),\, T_3(u^*),\, \dots,\, T_{(n\wedge m)}(u^*)\big)$ *is a global minimizer of* $(\mathrm{P})_1$ *as well as of* $(\mathrm{P})_2$. *Conversely, if* (x^*, u^*, w^*) *is a global minimizer of* $(\mathrm{P})_1$ *or* $(\mathrm{P})_2$ *then* (x^*, u^*) *is a global minimizer of* $(\mathrm{P})_0$.

Proof. Let (x^*, u^*) be a global minimizer of $(\mathrm{P})_0$ and assume that (x, u, w) is a feasible triple within $(\mathrm{P})_1$. Then, by definition of G, $G(x, u, w) = F(x, u) \geqslant F(x^*, u^*) = G(x^*, u^*, w^*)$ with $w^* = \big(T_2(u^*),\, T_3(u^*),\, \dots,\, T_{(n\wedge m)}(u^*)\big)$, and (x^*, u^*, w^*) is a global minimizer of $(\mathrm{P})_1$. On the other hand, let (x^*, u^*, w^*) be a global minimizer of $(\mathrm{P})_1$ and assume that (x, u) is feasible in $(\mathrm{P})_0$. Then, again by definition of G, we have $F(x, u) = G(x, u, w) \geqslant G(x^*, u^*, w^*) = F(x^*, u^*)$ where $w = \big(T_2(u),\, T_3(u),\, \dots,\, T_{(n\wedge m)}(u)\big)$, and (x^*, u^*) is a global minimizer of $(\mathrm{P})_0$. The same is true if $(\mathrm{P})_1$ is replaced by $(\mathrm{P})_2$ since the feasible domains of both problems coincide. ∎

c) Existence of global minimizers.

Assumptions 3.1. have been chosen in such a way that the existence of a global minimizer of problem $(\mathrm{P})_0$ and, consequently, of problems $(\mathrm{P})_1$ and $(\mathrm{P})_2$ as well, can be guaranteed.

Theorem 3.3. (Existence of global minimizers for $(\mathrm{P})_0 - (\mathrm{P})_2$**)** *Consider problem* $(\mathrm{P})_0$ *under Assumptions 3.1. Then there exists a global minimizer* (x^*, u^*) *of* $(\mathrm{P})_0$. *Consequently, there exists a global minimizer* (x^*, u^*, w^*) *of* $(\mathrm{P})_1$ *and* $(\mathrm{P})_2$ *as well.*

Proof. The growth condition (3.5) implies that

$$\begin{aligned} \big|\, f(s, \xi, v)\,\big| \;&=\; \big|\, g(\, s, \xi, v, T_2(v), T_3(v),\, \dots, T_{(n\wedge m)}(v)\,)\,\big| \qquad (3.18) \\ &\leqslant\; A_0(s) \,+\, B_0(\xi) \,+\, C_0\, \Big(\, 1 + \big|\, v\,\big|^p + \sum_{r=2}^{(n\wedge m)} \big|\, T_r(v)\,\big|^{p/r} \Big) \quad (\forall)\, s \in \Omega \quad \forall\, (\xi, v) \in \mathbb{R}^n \times \mathrm{K} \end{aligned}$$

for almost all $s \in \Omega$ and arbitrary $(\xi, v) \in \mathbb{R}^n \times \mathrm{K}$ where the sum of the second and third term is a bounded function on every bounded subset of $\mathbb{R}^n \times \mathrm{K}$. Consequently, the function $\widetilde{f}(s, \xi, v)\colon\, \Omega \,\times\, \mathbb{R}^n \times \mathbb{R}^{nm} \to \mathbb{R} \cup \{\, +(\infty)\,\}$ obtained as

$$\widetilde{f}(s, \xi, v) \;=\; f(s, \xi, v) + \begin{cases} \;0 & \big|\; (s, \xi, v) \in \Omega \,\times\, \mathbb{R}^n \times \mathrm{K}\,; \\ (+\infty) & \big|\; (s, \xi, v) \in \Omega \,\times\, \mathbb{R}^n \times (\mathbb{R}^{nm} \setminus \mathrm{K}) \end{cases} \qquad (3.19)$$

belongs to the function class $\widetilde{\mathfrak{F}}_{\mathrm{K}}$ described in [Wagner 11], p. 191, Definition 1.1., 2), and we may apply the existence theorem [Wagner 11], p. 193, Theorem 1.5., in order to ensure the existence of a global minimizer of $(\mathrm{P})_0$ while using the modified integrand $\widetilde{f}$ instead of f. By Proposition 3.2., $(\mathrm{P})_1$ and $(\mathrm{P})_2$ admit global minimizers as well. ∎

d) Remarks and generalizations.

Remark 3.4. If the convex representative $g(s, \xi, v, \omega)$ does not depend explicitly on certain components of $\omega \in \mathbb{R}^{\sigma(2)} \times \mathbb{R}^{\sigma(3)} \times \dots \times \mathbb{R}^{\sigma(n\wedge m)}$ then, obviously, the equations referring to these components may be omitted from (3.9) – (3.11), and the feasible domain of $(\mathrm{P})_1$ and $(\mathrm{P})_2$ may be considered as a subset of an accordingly smaller space.

Remark 3.5. Theorem 3.3. remains true if Assumptions 3.1., 3) and 4) are replaced by the following weaker conditions: 3)$'$ $f(s, \xi, v)$ is Borel measurable with respect to s, continuous with respect to ξ and v and polyconvex as a function of v for all fixed $(\hat{s}, \hat{\xi}) \in \big(\, \Omega \setminus \mathrm{N}\,\big) \times \mathbb{R}^n$ where $\mathrm{N} \subset \Omega$ is a m-dimensional Lebesgue null set, and 4)$'$ the convex representative $g(s, \xi, v, \omega)$ is Borel measurable with respect to s, continuously differentiable with respect to ξ, v and ω while still satisfying (3.5). Then the integrand f still fits into the framework described in [Wagner 11] . Note that Proposition 3.2. remains unaffected as well.

4. Pontryagin's principle for polyconvex integrands.

a) The special case $n = m = 2$.

First, let us illustrate the assertions of our main theorems in the simplest case with dimensions $n = m = 2$. Then a global minimizer of $(\mathrm{P})_0$ must satisfy the following first-order necessary optimality conditions.

Theorem 4.1. (Pontryagin's principle for $(\mathrm{P})_0$ with $n = m = 2$) [15] *Consider the problem $(\mathrm{P})_0$ with $n = m = 2$ under Assumptions 3.1. mentioned above and choose for the polyconvex integrand $f(s, \xi, v)$ in $(\mathrm{P})_0$ a convex representative $g(s, \xi, v, \omega_2)$ in accordance with Assumption 3.1., 4). If (x^*, u^*) is a global minimizer of $(\mathrm{P})_0$ then there exist multipliers $\lambda_0 > 0$, $y^{(1)} \in L^{p/(p-1)}(\Omega, \mathbb{R}^4)$ and $y^{(2)} \in L^{p/(p-2)}(\Omega, \mathbb{R})$ such that the following conditions are satisfied:*

$$(\mathcal{M}) \quad \lambda_0 \int_\Omega \Big(g(\, s, x^*(s), u(s), w_2(s)\,) - g(\, s, x^*(s), u^*(s), \det u^*(s)\,) \Big)\, ds - \int_\Omega \big(u(s) - u^*(s) \big)^{\mathrm{T}}\, y^{(1)}(s)\, ds \quad (4.1)$$
$$+ \int_\Omega \big(w_2(s) - \det u^*(s) \big)\, y^{(2)}(s)\, ds - \int_\Omega \nabla_v \det\big(u^*(s) \big)^{\mathrm{T}} \big(u(s) - u^*(s) \big)\, y^{(2)}(s)\, ds \geqslant 0$$
$$\forall\, u \in \mathrm{U} = \{\, z_1 \in L^p(\Omega, \mathbb{R}^{nm}) \mid z_1(s) \in \mathrm{K}\ (\forall)\, s \in \Omega \,\} \quad \forall\, w_2 \in L^{p/2}(\Omega, \mathbb{R})\,;$$

$$(\mathcal{K}) \quad \lambda_0 \sum_{i=1}^{2} \int_\Omega \frac{\partial g}{\partial \xi_i}(\, s, x^*(s), u^*(s), \det u^*(s)\,) \big(x_i(s) - x_i^*(s) \big)\, ds \quad (4.2)$$
$$+ \sum_{i=1}^{2} \sum_{j=1}^{2} \int_\Omega \Big(\frac{\partial x_i}{\partial s_j}(s) - \frac{\partial x_i^*}{\partial s_j}(s) \Big)\, y_{ij}^{(1)}(s)\, ds = 0 \quad \forall\, x \in W_0^{1,p}(\Omega, \mathbb{R}^2)\,. \quad \blacksquare$$

Theorem 4.2. (Pointwise maximum condition for $(\mathrm{P})_0$ with $n = m = 2$) [16] *Consider the problem $(\mathrm{P})_0$ with $n = m = 2$ under the Assumptions 3.1. mentioned above and choose for the polyconvex integrand $f(s, \xi, v)$ in $(\mathrm{P})_0$ a convex representative $g(s, \xi, v, \omega_2)$ in accordance with Assumption 3.1., 4). If (x^*, u^*) is a global minimizer of $(\mathrm{P})_0$ then the maximum condition $(\mathcal{M})$ from Theorem 4.1. implies the following pointwise maximum condition:*

$$(\mathcal{MP}) \quad \lambda_0 \Big(g(\, s, x^*(s), v, \omega_2\,) - g(\, s, x^*(s), u^*(s), \det u^*(s)\,) \Big) - \sum_{i=1}^{2} \sum_{j=1}^{2} \big(v_{ij} - u_{ij}^*(s) \big)\, y_{ij}^{(1)}(s) \quad (4.3)$$
$$+ \big(\omega_2 - \det u^*(s) \big)\, y^{(2)}(s) - \sum_{i=1}^{2} \sum_{j=1}^{2} \frac{\partial}{\partial v_{ij}} \det(\, u^*(s)\,) \big(v_{ij} - u_{ij}^*(s) \big)\, y^{(2)}(s) \geqslant 0$$
$$(\forall)\, s \in \Omega \quad \forall\, v \in \mathrm{K} \quad \forall\, \omega_2 \in \mathbb{R}\,. \quad \blacksquare$$

Obviously, $(\mathcal{MP})$ can be decomposed into the separated conditions

$$(\mathcal{MP})_1 \quad \lambda_0 \Big(g(\, s, x^*(s), v, \det u^*(s)\,) - g(\, s, x^*(s), u^*(s), \det u^*(s)\,) \Big) \quad (4.4)$$
$$- \sum_{i=1}^{2} \sum_{j=1}^{2} \Big(y_{ij}^{(1)}(s) + \frac{\partial}{\partial v_{ij}} \det(\, u^*(s)\,)\, y^{(2)}(s) \Big) \big(v_{ij} - u_{ij}^*(s) \big) \geqslant 0 \quad (\forall)\, s \in \Omega \quad \forall\, v \in \mathrm{K}\,;$$

$$(\mathcal{MP})_2 \quad \lambda_0 \Big(g(\, s, x^*(s), u^*(s), \omega_2\,) - g(\, s, x^*(s), u^*(s), \det u^*(s)\,) \Big) \quad (4.5)$$
$$+ \big(\omega_2 - \det u^*(s) \big)\, y^{(2)}(s) \geqslant 0 \quad (\forall)\, s \in \Omega \quad \forall\, \omega_2 \in \mathbb{R}\,.$$

---

[15] Special case of Theorem 4.3. below.

[16] Special case of Theorem 4.4. below.

b) The main theorems in the general case $n \geqslant 2$, $m \geqslant 2$.

For general dimensions $n \geqslant 2$, $m \geqslant 2$, the first-order necessary optimality conditions for a global minimizer of the multidimensional control problem $(\mathrm{P})_0$ will be stated in the following main theorem.

Theorem 4.3. (Pontryagin's principle for $(\mathrm{P})_0$) *Consider the problem $(\mathrm{P})_0$ under Assumptions 3.1. and choose for the polyconvex integrand $f(s,\xi,v)$ in $(\mathrm{P})_0$ a convex representative $g(s,\xi,v,\omega)$ in accordance with Assumption 3.1., 4). If (x^*,u^*) is a global minimizer of $(\mathrm{P})_0$ then there exist multipliers $\lambda_0 > 0$, $y^{(1)} \in L^{p/(p-1)}(\Omega,\mathbb{R}^{nm})$, $y^{(2)} \in L^{p/(p-2)}(\Omega,\mathbb{R}^{\sigma(2)})$, $y^{(3)} \in L^{p/(p-3)}(\Omega,\mathbb{R}^{\sigma(3)})$, ... , $y^{(n\wedge m)} \in L^{p/(p-(n\wedge m))}(\Omega,\mathbb{R}^{\sigma(n\wedge m)})$ such that the following conditions are satisfied:*

$$(\mathcal{M})\quad \lambda_0 \int_\Omega \Big(g(\,s,x^*(s),u(s),w(s)\,) - g(\,s,x^*(s),u^*(s),w^*(s)\,)\Big)\,ds - \int_\Omega \big(\,u(s)-u^*(s)\,\big)^{\mathrm{T}} y^{(1)}(s)\,ds \tag{4.6}$$
$$+ \sum_{r=2}^{(n\wedge m)} \int_\Omega \big(\,w_r(s)-w_r^*(s)\,\big)^{\mathrm{T}} y^{(r)}(s)\,ds - \sum_{r=2}^{(n\wedge m)} \int_\Omega \nabla_v\,\mathrm{adj}_r(\,u^*(s)\,)\big(\,u(s)-u^*(s)\,\big)^{\mathrm{T}} y^{(r)}(s)\,ds \geqslant 0$$
$$\forall\, u \in \mathrm{U} = \{\, z_1 \in L^p(\Omega,\mathbb{R}^{nm}) \;\big|\; z_1(s) \in \mathrm{K}\ (\forall)\, s \in \Omega\,\}$$
$$\forall\, w_2 \in L^{p/2}(\Omega,\mathbb{R}^{\sigma(2)}) \quad \forall\, w_3 \in L^{p/3}(\Omega,\mathbb{R}^{\sigma(3)}) \quad \ldots \quad \forall\, w_{(n\wedge m)} \in L^{p/(n\wedge m)}(\Omega,\mathbb{R}^{\sigma(n\wedge m)})\,;$$

$$(\mathcal{K})\quad \lambda_0 \sum_{i=1}^{n} \int_\Omega \frac{\partial g}{\partial \xi_i}(\,s,x^*(s),u^*(s),w^*(s)\,)\big(\,x_i(s)-x_i^*(s)\,\big)\,ds \tag{4.7}$$
$$+ \sum_{i=1}^{n}\sum_{j=1}^{m} \int_\Omega \Big(\frac{\partial x_i}{\partial s_j}(s) - \frac{\partial x_i^*}{\partial s_j}(s)\Big)\, y_{ij}^{(1)}(s)\,ds = 0 \quad \forall\, x \in W_0^{1,p}(\Omega,\mathbb{R}^n)\,.$$

Note that the regular case always occurs, i. e. $\lambda_0 \neq 0$.

The following assertion shows that the condition $(\mathcal{M})$ from Theorem 4.3. implies a pointwise maximum condition.

Theorem 4.4. (Pointwise maximum condition for $(\mathrm{P})_0$) *Consider the problem $(\mathrm{P})_0$ under Assumptions 3.1. and choose for the polyconvex integrand $f(s,\xi,v)$ in $(\mathrm{P})_0$ a convex representative $g(s,\xi,v,\omega)$ in accordance with Assumption 3.1., 4). If (x^*,u^*) is a global minimizer of $(\mathrm{P})_0$ then the maximum condition $(\mathcal{M})$ from Theorem 4.3. implies the following pointwise maximum condition:*

$$(\mathcal{MP})\quad \lambda_0 \Big(g(\,s,x^*(s),v,\omega\,) - g(\,s,x^*(s),u^*(s),w^*(s)\,)\Big) - \big(\,v-u^*(s)\,\big)^{\mathrm{T}} y^{(1)}(s) \tag{4.8}$$
$$+ \sum_{r=2}^{(n\wedge m)} \big(\,\omega_r - w_r^*(s)\,\big)^{\mathrm{T}} y^{(r)}(s) - \sum_{r=2}^{(n\wedge m)} \nabla_v\,\mathrm{adj}_r(\,u^*(s)\,)\big(\,v-u^*(s)\,\big)^{\mathrm{T}} y^{(r)}(s) \geqslant 0$$
$$(\forall)\, s \in \Omega \quad \forall\, v \in \mathrm{K} \quad \forall\, \omega_2 \in \mathbb{R}^{\sigma(2)} \quad \forall\, \omega_3 \in \mathbb{R}^{\sigma(3)} \quad \ldots \quad \forall\, \omega_{(n\wedge m)} \in \mathbb{R}^{\sigma(n\wedge m)}\,.$$

Remark. Obviously, $(\mathcal{MP})$ can be further decomposed into the following set of separated conditions:

$$(\mathcal{MP})_1\quad \lambda_0 \Big(g(\,s,x^*(s),v,w^*(s)\,) - g(\,s,x^*(s),u^*(s),w^*(s)\,)\Big) \tag{4.9}$$
$$- \big(\,v-u^*(s)\,\big)^{\mathrm{T}} y^{(1)}(s) - \sum_{r=2}^{(n\wedge m)} \nabla_v\,\mathrm{adj}_r(\,u^*(s)\,)\big(\,v-u^*(s)\,\big)^{\mathrm{T}} y^{(r)}(s) \geqslant 0 \quad (\forall)\, s \in \Omega \quad \forall\, v \in \mathrm{K}\,;$$

$$(\mathcal{MP})_2\quad \lambda_0 \Big(g(\,s,x^*(s),u^*(s),\omega_2,w_3^*(s),\,\ldots\,,w_{(n\wedge m)}^*(s)\,) - g(\,s,x^*(s),u^*(s),w^*(s)\,)\Big) \tag{4.10}$$
$$+ \big(\,\omega_2 - w_2^*(s)\,\big)^{\mathrm{T}} y^{(2)}(s) \geqslant 0 \quad (\forall)\, s \in \Omega \quad \forall\, \omega_2 \in \mathbb{R}^{\sigma(2)}\,;$$

$$(\mathcal{MP})_3\quad \lambda_0 \Big(g(\,s,x^*(s),u^*(s),w_2^*(s),\omega_3,\,\ldots\,,w_{(n\wedge m)}^*(s)\,) - g(\,s,x^*(s),u^*(s),w^*(s)\,)\Big) \tag{4.11}$$
$$+ \big(\,\omega_3 - w_3^*(s)\,\big)^{\mathrm{T}} y^{(3)}(s) \geqslant 0 \quad (\forall)\, s \in \Omega \quad \forall\, \omega_3 \in \mathbb{R}^{\sigma(3)}\,;$$

$\vdots$

$$(\mathcal{MP})_{(n\wedge m)} \quad \lambda_0 \Big(g(\,s, x^*(s), u^*(s), w_2^*(s), w_3^*(s), \,\ldots\,, \omega_{(n\wedge m)}\,) - g(\,s, x^*(s), u^*(s), w^*(s)\,)\Big) \qquad (4.12)$$
$$+ \big(\omega_{(n\wedge m)} - w^*_{(n\wedge m)}(s)\big)^{\mathrm{T}}\, y^{(n\wedge m)}(s) \,\geqslant\, 0 \quad (\forall)\, s \in \Omega \quad \forall\, \omega_{(n\wedge m)} \in \mathbb{R}^{\sigma(n\wedge m)}\,.$$

c) Proof of Theorem 4.3.

Sketch of the proof. The proof of Theorem 4.3. is structured as follows: By Proposition 3.2., any given global minimizer (x^*, u^*) of $(\mathrm{P})_0$ corresponds to a global minimizer $(x^*, u^*, w^*) = \big(x^*, u^*, T_2(u^*), T_3(u^*), \ldots, T_{(n\wedge m)}(u^*)\big)$ of $(\mathrm{P})_2$. With reference to (x^*, u^*, w^*), we define a pair of convex variational sets C and D within the space $\mathbb{R} \times L^p(\Omega, \mathbb{R}^{nm}) \times L^{p/2}(\Omega, \mathbb{R}^{\sigma(2)}) \times L^{p/3}(\Omega, \mathbb{R}^{\sigma(3)}) \times \ldots \times L^{p/(n\wedge m)}(\Omega, \mathbb{R}^{\sigma(n\wedge m)})$ and show that C is closed and D has a nonempty interior (Step 1). Applying Lyusternik's theorem to the operators E_2, E_3, ... , $E_{(n\wedge m)}$, we prove that the nonnegativity of the first variation of the objective G of $(\mathrm{P})_2$ at (x^*, u^*, w^*) implies the relation $\mathrm{C} \cap \mathrm{D} = \emptyset$ (Steps $2-4$). Regardless of the failing regularity condition for E_1, the weak separation theorem yields a variational inequality, from which the claimed optimality conditions are obtained (Steps 5 and 6). The occurence of the regular case is a consequence of the fact that the set U of feasible controls within $(\mathrm{P})_0$ admits a nonempty interior with respect to the L^∞-norm topology (Step 7).

• **Step 1.** *The variational sets* C *and* D. Assume that a global minimizer (x^*, u^*) of $(\mathrm{P})_0$ is given. Then $(x^*, u^*, w^*) = \big(x^*, u^*, T_2(u^*), T_3(u^*), \ldots, T_{(n\wedge m)}(u^*)\big)$ is a global minimizer of $(\mathrm{P})_2$. Fixing a number $\alpha > 0$, we define the variational sets

$$\mathrm{C} \,=\, \big\{\, \big(\,\varrho,\, z_1,\, z_2,\, z_3,\, \ldots,\, z_{(n\wedge m)}\,\big) \qquad (4.13)$$
$$\in \mathbb{R} \times L^p(\Omega, \mathbb{R}^{nm}) \times L^{p/2}(\Omega, \mathbb{R}^{\sigma(2)}) \times L^{p/3}(\Omega, \mathbb{R}^{\sigma(3)}) \times \ldots \times L^{p/(n\wedge m)}(\Omega, \mathbb{R}^{\sigma(n\wedge m)}) \quad \text{with}$$
$$\varrho = \varepsilon + D_x G(x^*, u^*, w^*)(x - x^*) + D_u G(x^*, u^*, w^*)(u - u^*) + D_w G(x^*, u^*, w^*)(w - w^*)\,; \qquad (4.14)$$
$$z_1 = Jx - Jx^* - (u - u^*)\,; \qquad (4.15)$$
$$z_2 = (w_2 - w_2^*) - D_u T_2(u^*)(u - u^*)\,; \qquad (4.16)$$
$$z_3 = (w_3 - w_3^*) - D_u T_3(u^*)(u - u^*)\,; \qquad (4.17)$$
$$\vdots$$
$$z_{(n\wedge m)} = (w_{(n\wedge m)} - w^*_{(n\wedge m)}) - D_u T_{(n\wedge m)}(u^*)(u - u^*)\,; \qquad (4.18)$$
$$\varepsilon \geqslant 0\,,\ x \in W_0^{1,p}(\Omega, \mathbb{R}^n)\,, \qquad (4.19)$$
$$u \in \mathrm{U}\,,\ w_2 \in L^{p/2}(\Omega, \mathbb{R}^{\sigma(2)})\,,\ w_3 \in L^{p/3}(\Omega, \mathbb{R}^{\sigma(3)})\,, \ldots, w_{(n\wedge m)} \in L^{p/(n\wedge m)}(\Omega, \mathbb{R}^{\sigma(n\wedge m)})\,\big\}\,; \qquad (4.20)$$
$$\mathrm{D} \,=\, \big\{\, \big(\,\varrho,\, z_1,\, z_2,\, z_3,\, \ldots,\, z_{(n\wedge m)}\,\big) \qquad (4.21)$$
$$\in \mathbb{R} \times L^p(\Omega, \mathbb{R}^{nm}) \times L^{p/2}(\Omega, \mathbb{R}^{\sigma(2)}) \times L^{p/3}(\Omega, \mathbb{R}^{\sigma(3)}) \times \ldots \times L^{p/(n\wedge m)}(\Omega, \mathbb{R}^{\sigma(n\wedge m)}) \quad \text{with}$$
$$\varrho < -2\,K \Big(\, \|\, z_1 \,\|_{L^p} + \|\, z_2 \,\|_{L^{p/2}} + \|\, z_3 \,\|_{L^{p/3}} + \ldots + \|\, z_{(n\wedge m)} \,\|_{L^{p/(n\wedge m)}} \Big)\,; \qquad (4.22)$$
$$z_1 \in \mathrm{K}(\mathfrak{o}, \alpha) \subset L^p(\Omega, \mathbb{R}^{nm})\,; \qquad (4.23)$$
$$z_2 \in \mathrm{K}(\mathfrak{o}, \alpha) \subset L^{p/2}(\Omega, \mathbb{R}^{\sigma(2)})\,; \qquad (4.24)$$
$$z_3 \in \mathrm{K}(\mathfrak{o}, \alpha) \subset L^{p/3}(\Omega, \mathbb{R}^{\sigma(3)})\,; \qquad (4.25)$$
$$\vdots$$
$$z_{(n\wedge m)} \in \mathrm{K}(\mathfrak{o}, \alpha) \subset L^{p/(n\wedge m)}(\Omega, \mathbb{R}^{\sigma(n\wedge m)})\,\big\} \qquad (4.26)$$

The value of the constant $K > 0$ will be chosen according to (4.103) below.

Proposition 4.5. *1) The variational set* C *is nonempty, closed and convex.*

2) The variational set D *is convex with nonempty interior.*

Proof. 1) Obviously, C contains the origin and is convex since U is convex together with K. In order to prove closedness, assume that a sequence $\{\,(\,\varrho^N,\,z_1^N,\,z_2^N,\,z_3^N,\,\dots,\,z_{(n\wedge m)}^N\,)\,\}$, C with $\varrho^N \to \varrho_0$, $z_1^N \to z_1^0$, $z_2^N \to z_2^0$, $z_3^N \to z_3^0$, ... , $z_{(n\wedge m)}^N \to z_{(n\wedge m)}^0$ is given. The elements of this sequence are generated by numbers $\varepsilon^N \geqslant 0$ and functions $x^N \in W_0^{1,p}(\Omega,\mathbb{R}^n)$, $u^N \in$ U, $w_2^N \in L^{p/2}(\Omega,\mathbb{R}^{\sigma(2)})$, $w_3^N \in L^{p/3}(\Omega,\mathbb{R}^{\sigma(3)})$, ... , $w_{(n\wedge m)}^N \in L^{p/(n\wedge m)}(\Omega,\mathbb{R}^{\sigma(n\wedge m)})$. We check now whether the sequences $\{\,x^N\,\}$, $\{\,u^N\,\}$, $\{\,w_2^N\,\}$, $\{\,w_3^N\,\}$, ... , $\{\,w_{(n\wedge m)}^N\,\}$ remain bounded. First, the sequence $\{\,u^N\,\}$, U is bounded in L^∞-norm (thus in L^p-norm as well). Consequently, $z_2^N \to z_2^0$, $z_3^N \to z_3^0$, ... , $z_{(n\wedge m)}^N \to z_{(n\wedge m)}^0$ imply that the sequences $\{\,w_2^N\,\}$, $\{\,w_3^N\,\}$, ... , $\{\,w_{(n\wedge m)}^N\,\}$ remain bounded in $L^{p/2}$-, $L^{p/3}$-, ... , $L^{p/(n\wedge m)}$-norm, respectively. Further, from $z_1^N = Jx^N - u^N \to z_1^0$ and $x^N \in W_0^{1,p}(\Omega,\mathbb{R}^n)$ we may conclude that the sequence $\{\,x^N\,\}$ is bounded in $W^{1,p}$-norm. Assumption 3.1., 4) ensures that $D_xG(x^*,u^*,w^*)$, $D_uG(x^*,u^*,w^*)$ and $D_wG(x^*,u^*,w^*)$ act as linear, continuous functionals on the spaces $W_0^{1,p}(\Omega,\mathbb{R}^n)$, $L^p(\Omega,\mathbb{R}^{nm})$ and $L^{p/2}(\Omega,\mathbb{R}^{\sigma(2)}) \times L^{p/3}(\Omega,\mathbb{R}^{\sigma(3)}) \times \dots \times L^{p/(n\wedge m)}(\Omega,\mathbb{R}^{\sigma(n\wedge m)}$, respectively. Consequently, $\varrho^N \to \varrho_0$ implies the boundedness of the sequence $\{\,\varepsilon^N\,\}$.

As a consequence, $\{\,u^N\,\}$ contains a subsequence, which converges weak* (with respect to L^∞) or weak (with respect to L^p) to a limit element $u^0 \in L^p(\Omega,\mathbb{R}^{nm})$. Since $\mathrm{U} \subset L^\infty(\Omega,\mathbb{R}^{nm}) \subset L^p(\Omega,\mathbb{R}^{nm})$ is convex, u^0 still belongs to U. Further, within $\{\,w_2^N\,\}$, $\{\,w_3^N\,\}$, ... , $\{\,w_{(n\wedge m)}^N\,\}$ we find weakly convergent subsequences with limit elements $w_2^0 \in L^{p/2}(\Omega,\mathbb{R}^{\sigma(2)})$, $w_3^0 \in L^{p/3}(\Omega,\mathbb{R}^{\sigma(3)})$, ... , $w_{(n\wedge m)}^0 \in L^{p/(n\wedge m)}(\Omega,\mathbb{R}^{\sigma(n\wedge m)})$ satisfying

$$z_2^0 = (w_2^0 - w_2^*) - D_uT_2(u^*)(u^0 - u^*)\,; \tag{4.27}$$

$$z_3^0 = (w_3^0 - w_3^*) - D_uT_3(u^*)(u^0 - u^*)\,; \tag{4.28}$$

$$\vdots$$

$$z_{(n\wedge m)}^0 = (w_{(n\wedge m)}^0 - w_{(n\wedge m)}^*) - D_uT_{(n\wedge m)}(u^*)(u^0 - u^*)\,. \tag{4.29}$$

Moreover, $\{\,x^N\,\}$ contains a weakly convergent subsequence with limit element $x^0 \in W_0^{1,p}(\Omega,\mathbb{R}^n)$ as well. By Assumption 3.1., 1), this subsequence may be chosen according to the Rellich/Kondrachev theorem [17] such that $x^{N'} \rightrightarrows x^0$, and x^0 satisfies the zero boundary condition. Since the generalized derivative operator is weakly continuous, $z_1^N \to z_1^0$ implies that $Jx^0 = u^0$, and we get

$$z_1^0 = Jx^0 - Jx^* - (u^0 - u^*)\,. \tag{4.30}$$

Finally, $\{\,\varepsilon^N\,\}$ contains a convergent subsequence with limit element $\varepsilon^0 \geqslant 0$, and we obtain

$$\varrho^0 = \varepsilon^0 + D_xG(x^*,u^*,w^*)(x^0 - x^*) + D_uG(x^*,u^*,w^*)(u^0 - u^*) + D_wG(x^*,u^*,w^*)(w^0 - w^*)\,. \tag{4.31}$$

Consequently, $\big(\,\varrho^0,\,z_1^0,\,z_2^0,\,z_3^0,\,\dots,\,z_{(n\wedge m)}^0\,\big)$ belongs to C, and the set is closed.

2) D is the subgraph of a concave function over a convex range of definition in the space $L^p(\Omega,\mathbb{R}^{nm}) \times L^{p/2}(\Omega,\mathbb{R}^{\sigma(2)}) \times L^{p/3}(\Omega,\mathbb{R}^{\sigma(3)}) \times \dots \times L^{p/(n\wedge m)}(\Omega,\mathbb{R}^{\sigma(n\wedge m)}$ and, therefore, convex. Obviously, the point $\big(-2\,K,\,\mathfrak{o},\,\mathfrak{o},\,\mathfrak{o},\,\dots,\,\mathfrak{o}\big)$ belongs to the interior of D. ∎

• **Step 2.** *Definition of the set* C_0. Denote by

$$\mathrm{G} = \big\{\,z_1 \in L^p(\Omega,\mathbb{R}^{nm}) \;\big|\; \exists\, x \in W_0^{1,p}(\Omega,\mathbb{R}^n) \text{ such that } z_1 = Jx\,\big\} \tag{4.32}$$

the "gradient" subspace of $L^p(\Omega,\mathbb{R}^{nm})$ [18] and by

$$\mathrm{U}_0 = \mathrm{U} \cap \mathrm{G} \tag{4.33}$$

---

17) [Adams/Fournier 07], p. 168, Theorem 6.3.

18) If Ω admits even a C^1-boundary then $L^p(\Omega,\mathbb{R}^{nm})$ admits a direct decomposition into a "gradient" and a "curl" subspace for any $1 < p < \infty$, cf. [Wagner 09], p. 555, Theorem 3.1., (iii).

the subset of all admissible controls of $(\mathrm{P})_0$, which may be completed to feasible pairs for $(\mathrm{P})_0$. With the aid of U_0, we define the following set

$$\begin{aligned}
\mathrm{C}_0 \;=\; \big\{ \big(&\varrho,\, z_1,\, z_2,\, z_3,\, \dots,\, z_{(n\wedge m)} \big) \qquad (4.34)\\
&\in \mathbb{R}\times L^p(\Omega,\mathbb{R}^{nm})\times L^{p/2}(\Omega,\mathbb{R}^{\sigma(2)})\times L^{p/3}(\Omega,\mathbb{R}^{\sigma(3)})\times \dots \times L^{p/(n\wedge m)}(\Omega,\mathbb{R}^{\sigma(n\wedge m)}) \quad \text{with}\\
&\mathfrak{o} = Jx - Jx^* - (u-u^*)\,; \qquad (4.35)\\
&\mathfrak{o} = (w_2 - w_2^*) - D_uT_2(u^*)(u-u^*)\,; \qquad (4.36)\\
&\mathfrak{o} = (w_3 - w_3^*) - D_uT_3(u^*)(u-u^*)\,; \qquad (4.37)\\
&\vdots\\
&\mathfrak{o} = (w_{(n\wedge m)} - w_{(n\wedge m)}^*) - D_uT_{(n\wedge m)}(u^*)(u-u^*)\,; \qquad (4.38)\\
&x \in W_0^{1,p}(\Omega,\mathbb{R}^n)\,,\; u\in \mathrm{U}_0\,, \qquad (4.39)\\
&w_2 \in L^{p/2}(\Omega,\mathbb{R}^{\sigma(2)})\,,\; w_3\in L^{p/3}(\Omega,\mathbb{R}^{\sigma(3)})\,,\,\dots,\, w_{(n\wedge m)}\in L^{p/(n\wedge m)}(\Omega,\mathbb{R}^{\sigma(n\wedge m)})\,\big\}\,. \qquad (4.40)
\end{aligned}$$

• **Step 3. Proposition 4.6.** *The following implication holds:* $\big(\varrho,\, z_1,\, z_2,\, z_3,\, \dots,\, z_{(n\wedge m)}\big) \in \mathrm{C}_0 \cap \mathrm{C} \Longrightarrow \varrho \geqslant 0$.

Proof. The proof of Proposition 4.6. will be delivered in several steps.

• **Step 3.1.** Assume that an element $\big(\varrho^0,\, \mathfrak{o},\, \mathfrak{o},\, \mathfrak{o},\, \dots,\, \mathfrak{o}\big)$ is contained in $\mathrm{C}_0\cap\mathrm{C}$. Then we find a number $\varepsilon^0 \geqslant 0$ and functions $x^0\in W_0^{1,p}(\Omega,\mathbb{R}^n)$, $u^0\in \mathrm{U}_0 \subset L^p(\Omega,\mathbb{R}^{nm})$, $w_2^0 \in L^{p/2}(\Omega,\mathbb{R}^{\sigma(2)})$, $w_3^0\in L^{p/3}(\Omega,\mathbb{R}^{\sigma(3)})$, ... , $w^0_{(n\wedge m)}\in L^{p/(n\wedge m)}(\Omega,\mathbb{R}^{\sigma(n\wedge m)})$ satisfying

$$\begin{aligned}
\varrho^0 \;&=\; \varepsilon^0 + D_xG(x^*,u^*,w^*)(x^0-x^*) + D_uG(x^*,u^*,w^*)(u^0-u^*) + D_wG(x^*,u^*,w^*)(w^0-w^*)\,; \qquad (4.41)\\
\mathfrak{o} \;&=\; Jx^0 - Jx^* - (u^0-u^*)\,; \qquad (4.42)\\
\mathfrak{o} \;&=\; (w_2^0 - w_2^*) - D_uT_2(u^*)(u^0-u^*)\,; \qquad (4.43)\\
\mathfrak{o} \;&=\; (w_3^0 - w_3^*) - D_uT_3(u^*)(u^0-u^*)\,; \qquad (4.44)\\
&\vdots\\
\mathfrak{o} \;&=\; (w_{(n\wedge m)}^0 - w_{(n\wedge m)}^*) - D_uT_{(n\wedge m)}(u^*)(u^0-u^*)\,. \qquad (4.45)
\end{aligned}$$

• **Step 3.2.** Let us invoke now Lyusternik's theorem, which reads as follows:

Theorem 4.7. (Ljusternik's theorem) [19] *Consider Banach spaces* X, Y, *the (possibly nonlinear) operator* $M\colon \mathrm{X}\to\mathrm{Y}$ *and its kernel* $\mathcal{M} = \{\, r\in \mathrm{X} \mid M(r) = \mathfrak{o}\,\}$. *If* $r^*\in\mathcal{M}$, M *is continuously Fréchet differentiable in a neighbourhood of* r^* *and* $DM(r^*)$ *maps onto* Y *then the set of the tangential vectors for* $\mathcal{M}$ *at the point* r^* *coincides with the kernel* $\{\, r\in\mathrm{X} \mid DM(r^*)(r) = \mathfrak{o}\,\}$.

Applying the theorem to the data

$$\begin{aligned}
\mathrm{X} \;&=\; L^p(\Omega,\mathbb{R}^{nm})\times L^{p/2}(\Omega,\mathbb{R}^{\sigma(2)})\times L^{p/3}(\Omega,\mathbb{R}^{\sigma(3)})\times\dots\times L^{p/(n\wedge m)}(\Omega,\mathbb{R}^{\sigma(n\wedge m)})\,; \qquad (4.46)\\
\mathrm{Y} \;&=\; L^{p/2}(\Omega,\mathbb{R}^{\sigma(2)})\times L^{p/3}(\Omega,\mathbb{R}^{\sigma(3)})\times\dots\times L^{p/(n\wedge m)}(\Omega,\mathbb{R}^{\sigma(n\wedge m)})\,; \qquad (4.47)\\
M \;&=\; \big(E_2\,,\,\dots,\, E_{(n\wedge m)}\big)\,; \qquad (4.48)\\
r^* \;&=\; (u^*, w^*)\,, \qquad (4.49)
\end{aligned}$$

[19] [Ioffe/Tichomirow 79], p. 42.

we observe that the Fréchet derivative $DM(u^*,w^*)\colon\ \mathrm{X}\times\mathrm{Y}\to\mathrm{Y}$, which is given through

$$DM(u^*,w^*)(\,u-u^*,w-w^*)\;=\;\begin{pmatrix} w_2-w_2^*-D_uT_2(u^*)(u-u^*)\\ w_3-w_3^*-D_uT_3(u^*)(u-u^*)\\ \vdots\\ w_{(n\wedge m)}-w^*_{(n\wedge m)}-D_uT_{(n\wedge m)}(u^*)(u-u^*)\end{pmatrix}, \tag{4.50}$$

maps onto Y. The continuity of DM with respect to the reference point is obvious. Consequently, (4.43) − (4.45) imply that (u^0-u^*,w^0-w^*) is a tangential vector for $\mathcal{M}=\{\,(u,w)\in\mathrm{X}\,\big|\,M(u,w)=\mathfrak{o}\,\}$ at (u^*,w^*), and there exist elements $\big(Q(u^0,\lambda)\,,\,R(w^0,\lambda)\big)\in\mathrm{X}$ such that

$$\big(u^*+\lambda\,(u^0-u^*)+Q(u^0,\lambda)\,,\;w^*+\lambda\,(w^0-w^*)+R(w^0,\lambda)\big)\;\in\;\mathcal{M}\quad\Longleftrightarrow \tag{4.51}$$

$$w_2^*(s)+\lambda\,(w_2^0-w_2^*)+R_2(w^0,\lambda)-\mathrm{adj}_2\,\big(u^*(s)+\lambda\,(u^0-u^*)+Q(u^0,\lambda)\big)\;=\;0\quad(\forall)\,s\in\Omega\,; \tag{4.52}$$

$$w_3^*(s)+\lambda\,(w_3^0-w_3^*)+R_3(w^0,\lambda)-\mathrm{adj}_3\,\big(u^*(s)+\lambda\,(u^0-u^*)+Q(u^0,\lambda)\big)\;=\;0\quad(\forall)\,s\in\Omega\,; \tag{4.53}$$

$$\vdots$$

$$w^*_{(n\wedge m)}(s)+\lambda\,(w^0_{(n\wedge m)}-w^*_{(n\wedge m)})+R_{(n\wedge m)}(w^0,\lambda) - \mathrm{adj}_{(n\wedge m)}\,\big(u^*(s)+\lambda\,(u^0-u^*)+Q(u^0,\lambda)\big)\;=\;0\quad(\forall)\,s\in\Omega \tag{4.54}$$

for all sufficiently small $\lambda\geqslant 0$ where

$$\lim_{\lambda\to 0+0}\ \lambda^{-1}\,\|\,Q(u^0,\lambda)\,\|_{L^p}\;=\;0\,; \tag{4.55}$$

$$\lim_{\lambda\to 0+0}\ \lambda^{-1}\,\|\,R_2(w^0,\lambda)\,\|_{L^{p/2}}\;=\;0\,; \tag{4.56}$$

$$\lim_{\lambda\to 0+0}\ \lambda^{-1}\,\|\,R_3(w^0,\lambda)\,\|_{L^{p/3}}\;=\;0\,; \tag{4.57}$$

$$\vdots$$

$$\lim_{\lambda\to 0+0}\ \lambda^{-1}\,\|\,R_{(n\wedge m)}(w^0,\lambda)\,\|_{L^{p/(n\wedge m)}}\;=\;0\,. \tag{4.58}$$

• **Step 3.3.** Let us decompose

$$\mathrm{adj}_2\,\big(u^*(s)+\lambda\,(u^0-u^*)+Q(u^0,\lambda)\big)\;=\;\mathrm{adj}_2\,\big(u^*(s)+\lambda\,(u^0-u^*)\big)+S_2(u^*,u^0,\lambda)\,; \tag{4.59}$$

$$\mathrm{adj}_3\,\big(u^*(s)+\lambda\,(u^0-u^*)+Q(u^0,\lambda)\big)\;=\;\mathrm{adj}_3\,\big(u^*(s)+\lambda\,(u^0-u^*)\big)+S_3(u^*,u^0,\lambda)\,; \tag{4.60}$$

$$\vdots$$

$$\mathrm{adj}_{(n\wedge m)}\,\big(u^*(s)+\lambda\,(u^0-u^*)+Q(u^0,\lambda)\big)=\mathrm{adj}_{(n\wedge m)}\,\big(u^*(s)+\lambda\,(u^0-u^*)\big)+S_{(n\wedge m)}(u^*,u^0,\lambda)\,. \tag{4.61}$$

Consequently, we have

$$w_2^*(s)+\lambda\,(w_2^0-w_2^*)+R_2(w^0,\lambda)-S_2(u^*,u^0,\lambda)-\mathrm{adj}_2\,\big(u^*(s)+\lambda\,(u^0-u^*)\big)\;=\;0\,; \tag{4.62}$$

$$w_3^*(s)+\lambda\,(w_3^0-w_3^*)+R_3(w^0,\lambda)-S_3(u^*,u^0,\lambda)-\mathrm{adj}_3\,\big(u^*(s)+\lambda\,(u^0-u^*)\big)\;=\;0\,; \tag{4.63}$$

$$\vdots$$

$$w^*_{(n\wedge m)}(s)+\lambda\,(w^0_{(n\wedge m)}-w^*_{(n\wedge m)})+R_{(n\wedge m)}(w^0,\lambda)-S_{(n\wedge m)}(u^*,u^0,\lambda) - \mathrm{adj}_{(n\wedge m)}\,\big(u^*(s)+\lambda\,(u^0-u^*)\big)\;=\;0\,, \tag{4.64}$$

and the triples

$$\Big(x^*+\lambda\,(x^0-x^*)\,,\;u^*+\lambda\,(u^0-u^*)\,,\;w^*+\lambda\,(w^0-w^*)+R(w^0,\lambda)-S(u^*,u^0,\lambda)\Big) \tag{4.65}$$

are feasible in $(\mathrm{P})_2$ for all sufficiently small $\lambda > 0$. Moreover, the expressions $S(u^*, u^0, \lambda)$ satisfy limit relations analogous to $R(w^0, \lambda)$.

• **Step 3.4. Lemma 4.8.** *It holds that* $\lim\limits_{\lambda \to 0+0} \lambda^{-1} \| S_2(u^*, u^0, \lambda) \|_{L^{p/2}} = 0$, $\lim\limits_{\lambda \to 0+0} \lambda^{-1} \| S_3(u^*, u^0, \lambda) \|_{L^{p/3}} = 0$, ... , $\lim\limits_{\lambda \to 0+0} \lambda^{-1} \| S_{(n\wedge m)}(u^*, u^0, \lambda) \|_{L^{p/(n\wedge m)}} = 0$.

Proof. Expanding (4.59), to every index $1 \leqslant l \leqslant \sigma(2)$ correspond indices $1 \leqslant i < k \leqslant n$, $1 \leqslant j < r \leqslant m$ such that

$$S_{2,l}(u^*, u^0, \lambda) = \big(u^*_{ij} + \lambda\, (u^0_{ij} - u^*_{ij}) \big)\, Q_{kr}(u^0, \lambda) - \big(u^*_{kj} + \lambda\, (u^0_{kj} - u^*_{kj}) \big)\, Q_{ir}(u^0, \lambda) \tag{4.66}$$
$$+ Q_{ij}(u^0, \lambda)\, \big(u^*_{kr} + \lambda\, (u^0_{kr} - u^*_{kr}) \big) - Q_{kj}(u^0, \lambda)\, \big(u^*_{ir} + \lambda\, (u^0_{ir} - u^*_{ir}) \big)$$
$$+ Q_{ij}(u^0, \lambda)\, Q_{kr}(u^0, \lambda) - Q_{kj}(u^0, \lambda)\, Q_{ir}(u^0, \lambda) \quad \Longrightarrow$$

$$\int_\Omega \big| S_{2,l}(u^*, u^0, \lambda) \big|^{p/2} ds \leqslant C \Big(\int_\Omega \big| Q_{kr}(u^0, \lambda) \big|^{p/2} ds + \int_\Omega \big| Q_{ir}(u^0, \lambda) \big|^{p/2} ds + \int_\Omega \big| Q_{ij}(u^0, \lambda) \big|^{p/2} ds \tag{4.67}$$
$$+ \int_\Omega \big| Q_{kj}(u^0, \lambda) \big|^{p/2} ds + \int_\Omega \big| Q_{ij}(u^0, \lambda)\, Q_{kr}(u^0, \lambda) \big|^{p/2} ds + \int_\Omega \big| Q_{kj}(u^0, \lambda)\, Q_{ir}(u^0, \lambda) \big|^{p/2} ds \Big)$$

since $\big(u^*_{ij}(s) + \lambda\, (u^0_{ij}(s) - u^*_{ij}(s)) \big)$, $\big(u^*_{kj}(s) + \lambda\, (u^0_{kj}(s) - u^*_{kj}(s)) \big)$, $\big(u^*_{kr}(s) + \lambda\, (u^0_{kr}(s) - u^*_{kr}(s)) \big)$ and $\big(u^*_{ir}(s) + \lambda\, (u^0_{ir}(s) - u^*_{ir}(s)) \big)$ belong to the compact set K for almost all $s \in \Omega$. Consequently, we get

$$\| S_{2,l}(u^*, u^0, \lambda) \|_{L^{p/2}(\Omega)} \tag{4.68}$$
$$\leqslant C \Big(\| Q_{kr}(u^0, \lambda) \|_{L^{p/2}(\Omega)} + \| Q_{ir}(u^0, \lambda) \|_{L^{p/2}(\Omega)} + \| Q_{ij}(u^0, \lambda) \|_{L^{p/2}(\Omega)} + \| Q_{kj}(u^0, \lambda) \|_{L^{p/2}(\Omega)}$$
$$+ \| Q_{ij}(u^0, \lambda) \|_{L^p(\Omega)} \cdot \| Q_{kr}(u^0, \lambda) \|_{L^p(\Omega)} + \| Q_{kj}(u^0, \lambda) \|_{L^p(\Omega)} \cdot \| Q_{ir}(u^0, \lambda) \|_{L^p(\Omega)} \Big)$$
$$\leqslant \widetilde{C} \Big(\| Q_{kr}(u^0, \lambda) \|_{L^p(\Omega)} + \| Q_r(u^0, \lambda) \|_{L^p(\Omega)} + \| Q_{ij}(u^0, \lambda) \|_{L^p(\Omega)} + \| Q_{kj}(u^0, \lambda) \|_{L^p(\Omega)} \tag{4.69}$$
$$+ \| Q_{ij}(u^0, \lambda) \|_{L^p(\Omega)} \cdot \| Q_{kr}(u^0, \lambda) \|_{L^p(\Omega)} + \| Q_{kj}(u^0, \lambda) \|_{L^p(\Omega)} \cdot \| Q_{ir}(u^0, \lambda) \|_{L^p(\Omega)} \Big) \quad \Longrightarrow$$

$$\lim_{\lambda \to 0+0} \lambda^{-1} \| S_{2,l}(u^*, u^0, \lambda) \|_{L^{p/2}} \tag{4.70}$$
$$\leqslant \widetilde{C} \Big(\lim_{\lambda \to 0+0} \lambda^{-1} \| Q_{kr}(u^0, \lambda) \|_{L^p(\Omega)} + \lim_{\lambda \to 0+0} \lambda^{-1} \| Q_{ir}(u^0, \lambda) \|_{L^p(\Omega)}$$
$$+ \lim_{\lambda \to 0+0} \lambda^{-1} \| Q_{ij}(u^0, \lambda) \|_{L^p(\Omega)} + \lim_{\lambda \to 0+0} \lambda^{-1} \| Q_{kj}(u^0, \lambda) \|_{L^p(\Omega)}$$
$$+ \lim_{\lambda \to 0+0} \lambda^{-1} \| Q_{ij}(u^0, \lambda) \|_{L^p(\Omega)} \cdot \lim_{\lambda \to 0+0} \| Q_{kr}(u^0, \lambda) \|_{L^p(\Omega)}$$
$$+ \lim_{\lambda \to 0+0} \lambda^{-1} \| Q_{kj}(u^0, \lambda) \|_{L^p(\Omega)} \cdot \lim_{\lambda \to 0+0} \| Q_{ir}(u^0, \lambda) \|_{L^p(\Omega)} \Big) = 0 \tag{4.71}$$

by assumption about $Q(u^0, \lambda)$. Analogously, we may confirm that $\lambda^{-1} \| S_{3,l}(u^*, u^0, \lambda) \|_{L^{p/3}} \to 0$ for all $1 \leqslant l \leqslant \sigma(3)$, ... , $\lambda^{-1} \| S_{(n\wedge m),l}(u^*, u^0, \lambda) \|_{L^{p/(n\wedge m)}} \to 0$ for all $1 \leqslant l \leqslant \sigma(n \wedge m)$. ■

• **Step 3.5.** We compute the limit

$$0 \leqslant \lim_{\lambda \to 0+0} \frac{1}{\lambda} \Big(G\big(x^* + \lambda\, (x^0 - x^*)\, ,\, u^* + \lambda\, (u^0 - u^*)\, ,\, w^* + \lambda\, (w^0 - w^*) + R(w^0, \lambda) - S(u^*, u^0, \lambda) \big) \tag{4.72}$$
$$- G(x^*, u^*, w^*) \Big)$$

$$= \lim_{\lambda\to 0+0} \frac{1}{\lambda} \Big(G\big(x^* + \lambda\,(x^0 - x^*)\,,\, u^* + \lambda\,(u^0 - u^*)\,,\, w^* + \lambda\,(w^0 - w^*) + R(w^0,\lambda) - S(u^*,u^0,\lambda)\big) \qquad (4.73)$$

$$- G\big(x^*\,,\, u^* + \lambda\,(u^0 - u^*)\,,\, w^* + \lambda\,(w^0 - w^*) + R(w^0,\lambda) - S(u^*,u^0,\lambda)\big)\Big)$$

$$+ \lim_{\lambda\to 0+0} \frac{1}{\lambda} \Big(G\big(x^*\,,\, u^* + \lambda\,(u^0 - u^*)\,,\, w^* + \lambda\,(w^0 - w^*) + R(w^0,\lambda) - S(u^*,u^0,\lambda)\big)$$

$$- G\big(x^*\,,\, u^*\,,\, w^* + \lambda\,(w^0 - w^*) + R(w^0,\lambda) - S(u^*,u^0,\lambda)\big)\Big)$$

$$+ \lim_{\lambda\to 0+0} \frac{1}{\lambda} \Big(G\big(x^*\,,\, u^*\,,\, w^* + \lambda\,(w^0 - w^*) + R(w^0,\lambda) - S(u^*,u^0,\lambda)\big) - G(x^*,u^*,w^*)\Big)$$

$$= D_xG(x^*,u^*,w^*)\,(x^0 - x^*) + D_uG(x^*,u^*,w^*)\,(u^0 - u^*) + D_wG(x^*,u^*,w^*)\,(w^0 - w^*) \qquad (4.74)$$

$$= \sum_{i=1}^{n} \int_\Omega \frac{\partial g}{\partial \xi_i}(\,\dots)\,(x_i^0 - x_i^*)\,ds + \sum_{l=1}^{\sigma(1)} \int_\Omega \frac{\partial g}{\partial v_l}(\,\dots)\,(u_l^0 - u_l^*)\,ds \qquad (4.75)$$

$$+ \sum_{l=1}^{\sigma(2)} \int_\Omega \frac{\partial g}{\partial \omega_{2,l}}(\,\dots)\,(w_{2,l}^0 - w_{2,l}^*)\,ds + \,\dots\, + \sum_{l=1}^{\sigma(n\wedge m)} \int_\Omega \frac{\partial g}{\partial \omega_{(n\wedge m),l}}(\,\dots)\,(w_{(n\wedge m),l}^0 - w_{(n\wedge m),l}^*)\,ds$$

as a consequence of Assumption 3.1., 4). Consequently, we get

$$\varrho^0 \;=\; \varepsilon^0 + D_xG(x^*,u^*,w^*)\,(x^0 - x^*) + D_uG(x^*,u^*,w^*)\,(u^0 - u^*) + D_wG(x^*,u^*,w^*)\,(w^0 - w^*) \;\geqslant\; 0\,, \qquad (4.76)$$

and the proof of Proposition 4.6. is complete. ∎

• **Step 4.** *Definition and properties of the sets* C_η. To every $\eta > 0$, we associate the set

$$\mathrm{C}_\eta \;=\; \big\{\, \big(\, \varrho,\, z_1,\, z_2,\, z_3,\, \dots,\, z_{(n\wedge m)}\,\big) \qquad (4.77)$$

$$\in \mathbb{R} \times L^p(\Omega,\mathbb{R}^{nm}) \times L^{p/2}(\Omega,\mathbb{R}^{\sigma(2)}) \times L^{p/3}(\Omega,\mathbb{R}^{\sigma(3)}) \times \,\dots\, \times L^{p/(n\wedge m)}(\Omega,\mathbb{R}^{\sigma(n\wedge m)}) \quad \text{with}$$

$$z_1 = Jx - Jx^* - (u - u^*)\,,\;\; \|\, z_1 \,\|_{L^p} \leqslant \eta\,; \qquad (4.78)$$

$$z_2 = (w_2 - w_2^*) - D_uT_2(u^*)(u - u^*)\,,\;\; \|\, z_2 \,\|_{L^{p/2}} \leqslant \eta\,; \qquad (4.79)$$

$$z_3 = (w_3 - w_3^*) - D_uT_3(u^*)(u - u^*)\,,\;\; \|\, z_3 \,\|_{L^{p/3}} \leqslant \eta\,; \qquad (4.80)$$

$$\vdots$$

$$z_{(n\wedge m)} = (w_{(n\wedge m)} - w_{(n\wedge m)}^*) - D_uT_{(n\wedge m)}(u^*)(u - u^*)\,,\;\; \|\, z_{(n\wedge m)} \,\|_{L^{p/(n\wedge m)}} \leqslant \eta\,; \qquad (4.81)$$

$$x \in W_0^{1,p}(\Omega,\mathbb{R}^n)\,,\;\; u \in \mathrm{U}_0 + \mathrm{K}(\mathfrak{o},\eta) \subset L^p(\Omega,\mathbb{R}^{nm})\,, \qquad (4.82)$$

$$w_2 \in L^{p/2}(\Omega,\mathbb{R}^{\sigma(2)})\,,\;\; w_3 \in L^{p/3}(\Omega,\mathbb{R}^{\sigma(3)})\,,\, \dots\,,\, w_{(n\wedge m)} \in L^{p/(n\wedge m)}(\Omega,\mathbb{R}^{\sigma(n\wedge m)})\,\big\}\,. \qquad (4.83)$$

Proposition 4.9. *For arbitrary $\eta > 0$, the following implication holds:* $\big(\, \varrho,\, z_1,\, z_2,\, z_3,\, \dots,\, z_{(n\wedge m)}\,\big) \in \mathrm{C}_\eta \cap \mathrm{C} \Longrightarrow \varrho \geqslant -K\,\eta$ *with a constant $K > 0$ independent on η.*

Proof. Assume that an element $\big(\, \varrho,\, z_1,\, z_2,\, z_3,\, \dots,\, z_{(n\wedge m)}\,\big)$ belongs to $\mathrm{C}_\eta \cap \mathrm{C}$. Consequently, we find a number $\varepsilon^0 \geqslant 0$ and functions $x^0 \in W_0^{1,p}(\Omega,\mathbb{R}^n)$, $u^0 \in \mathrm{U} \cap \big(\mathrm{U}_0 + \mathrm{K}(\mathfrak{o},\eta)\big) \subset L^p(\Omega,\mathbb{R}^{nm})$, $w_2^0 \in L^{p/2}(\Omega,\mathbb{R}^{\sigma(2)})$, $w_3^0 \in L^{p/3}(\Omega,\mathbb{R}^{\sigma(3)})$, ... , $w_{(n\wedge m)}^0 \in L^{p/(n\wedge m)}(\Omega,\mathbb{R}^{\sigma(n\wedge m)})$ such that

$$\varrho^0 \;=\; \varepsilon^0 + D_xG(x^*,u^*,w^*)(x^0 - x^*) + D_uG(x^*,u^*,w^*)(u^0 - u^*) + D_wG(x^*,u^*,w^*)(w^0 - w^*)\,; \qquad (4.84)$$

$$z_1 \;=\; Jx^0 - Jx^* - (u^0 - u^*)\,; \qquad (4.85)$$

$$z_2 \;=\; (w_2^0 - w_2^*) - D_uT_2(u^*)(u^0 - u^*)\,; \qquad (4.86)$$

$$z_3 \;=\; (w_3^0 - w_3^*) - D_u T_3(u^*)(u^0 - u^*)\,; \tag{4.87}$$

$$\vdots$$

$$z_{(n\wedge m)} \;=\; (w_{(n\wedge m)}^0 - w_{(n\wedge m)}^*) - D_u T_{(n\wedge m)}(u^*)(u^0 - u^*) \quad \text{and} \tag{4.88}$$

$$\|\, z_1 \,\|_{L^p} \leqslant \eta\,,\ \|\, z_2 \,\|_{L^{p/2}} \leqslant \eta\,,\ \|\, z_3 \,\|_{L^{p/3}} \leqslant \eta\,,\ \dots,\ \|\, z_{(n\wedge m)} \,\|_{L^{p/(n\wedge m)}} \leqslant \eta\,. \tag{4.89}$$

Since $u^0 \in \mathrm{U}_0 + \mathrm{K}(\mathfrak{o},\eta)$, we find $\tilde u \in \mathrm{U}_0$ with $\tilde u = J\tilde x$, $\tilde x \in W_0^{1,p}(\Omega,\mathbb{R}^n)$, and $\|\, u^0 - \tilde u \,\| \leqslant \eta$. Thus we get

$$\|\, Jx^0 - J\tilde x \,\|_{L^p} \;=\; \|\, Jx^0 - \tilde u \,\|_{L^p} \;\leqslant\; \|\, Jx^0 - u^0 \,\|_{L^p} + \|\, u^0 - \tilde u \,\|_{L^p} \;\leqslant\; 2\,\eta\,, \tag{4.90}$$

and the Poincaré inequality [20] implies

$$\|\, x^0 - \tilde x \,\|_{W_0^{1,p}} \;\leqslant\; C_1 \,\|\, Jx^0 - J\tilde x \,\|_{L^p} \;\leqslant\; 2\, C_1\, \eta\,. \tag{4.91}$$

Further, we get

$$w_2^0 - w_2^* \;=\; D_u T_2(u^*)(u^0 - u^*) + z_2 \;=\; D_u T_2(u^*)(u^0 - \tilde u) + D_u T_2(u^*)(\tilde u - u^*) + z_2 \quad \Longrightarrow \tag{4.92}$$

$$\big(\, w_2^0 - D_u T_2(u^*)(u^0 - \tilde u) - z_2 \,\big) - w_2^* \;=\; D_u T_2(u^*)(\tilde u - u^*)\,. \tag{4.93}$$

Abbreviating now $w_2^0 - D_u T_2(u^*)(u^0 - \tilde u) - z_2 = \tilde w_2 \in L^{p/2}(\Omega, \mathbb{R}^{\sigma(2)})$, we see that

$$\|\, \tilde w_2 - w_2^0 \,\|_{L^{p/2}} \;\leqslant\; \|\, D_u T_2(u^*) \,\|_{\mathcal{L}(L^p, L^{p/2})} \cdot \|\, u^0 - \tilde u \,\|_{L^p} + \|\, z_2 \,\|_{L^{p/2}} \;\leqslant\; (1 + C_2)\, \eta\,. \tag{4.94}$$

Analogously, we find elements $\tilde w_3 \in L^{p/3}(\Omega, \mathbb{R}^{\sigma(3)})$, ... , $\tilde w_{(n\wedge m)} \in L^{p/(n\wedge m)}(\Omega, \mathbb{R}^{\sigma((n\wedge m))})$ such that

$$\tilde w_3 - w_3^* \;=\; D_u T_3(u^*)(\tilde u - u^*)\,,\ \dots,\ \tilde w_{(n\wedge m)} - w_{(n\wedge m)}^* \;=\; D_u T_{(n\wedge m)}(u^*)(\tilde u - u^*) \quad \text{and} \tag{4.95}$$

$$\|\, \tilde w_3 - w_3^0 \,\|_{L^{p/3}} \;\leqslant\; \|\, D_u T_3(u^*) \,\|_{\mathcal{L}(L^p, L^{p/3})} \cdot \|\, u^0 - \tilde u \,\|_{L^p} + \|\, z_3 \,\|_{L^{p/3}} \;\leqslant\; (1 + C_3)\, \eta\,; \tag{4.96}$$

$$\vdots$$

$$\begin{aligned} \|\, \tilde w_{(n\wedge m)} - w_{(n\wedge m)}^0 \,\|_{L^{p/(n\wedge m)}} \;&\leqslant\; \|\, D_u T_{(n\wedge m)}(u^*) \,\|_{\mathcal{L}(L^p, L^{p/(n\wedge m)})} \cdot \|\, u^0 - \tilde u \,\|_{L^p} + \|\, z_{(n\wedge m)} \,\|_{L^{p/(n\wedge m)}} \\ &\leqslant\; (1 + C_{(n\wedge m)})\, \eta\,. \end{aligned} \tag{4.97}$$

From Proposition 4.6., we conclude further that

$$\varepsilon^0 + D_x G(x^*, u^*, w^*)(\tilde x - x^*) + D_u G(x^*, u^*)(\tilde u - u^*) + \sum_{r=2}^{(n\wedge m)} D_{w_r} G(x^*, u^*, w^*)(\tilde w_r - w_r^*) \geqslant 0\,, \tag{4.98}$$

and we may estimate

$$\varrho^0 \;=\; \varepsilon^0 + D_x G(x^*, u^*, w^*)(x^0 - x^*) + D_u G(x^*, u^*, w^*)(u^0 - u^*) + \sum_{r=2}^{(n\wedge m)} D_{w_r} G(x^*, u^*, w^*)(w_r^0 - w_r^*) \tag{4.99}$$

$$\begin{aligned} =\; &\varepsilon^0 + D_x G(x^*, u^*, w^*)(x^0 - \tilde x) + D_u G(x^*, u^*, w^*)(u^0 - \tilde u) + \sum_{r=2}^{(n\wedge m)} D_{w_r} G(x^*, u^*, w^*)(w_r^0 - \tilde w_r) \\ &+ D_x G(x^*, u^*, w^*)(\tilde x - x^*) + D_u G(x^*, u^*, w^*)(\tilde u - u^*) + \sum_{r=2}^{(n\wedge m)} D_{w_r} G(x^*, u^*, w^*)(\tilde w_r - w_r^*) \end{aligned} \tag{4.100}$$

$$\begin{aligned} \geqslant\; &\varepsilon^0 - \|\, D_x G(x^*, u^*, w^*) \,\| \cdot \|\, x^0 - \tilde x \,\| - \|\, D_u G(x^*, u^*, w^*) \,\| \cdot \|\, u^0 - \tilde u \,\| \\ &- \sum_{r=2}^{(n\wedge m)} \|\, D_{w_r} G(x^*, u^*, w^*) \,\| \cdot \|\, w_r^0 - \tilde w_r \,\| \end{aligned} \tag{4.101}$$

$$\geqslant\; -\Big(\, 2\, C_1 \,\|\, D_x G(x^*, u^*, w^*) \,\| + \|\, D_u G(x^*, u^*, w^*) \,\| + \sum_{r=2}^{(n\wedge m)} (1 + C_r) \,\|\, D_{w_r} G(x^*, u^*, w^*) \,\| \,\Big)\, \eta\,. \tag{4.102}$$

[20] [Adams/Fournier 07] , p. 184, Corollary 6.31.

Consequently, the claimed implication is true with

$$K \;=\; 2\,C_1\,\|\,D_xG(x^*,u^*,w^*)\,\| + \|\,D_uG(x^*,u^*,w^*)\,\| + \sum_{r=2}^{(n\wedge m)} (1+C_r)\,\|\,D_{w_r}G(x^*,u^*,w^*)\,\|\,. \quad \blacksquare \tag{4.103}$$

Proposition 4.10. *The variational set* D *is a subset of* C_α.

Proof. We must convince ourselves that the components of a given element $\big(\varrho,\, z_1,\, z_2,\, z_3,\, \dots,\, z_{(n\wedge m)}\big) \in \mathrm{D}$ admit representations

$$z_1 = Jx^0 - Jx^* - (u^0-u^*)\,; \tag{4.104}$$
$$z_2 = (w_2^0 - w_2^*) - D_uT_2(u^*)(u^0-u^*)\,; \tag{4.105}$$
$$z_3 = (w_3^0 - w_3^*) - D_uT_3(u^*)(u^0-u^*)\,; \tag{4.106}$$
$$\vdots$$
$$z_{(n\wedge m)} = (w_{(n\wedge m)}^0 - w_{(n\wedge m)}^*) - D_uT_{(n\wedge m)}(u^*)(u^0-u^*) \tag{4.107}$$

with functions $x^0 \in W_0^{1,p}(\Omega,\mathbb{R}^n)$, $u^0 \in \mathrm{U}_0 + \mathrm{K}(\mathfrak{o},\alpha) \subset L^p(\Omega,\mathbb{R}^{nm})$, $w_2^0 \in L^{p/2}(\Omega,\mathbb{R}^{\sigma(2)})$, $w_3^0 \in L^{p/3}(\Omega,\mathbb{R}^{\sigma(3)})$, ... , $w^0_{(n\wedge m)} \in L^{p/(n\wedge m)}(\Omega,\mathbb{R}^{\sigma(n\wedge m)})$. Indeed, since $\mathfrak{o} \in \mathrm{U}_0 \subset L^p(\Omega,\mathbb{R}^{nm})$ and $\|\,z_1\,\|_{L^p} \leqslant \alpha$, we may choose $x^0 = \mathfrak{o} \in W_0^{1,p}(\Omega,\mathbb{R}^n)$, $u^0 = \mathfrak{o} + z_1 \in \mathrm{U}_0 + \mathrm{K}(\mathfrak{o},\alpha) \subset L^p(\Omega,\mathbb{R}^{nm})$, $w_2^0 = z_2 + D_uT_2(u^*)z_1 \in L^{p/2}(\Omega,\mathbb{R}^{\sigma(2)})$, $w_3^0 = z_3 + D_uT_3(u^*)z_1 \in L^{p/3}(\Omega,\mathbb{R}^{\sigma(3)})$, ... , $w^0_{(n\wedge m)} = z_{(n\wedge m)} + D_uT_{(n\wedge m)}(u^*)z_1 \in L^{p/(n\wedge m)}(\Omega,\mathbb{R}^{\sigma(n\wedge m)})$, thus obtaining the claimed representation. ∎

• **Step 5.** *Separation of* C *and* D. From Propositions 4.9. and 4.10., we see now that the convex sets C and D are disjoint while int (D) $\neq \emptyset$. Consequently, application of the weak separation theorem [21] yields the existence of a nontrivial linear, continuous functional $(\,\lambda_0,\, y^{(1)},\, y^{(2)},\, y^{(3)},\, \dots,\, y^{(n\wedge m)}) \in \mathbb{R} \times L^{p/(p-1)}(\Omega,\mathbb{R}^{nm}) \times L^{p/(p-2)}(\Omega,\mathbb{R}^{\sigma(2)}) \times L^{p/(p-3)}(\Omega,\mathbb{R}^{\sigma(3)}) \times \dots \times L^{p/(p-(n\wedge m))}(\Omega,\mathbb{R}^{\sigma(n\wedge m)})$, which separates C and D properly. As a result, we obtain the variational inequality

$$\lambda_0\,\varrho' + \langle\, y^{(1)},\, z_1'\,\rangle + \langle\, y^{(2)},\, z_2'\,\rangle + \langle\, y^{(3)},\, z_3'\,\rangle + \dots + \langle\, y^{(n\wedge m)},\, z'_{(n\wedge m)}\,\rangle \tag{4.108}$$
$$\geqslant \lambda_0\,\varrho'' + \langle\, y^{(1)},\, z_1''\,\rangle + \langle\, y^{(2)},\, z_2''\,\rangle + \langle\, y^{(3)},\, z_3''\,\rangle + \dots + \langle\, y^{(n\wedge m)},\, z''_{(n\wedge m)}\,\rangle$$
$$\forall\,\big(\,\varrho',\, z_1',\, z_2',\, z_3',\, \dots,\, z'_{(n\wedge m)}\big) \in \mathrm{C} \quad \forall\,\big(\,\varrho'',\, z_1'',\, z_2'',\, z_3'',\, \dots,\, z''_{(n\wedge m)}\big) \quad \text{with}$$
$$\|\,z_1''\,\|_{L^p} \leqslant \alpha\,,\ \|\,z_2''\,\|_{L^{p/2}} \leqslant \alpha\,,\ \|\,z_3''\,\|_{L^{p/3}} \leqslant \alpha\,, \dots,\ \|\,z''_{(n\wedge m)}\,\|_{L^{p/(n\wedge m)}} \leqslant \alpha \quad \text{and} \tag{4.109}$$
$$\varrho'' < -K\,\big(\,\|\,z_1''\,\|_{L^p} + \|\,z_2''\,\|_{L^{p/2}} + \|\,z_3\,\|_{L^{p/3}} + \dots + \|\,z_{(n\wedge m)}\,\|_{L^{p/(n\wedge m)}}\,\big)\,. \tag{4.110}$$

• **Step 6.** *Derivation of the optimality conditions from the variational inequality* (4.108).

a) Nonnegativity. Inserting $(\,1,\, \mathfrak{o},\, \mathfrak{o},\, \mathfrak{o},\, \dots,\, \mathfrak{o}\,) \in \mathrm{C}$ (generated with $\varepsilon = 1$, $x = x^*$, $u = u^*$ and $w = w^*$) and $(\,-1,\, \mathfrak{o},\, \mathfrak{o},\, \mathfrak{o},\, \dots,\, \mathfrak{o}\,) \in \mathrm{D}$ into the inequality, we get $\lambda_0 \geqslant 0$.

b) Derivation of $(\mathcal{M})$. We insert into the inequality elements of C generated with $\varepsilon = 0$, $x = x^*$ and arbitrary $u \in \mathrm{U}$ as well as with arbitrary $w \in L^{p/2}(\Omega,\mathbb{R}^{\sigma(2)}) \times L^{p/3}(\Omega,\mathbb{R}^{\sigma(3)}) \times \dots \times L^{p/(n\wedge m)}(\Omega,\mathbb{R}^{\sigma(n\wedge m)})$ and $(\,0,\, \mathfrak{o},\, \mathfrak{o},\, \mathfrak{o},\, \dots,\, \mathfrak{o}\,) \in \mathrm{cl}\,(\mathrm{D})$. Then we obtain

$$\lambda_0\,\big(\,G(x^*,u,w) - G(x^*,u^*,w^*)\,\big) - \langle\, y^{(1)},\, u-u^*\,\rangle \tag{4.111}$$
$$+ \sum_{r=2}^{(n\wedge m)} \langle\, y^{(r)},\, w_r - w_r^*\,\rangle - \sum_{r=2}^{(n\wedge m)} \langle\, y^{(r)},\, D_uT_r(u^*)\,(u-u^*)\,\rangle \geqslant 0\,.$$

[21] [Ioffe/Tichomirow 79] , p. 152, Theorem 1.

c) Derivation of ($\mathcal{K}$). Now we insert elements of C generated with $\varepsilon = 0$, $u = u^*$, $w = w^*$ and arbitrary $x \in W_0^{1,p}(\Omega, \mathbb{R}^n)$ and $(0, \mathfrak{o}, \mathfrak{o}, \mathfrak{o}, \ldots, \mathfrak{o}) \in \mathrm{cl}(\mathrm{D})$. This yields

$$\lambda_0 \, D_x G(x^*, u^*, w^*)\,(x - x^*) + \langle\, y^{(1)},\, Jx - Jx^* \,\rangle \;\geqslant\; 0\,. \tag{4.112}$$

Inserting at the same time the element of C generated with $\varepsilon = 0$, $u = u^*$, $w = w^*$ and $(2x^* - x) \in W_0^{1,p}(\Omega, \mathbb{R}^n)$ instead of x, we obtain the reverse inequality

$$\lambda_0 \, D_x G(x^*, u^*, w^*)\,(x - x^*) + \langle\, y^{(1)},\, Jx - Jx^* \,\rangle \;\leqslant\; 0\,, \tag{4.113}$$

and we arrive at ($\mathcal{K}$).

• **Step 7.** *Occurrence of the regular case* $\lambda_0 > 0$. Let us assume, on the contrary, that $\lambda_0 = 0$. Inserting $u = u^*$ into the maximum condition ($\mathcal{M}$), this implies that

$$\sum_{r=2}^{(n\wedge m)} \langle\, y^{(r)},\, w_r - w_r^* \,\rangle \;\geqslant\; 0 \tag{4.114}$$

for all $w \in L^{p/2}(\Omega, \mathbb{R}^{\sigma(2)}) \times L^{p/3}(\Omega, \mathbb{R}^{\sigma(3)}) \times \ldots \times L^{p/(n\wedge m)}(\Omega, \mathbb{R}^{\sigma(n\wedge m)})$ which is only possible if $y^{(2)}$, $y^{(3)}$, ... , $y^{(n\wedge m)} = \mathfrak{o}$. Further, condition ($\mathcal{K}$) reduces to

$$\langle\, y^{(1)}\,,\, Jx \,\rangle \;=\; \langle\, y^{(1)}\,,\, Jx^* \,\rangle \quad \forall\, x \in W_0^{1,p}(\Omega, \mathbb{R}^n)\,, \tag{4.115}$$

and this implies $\langle\, y^{(1)}\,,\, Jx^* \,\rangle = \langle\, y^{(1)}\,,\, u^* \,\rangle = 0$. Within the maximum condition, we obtain

$$-\,\langle\, y^{(1)}\,,\, u - u^* \,\rangle \;=\; -\,\langle\, y^{(1)}\,,\, u \,\rangle \;\geqslant\; 0 \quad \forall\, u \in \mathrm{U}\,. \tag{4.116}$$

Since $\mathfrak{o} \in \mathrm{int}\,(\mathrm{K})$ by assumption, U contains some $L^\infty(\Omega, \mathbb{R}^{nm})$-norm ball V, and we conclude that $\langle\, y^{(1)}\,,\, u \,\rangle = 0$ for all $u \in \mathrm{U} \cap \mathrm{V}$. Consequently, $y^{(1)}$ vanishes on all functions $z \in C_0^\infty(\Omega, \mathbb{R}^{nm}) \cap L^p(\Omega, \mathbb{R}^{nm})$ and thus on the whole space $L^p(\Omega, \mathbb{R}^{nm})$, cf. [Adams/Fournier 07] , p. 38, Corollary 2.30. Summing up, we see that $\lambda_0 = 0$ implies $y^{(1)}$, $y^{(2)}$, $y^{(3)}$, ... , $y^{(n\wedge m)} = \mathfrak{o}$, and we get a contradiction since the separating hyperplane between C and D was described by a nontrivial functional. We obtain $\lambda_0 > 0$, and the proof of Theorem 4.3. is complete. ■

d) Proof of Theorem 4.4.

Proof. The countable subset $\mathrm{K}^0 = \big(\mathrm{K} \cap \mathbb{Q}^{nm}\big) \times \mathbb{Q}^{\sigma(2)} \times \mathbb{Q}^{\sigma(3)} \times \ldots \times \mathbb{Q}^{\sigma(n\wedge m)}$ lies dense in $\mathrm{K} \times \mathbb{R}^{\sigma(2)} \times \mathbb{R}^{\sigma(3)} \times \ldots \times \mathbb{R}^{\sigma(n\wedge m)}$. Let us consider the null sets of the non-Lebesgue points of the integrable functions $g(\,\cdot\,, x^*(\,\cdot\,), u^*(\,\cdot\,), w^*(\,\cdot\,))$, $g(\,\cdot\,, x^*(\,\cdot\,), v^0, \omega^0)$, $\big(v^0 - u^*(\,\cdot\,)\big)^{\mathrm{T}} y^{(1)}(\,\cdot\,)$, $\big(\omega_r^0 - w_r^*(\,\cdot\,)\big)^{\mathrm{T}} y^{(r)}(\,\cdot\,)$, $\nabla_v \,\mathrm{adj}_r(\, u^*(\,\cdot\,)\,)\,\big(v^0 - u^*(\,\cdot\,)\big)^{\mathrm{T}} \cdot y^{(r)}(\,\cdot\,)$, $2 \leqslant r \leqslant (n \wedge m)$ for $(v^0, \omega^0) \in \mathrm{K}^0$. The countable union N of these null sets is still a null set. Since $\Omega \subset \mathbb{R}^m$ is the closure of a strongly Lipschitz domain, $\partial\Omega$ is a null set as well. [22] Let us fix a point $s^0 \in \mathrm{int}\,(\Omega) \setminus \mathrm{N}$ as well as a pair $(v^0, \omega^0) \in \mathrm{K}^0$. Then a closed ball $\mathrm{B} = \mathrm{K}(s^0, \varepsilon)$ with sufficiently small radius $\varepsilon > 0$ is contained in $\mathrm{int}\,(\Omega)$, and the function pair (u, w) with

$$u(s) \;=\; \mathbb{1}_{\mathrm{B}}(s)\,\Big(\frac{\mathrm{Dist}\,(s, \partial \mathrm{B})}{\mathrm{Dist}\,(s^0, \partial B)} \cdot v^0 + \frac{\big(\mathrm{Dist}\,(s^0, \partial B) - \mathrm{Dist}\,(s, \partial B)\big)}{\mathrm{Dist}\,(s^0, \partial B)} \cdot u^*(s)\Big) + \mathbb{1}_{(\Omega\,\setminus\,\mathrm{B})}(s)\, u^*(s)\,; \tag{4.117}$$

$$w(s) \;=\; \mathbb{1}_{\mathrm{B}}(s)\,\Big(\frac{\mathrm{Dist}\,(s, \partial \mathrm{B})}{\mathrm{Dist}\,(s^0, \partial B)} \cdot \omega^0 + \frac{\big(\mathrm{Dist}\,(s^0, \partial B) - \mathrm{Dist}\,(s, \partial B)\big)}{\mathrm{Dist}\,(s^0, \partial B)} \cdot w^*(s)\Big) + \mathbb{1}_{(\Omega\,\setminus\,\mathrm{B})}(s)\, w^*(s) \tag{4.118}$$

[22] [Wagner 06] , p. 122, Lemma 9.2.

belongs to $\mathrm{U} \times L^{p/2}(\Omega, \mathbb{R}^{\sigma(2)}) \times L^{p/3}(\Omega, \mathbb{R}^{\sigma(3)}) \times \ldots \times L^{p/(n\wedge m)}(\Omega, \mathbb{R}^{\sigma(n\wedge m)})$. Since the functions mentioned above are continuous with respect to v and ω and $(u(s^0), w(s^0)) = (v^0, \omega^0)$, s^0 is a Lebesgue point of $g(\,\cdot\,, x^*(\,\cdot\,), u(\,\cdot\,), w(\,\cdot\,))$, $\big(u(\,\cdot\,) - u^*(\,\cdot\,)\big)^{\mathrm{T}} y^{(1)}(\,\cdot\,)$, $\big(w_r(\,\cdot\,) - w_r^*(\,\cdot\,)\big)^{\mathrm{T}} y^{(r)}(\,\cdot\,)$, $\frac{\partial}{\partial r}\,\mathrm{adj}_r(\,u^*(\,\cdot\,)\,)\,\big(u(\,\cdot\,) - u^*(\,\cdot\,)\big)^{\mathrm{T}} \cdot y^{(r)}(\,\cdot\,)$, $2 \leqslant r \leqslant (n \wedge m)$, as well, and we are allowed to form the Lebesgue derivative of $(\mathcal{M})$ at the point s^0 after inserting (u, w) into the inequality.

Consider now a Vitali covering of Ω [23] and specify therein some decreasing sequence $\{\,\Omega^N\,\}$ of closed subsets of $\Omega \cap \mathrm{B}$ with $\bigcap_N \Omega^N = \{\,s^0\,\}$. Together with (u, w), all function pairs (u^N, w^N) with

$$u^N(s) \;=\; \mathbb{1}_{\Omega^N}(s)\,u(s) + \mathbb{1}_{(\Omega\,\setminus\,\Omega^N)}(s)\,u^*(s)\,; \tag{4.119}$$

$$w^N(s) \;=\; \mathbb{1}_{\Omega^N}(s)\,w(s) + \mathbb{1}_{(\Omega\,\setminus\,\Omega^N)}(s)\,w^*(s) \tag{4.120}$$

form admissible controls, and we get

$$\lim_{N\to\infty} \frac{1}{|\,\Omega^N\,|} \int_{\Omega^N} \lambda_0 \Big(g(\,s, x^*(s), u^N(s), w^N(s)\,) - g(\,s, x^*(s), u^*(s), w^*(s)\,)\,\Big)\,ds \tag{4.121}$$

$$- \lim_{N\to\infty} \frac{1}{|\,\Omega^N\,|} \int_{\Omega^N} \big(\,u^N(s) - u^*(s)\,\big)^{\mathrm{T}} y^{(1)}(s)\,ds + \sum_{r=2}^{(n\wedge m)} \lim_{N\to\infty} \frac{1}{|\,\Omega^N\,|} \int_{\Omega^N} \big(\,w_r^N(s) - w_r^*(s)\,\big)^{\mathrm{T}} y^{(r)}(s)\,ds$$

$$- \sum_{r=2}^{(n\wedge m)} \lim_{N\to\infty} \frac{1}{|\,\Omega^N\,|} \int_{\Omega^N} \nabla_v\,\mathrm{adj}_r(\,u^*(s)\,)\,\big(\,u^N(s) - u^*(s)\,\big)^{\mathrm{T}} y^{(r)}(s)\,ds$$

$$= \lambda_0 \Big(g(\,s, x^*(s), v^0, \omega^0\,) - g(\,s, x^*(s), u^*(s), w^*(s)\,)\Big) \;-\; \big(\,v^0 - u^*(s)\,\big)^{\mathrm{T}} y^{(1)}(s) \tag{4.122}$$

$$+ \sum_{r=2}^{(n\wedge m)} \big(\,\omega_r^0 - w_r^*(s)\,\big)^{\mathrm{T}} y^{(r)}(s) \;-\; \sum_{r=2}^{(n\wedge m)} \nabla_v\,\mathrm{adj}_r(\,u^*(s)\,)\,\big(\,v^0 - u^*(s)\,\big)^{\mathrm{T}} y^{(r)}(s) \;\geqslant\; 0\,.$$

This inequality holds for fixed $s^0 \in \mathrm{int}\,(\Omega) \setminus \mathrm{N}$ for arbitrary $(v^0, \omega^0) \in \mathrm{K}^0$. Since its left-hand side is a continuous function of (v, ω), it may be extended to the whole set $\mathrm{K} \times \mathbb{R}^{\sigma(2)} \times \mathbb{R}^{\sigma(3)} \times \ldots \times \mathbb{R}^{\sigma(n\wedge m)}$, and the proof is complete. ∎

e) Remarks and generalizations.

Theorem 4.11. *Let us replace Assumptions 3.1., 3) and 4) by the weaker conditions 3)′ and 4)′ from Remark 3.5. Then Theorems 4.3. and 4.4. remain true provided that the partial derivatives of g satisfy the additional growth conditions*

$$\Big|\,\frac{\partial g}{\partial \xi_i}(s, \xi, v, \omega)\,\Big| \;\leqslant\; A_i(s) \;+\; B_i(\xi, v, \omega) \tag{4.123}$$

$$(\forall)\, s \in \Omega \quad \forall\,(\xi, v, \omega) \in \mathbb{R}^n \times \mathbb{R}^{nm} \times \big(\,\mathbb{R}^{\sigma(2)} \times \mathbb{R}^{\sigma(3)} \times \ldots \times \mathbb{R}^{\sigma(n\wedge m)}\,\big)$$

where $A_i \in L^1(\Omega, \mathbb{R})$ and B_i is measurable and bounded on every bounded subset of $\mathbb{R}^n \times \mathbb{R}^{nm} \times \big(\,\mathbb{R}^{\sigma(2)} \times \mathbb{R}^{\sigma(3)} \times \ldots \times \mathbb{R}^{\sigma(n\wedge m)}\,\big)$, $1 \leqslant i \leqslant n$;

$$\Big|\,\frac{\partial g}{\partial v_l}(s, \xi, v, \omega)\,\Big| \;\leqslant\; A_l^{(1)}(s) \;+\; B_l^{(1)}(\xi, v, \omega) \tag{4.124}$$

$$(\forall)\, s \in \Omega \quad \forall\,(\xi, v, \omega) \in \mathbb{R}^n \times \mathbb{R}^{nm} \times \big(\,\mathbb{R}^{\sigma(2)} \times \mathbb{R}^{\sigma(3)} \times \ldots \times \mathbb{R}^{\sigma(n\wedge m)}\,\big)$$

where $A_l^{(1)} \in L^{p/(p-1)}(\Omega, \mathbb{R})$, $B_l^{(1)}$ is measurable and bounded on every bounded subset of $\mathbb{R}^n \times \mathbb{R}^{nm} \times \big(\,\mathbb{R}^{\sigma(2)} \times \mathbb{R}^{\sigma(3)} \times \ldots \times \mathbb{R}^{\sigma(n\wedge m)}\,\big)$, $1 \leqslant l \leqslant nm$;

$$\Big|\,\frac{\partial g}{\partial \omega_{2,l}}(s, \xi, v, \omega)\,\Big| \;\leqslant\; A_l^{(2)}(s) \;+\; B_l^{(2)}(\xi, v, \omega) \tag{4.125}$$

$$(\forall)\, s \in \Omega \quad \forall\,(\xi, v, \omega) \in \mathbb{R}^n \times \mathbb{R}^{nm} \times \big(\,\mathbb{R}^{\sigma(2)} \times \mathbb{R}^{\sigma(3)} \times \ldots \times \mathbb{R}^{\sigma(n\wedge m)}\,\big)$$

---

[23] [Dunford/Schwartz 88], p. 212, Definition 2.

where $A_l^{(2)} \in L^{p/(p-2)}(\Omega, \mathbb{R})$, $B_l^{(2)}$ *is measurable and bounded on every bounded subset of* $\mathbb{R}^n \times \mathbb{R}^{nm} \times \big(\mathbb{R}^{\sigma(2)} \times \mathbb{R}^{\sigma(3)} \times ... \times \mathbb{R}^{\sigma(n\wedge m)} \big)$, $1 \leqslant l \leqslant \sigma(2)$;

$$\vdots$$

$$\Big| \frac{\partial g}{\partial \omega_{(n\wedge m),l}}(\, s, \xi, v, \omega) \Big| \;\leqslant\; A_l^{(n\wedge m)}(s) \,+\, B_l^{(n\wedge m)}(\xi, v, \omega) \tag{4.126}$$

$$(\forall)\, s \in \Omega \quad \forall\, (\xi, v, \omega) \in \mathbb{R}^n \,\times\, \mathbb{R}^{nm} \times \big(\mathbb{R}^{\sigma(2)} \times \mathbb{R}^{\sigma(3)} \times ... \times \mathbb{R}^{\sigma(n\wedge m)} \big)$$

where $A_l^{(n\wedge m)} \in L^{p/(p-(n\wedge m))}(\Omega, \mathbb{R})$, $B_l^{(n\wedge m)}$ *is measurable and bounded on every bounded subset of* $\mathbb{R}^n \times \mathbb{R}^{nm} \times \big(\mathbb{R}^{\sigma(2)} \times \mathbb{R}^{\sigma(3)} \times ... \times \mathbb{R}^{\sigma(n\wedge m)} \big)$, $1 \leqslant l \leqslant \sigma(n \wedge m)$.

Proof. An inspection of the proof of Theorem 4.3. reveals that conditions (4.123) – (4.126) are sufficient in order to ensure that $D_x G(x^*, u^*, w^*)$, $D_u G(x^*, u^*, w^*)$ and $D_w G(x^*, u^*, w^*)$ act as linear, continuous functionals on the spaces $W_0^{1,p}(\Omega, \mathbb{R}^n)$, $L^p(\Omega, \mathbb{R}^{nm})$ and $L^{p/2}(\Omega, \mathbb{R}^{\sigma(2)}) \times L^{p/3}(\Omega, \mathbb{R}^{\sigma(3)}) \times ... \times L^{p/(n\wedge m)}(\Omega, \mathbb{R}^{\sigma(n\wedge m)}$, respectively. Consequently, the first variation of G can be expressed as in (4.75). ∎

This generalization opens the way to the application of Pontryagin's principle to problems from mathematical image processing where, in general, the objectives depend on image data $I(s)$ being measurable and essentially bounded instead of continuous.

Remark 4.12. Remark 3.4. from above applies accordingly to Theorems 4.3., 4.4. and 4.11. Consequently, only those components of $w \in L^{p/2}(\Omega, \mathbb{R}^{\sigma(2)}) \times L^{p/3}(\Omega, \mathbb{R}^{\sigma(3)}) \times ... \times L^{p/(n\wedge m)}(\Omega, \mathbb{R}^{\sigma(n\wedge m)})$, which appear explicitly within the objective of $(\mathrm{P})_1$, must be paired with multipliers and incorporated into the conditions $(\mathcal{M})$ and $(\mathcal{MP})$, respectively.

Remark 4.13. With obvious adaptations, the proof of Theorem 4.3. applies to [Wagner 09] , p. 549, Theorem 2.2. as well. Consequently, the error occuring in the proof of this theorem ibid., p. 552, Step 3, can be completely removed. Analogously, the proof of Theorem 4.4. applies to ibid., p. 550, Theorem 2.3., thus fixing an error in the proof of this theorem ibid., p. 553, (33).

5. Application to hyperelastic image registration.

a) Unimodal image registration.

Assume that on a two- or three-dimensional domain $\Omega \subset \mathbb{R}^m$, $m \in \{2, 3\}$, two greyscale images are given, which will be identified with at least measurable functions I_0, $I_1\colon\ \Omega \to [\,0\,,\,1\,]$. Considering I_0 as reference image, one searches for a deformation field $x\colon\ \Omega \to \mathbb{R}^m$ satisfying the condition $I_1(s - x(s)) \approx I_0(s)$, thus modifying the template image I_1 such that it matches the reference image I_0 in a best possible way. In this abstract formulation of the registration problem, the single assumption is required that there is an overall correlation between the greyscale intensity distributions as well as the geometrical properties of the template and reference image. For the practical determination of a possible deformation field x as well as for a reliable interpretation of the result, more information about the pictured objects and their motion behaviour is needed. [24)]

In numerous situations, a reasonable approach to unimodal registration is to attribute the changes in I_1 with respect to the reference image I_0 to an *elastic deformation* of the pictured objects. This is particularly true

---

24) A detailed introduction to the registration problem may be found in [Hintermüller/Keeling 09] , [Modersitzki 04] and [Modersitzki 09] .

for the imaging of living tissue, which behaves according to hyperelastic material laws. [25] Consequently, a large part of the literature is concerned with variational or PDE methods where x is sought as a linear-elastic [26] or hyperelastic deformation. [27] In the latter case, the problems involve polyconvex stored-energy functions. [28]

The interest in an optimal control access to the elastic registration problem is caused by the fact that the validity of the underlying elasticity models depends crucially on the uniform boundedness of the maximal shear stress generated by the deformation x. [29] Consequently, it is advisable to incorporate restrictions for the partial derivatives of x into the given variational models. In the present paper, we confine ourselves to convex restriction sets. However, in a forthcoming publication we will extend our approach to polyconvex control restrictions as $0 < R_0 \leqslant \det(J(x)) \leqslant R_1 < \infty$. In the following, we reformulate a two-dimensional registration problem within the framework of optimal control and provide the necessary optimality conditions for the problem.

b) A two-dimensional registration problem with polyconvex regularizer.

Let us consider the following two-dimensional image registration problem with polyconvex regularizer: [30]

$$(\mathrm{R})_2: \quad F(x) = \int_\Omega \Big(I_1(\, s - x(s)\,) - I_0(s) \Big)^2 ds \, + \, \mu \cdot \int_\Omega \Big(c_1 \, \| \, Jx(s) \, \|^{\,p} + c_2 \left(\det\left(E_2 - Jx(s) \right) - 1 \right)^2 \Big) \, ds \, \longrightarrow \, \inf! \, ; \tag{5.1}$$

$$x \in W_0^{1,p}(\Omega, \mathbb{R}^2)\,; \; Jx(s) \in \mathrm{K} \subset \mathbb{R}^{2\times 2} \;\; (\forall)\, s \in \Omega\,. \tag{5.2}$$

As discussed in [Burger/Modersitzki/Ruthotto 13] and [Droske/Rumpf 04], the objective can be regarded as a stored-energy functional, which is connected with a generic hyperelastic model. We assume that $4 \leqslant p < \infty$, μ, c_1, $c_2 > 0$. The image data I_0 and I_1 belong to $L^\infty(\Omega, \mathbb{R})$ and $C_0^1(\Omega, \mathbb{R})$, respectively. $\mathrm{K} \subset \mathbb{R}^{2\times 2}$ is a convex body with $\mathfrak{o} \in \mathrm{int}\,(\mathrm{K})$. We use the matrix norm $\| \left(\begin{smallmatrix} v_1 & v_2 \\ v_3 & v_4 \end{smallmatrix}\right) \|^p = |\, v_1 \,|^p + |\, v_2 \,|^p + |\, v_3 \,|^p + |\, v_4 \,|^p$, which is continuously differentiable with respect to its arguments since $p \geqslant 4$. E_2 denotes the $(2, 2)$-unit matrix.

In [Wagner 10] and [Wagner 12], after an appropriate approximation of the data term, a direct method was employed for the numerical solution of this problem. In [Angelov 11] and [Angelov/Wagner 12], p. 5 f., the problem $(\mathrm{R})_2$ was incorporated into a larger scheme for hyperelastic registration of multimodal image data. Applying Theorems 4.1. and 4.2. to $(\mathrm{R})_2$, we obtain the following set of necessary optimality conditions:

Proposition 5.1. (Pontryagin's principle for $(\mathrm{R})_2$ **)** *Consider* $(\mathrm{R})_2$ *under the analytical assumptions mentioned above. If* (x^*, u^*) *is a global minimizer of* $(\mathrm{R})_2$ *then there exist multipliers* $\lambda_0 > 0$, $y^{(1)} \in$

---

[25] See e. g. [Ogden 03].

[26] We refer e. g. to [Fischer/Modersitzki 03], [Haber/Modersitzki 04], [Henn/Witsch 00], [Henn/Witsch 01] and [Modersitzki 04], pp. 77 ff.

[27] See e.g. [Burger/Modersitzki/Ruthotto 13], [Droske/Rumpf 04], [Droske/Rumpf 07] and [Le Guyader/Vese 09].

[28] Examples may be found in [Balzani/Neff/Schröder/Holzapfel 06].

[29] This is even true for living tissue, cf. [Gasser/Holzapfel 02], p. 340 f., and the literature cited therein.

[30] Sligthly modified from [Wagner 11], p. 218, (4.15), and [Wagner 10], p. 5, (2.16) − (2.19). Note that the reference points within the regularization term must be chosen in accordance with the deviation of $s - x(s)$ from the identity.

$L^{p/(p-1)}(\Omega,\mathbb{R}^4)$ *and* $y^{(2)} \in L^{p/(p-2)}(\Omega,\mathbb{R})$ *such that the following conditions are satisfied:*

$$(\mathcal{MP})_1 \quad \lambda_0\,\mu\,c_1 \sum_{i=1}^{2}\sum_{j=1}^{2}\Big(\,|\,v_{ij}\,|^p - |\,u^*_{ij}(s)\,|^p\,\Big) + \lambda_0\,\mu\,c_2\Big(\big(\,v_{11}+v_{22}\,\big)^2 - \big(\,u^*_{11}(s)+u^*_{22}(s)\,\big)^2 \tag{5.3}$$

$$+\ 2\,\det u^*(s)\,\big(\,v_{11}+v_{22}-u^*_{11}(s)-u^*_{22}(s)\,\big)\Big)$$

$$-\sum_{i=1}^{2}\sum_{j=1}^{2}\Big(\,y^{(1)}_{ij}(s)+\frac{\partial}{\partial v_{ij}}\det\big(\,u^*(s)\,\big)\,y^{(2)}(s)\Big)\big(\,v_{ij}-u^*_{ij}(s)\,\big) \geqslant 0 \quad (\forall)\,s\in\Omega \quad \forall\,v\in\mathrm{K}\,;$$

$$(\mathcal{MP})_2 \quad \lambda_0\,\mu\,c_2\Big(\big(\,\omega_2\,\big)^2 - \big(\,\det u^*(s)\,\big)^2 - 2\,\big(\,u^*_{11}(s)+u^*_{22}(s)\,\big)\big(\,\omega_2-\det u^*(s)\,\big)\Big) \tag{5.4}$$

$$+\big(\,\omega_2-\det u^*(s)\,\big)\,y^{(2)}(s) \geqslant 0 \quad (\forall)\,s\in\Omega \quad \forall\,\omega_2\in\mathbb{R}\,;$$

$$(\mathcal{K}) \quad -\lambda_0\int_\Omega\Big(\frac{\partial I_1}{\partial s_1}(\,s-x^*(s)\,)\,\big(\,x_1(s)-x^*_1(s)\,\big)+\frac{\partial I_1}{\partial s_2}(\,s-x^*(s)\,)\,\big(\,x_2(s)-x^*_2(s)\,\big)\Big)\,ds \tag{5.5}$$

$$+\sum_{i=1}^{2}\sum_{j=1}^{2}\int_\Omega\Big(\frac{\partial x_i}{\partial s_j}(s)-\frac{\partial x^*_i}{\partial s_j}(s)\Big)\,y^{(1)}_{ij}(s)\,ds \;=\; 0 \quad \forall\,x\in W^{1,p}_0(\Omega,\mathbb{R}^2)\,.$$

Proof. In order to apply Theorems 4.11., 4.1. and 4.2. to $(\mathrm{R})_2$, we must to verify that the data of the problem satisfy assumptions 3)′ and 4)′ from Remark 3.5. as well as the growth conditions (4.123) – (4.125). Obviously, assumption 3)′ from Remark 3.5. is satisfied. For the polyconvex integrand $f(s,\xi,v)$, we choose the convex representative $g\colon\ \Omega\times\mathbb{R}^2\times\mathbb{R}^4\times\mathbb{R}\to\mathbb{R}$ defined as

$$g(s,\xi,v,\omega_2) \;=\; \big(\,I_1(s-\xi)-I_0(s)\,\big)^2 + \mu\,c_1\sum_{i=1}^{2}\sum_{j=1}^{2}|\,v_{ij}\,|^p + \mu\,c_2\,\big(\,\omega_2-v_{11}-v_{22}\,\big)^2 \tag{5.6}$$

with the partial derivatives

$$\frac{\partial g}{\partial \xi_i}(s,\xi,v,\omega_2) \;=\; -2\,\big(\,I_1(s-\xi)-I_0(s)\,\big)\,\frac{\partial I_1}{\partial s_i}(s-\xi)\,,\ 1\leqslant i\leqslant 2\,; \tag{5.7}$$

$$\frac{\partial g}{\partial v_{11}}(s,\xi,v,\omega_2) \;=\; p\,\mu\,c_1\,|\,v_{11}\,|^{p-1} - 2\,\mu\,c_2\,\big(\,\omega_2-v_{11}-v_{22}\,\big)\,; \tag{5.8}$$

$$\frac{\partial g}{\partial v_{12}}(s,\xi,v,\omega_2) \;=\; p\,\mu\,c_1\,|\,v_{12}\,|^{p-1}\,; \tag{5.9}$$

$$\frac{\partial g}{\partial v_{21}}(s,\xi,v,\omega_2) \;=\; p\,\mu\,c_1\,|\,v_{21}\,|^{p-1}\,; \tag{5.10}$$

$$\frac{\partial g}{\partial v_{22}}(s,\xi,v,\omega_2) \;=\; p\,\mu\,c_1\,|\,v_{22}\,|^{p-1} - 2\,\mu\,c_2\,\big(\,\omega_2-v_{11}-v_{22}\,\big)\,; \tag{5.11}$$

$$\frac{\partial g}{\partial \omega_2}(s,\xi,v,\omega_2) \;=\; 2\,\mu\,c_2\,\big(\,\omega_2-v_{11}-v_{22}\,\big)\,. \tag{5.12}$$

Let us confirm the growth condition (3.5), thus establishing assumption 4)′ from Remark 3.5. Since $I_0(s)$ is essentially bounded on Ω and $I_1(s)$, after extension by zero to $\mathbb{R}^2\setminus\Omega$, is bounded on $\mathbb{R}^2$, we get for almost all $s\in\Omega$ and for all $(\xi,v,\omega_2)\in\mathbb{R}^2\times\mathbb{R}^4\times\mathbb{R}$ (5.13)

$$|\,g(s,\xi,v,\omega_2)\,| \;\leqslant\; C_1\Big(\,|\,I_0(s)\,|^2+|\,I_1(s-\xi)\,|^2\Big) + \mu\,c_1\sum_{i=1}^{2}\sum_{j=1}^{2}|\,v_{ij}\,|^p + \mu\,c_2\,C_2\Big(\,|\,v_{11}\,|^2+|\,v_{22}\,|^2+|\,\omega_2\,|^2\Big)$$

$$\leqslant\; 2\,C_3 + 4\,\mu\,c_1|\,v\,|^p + \mu\,c_2\,C_2\,\big(\,2\,|\,v\,|^2+|\,\omega_2\,|^2\,\big) \;\leqslant\; 2\,C_3 + C_4\,\big(\,1+|\,v\,|^p+|\,\omega_2\,|^{p/2}\,\big) \tag{5.14}$$

since $p\geqslant 4$. Thus (3.5) is satisfied with $A_0(s)\equiv C_3$ and $B_0(\xi)\equiv C_3$. Further, we have

$$|\,\frac{\partial g}{\partial \xi_i}(s,\xi,v,\omega_2)\,| \;\leqslant\; 2\,\big(\,|\,I_1(s-\xi)\,|+|\,I_0(s)\,|\,\big)\,\|\,I_1\,\|_{C^1} \;\leqslant\; 2\,C_5\,\|\,I_1\,\|_{C^1}\,,\ 1\leqslant i\leqslant 2\,, \tag{5.15}$$

and (4.123) is satisfied with $A_i(s) \equiv C_5 \parallel I_1 \parallel_{C^1}$ and $B_i(\xi, v, \omega_2) \equiv C_5 \parallel I_1 \parallel_{C^1}$, $1 \leqslant i \leqslant 2$. Concerning (4.124) and (4.125), we see that the right-hand sides in the inequalities

$$\Big| \frac{\partial g}{\partial v_{11}}(s, \xi, v, \omega_2) \Big| \leqslant p \, \mu \, c_1 \, | \, v_{11} \, |^{p-1} + 2 \, \mu \, c_2 \, \big(2 \, | \, v \, | + | \, \omega_2 \, | \big) \, , \tag{5.16}$$

$$\Big| \frac{\partial g}{\partial v_{12}}(s, \xi, v, \omega_2) \Big| = p \, \mu \, c_1 \, | \, v_{12} \, |^{p-1} \, , \tag{5.17}$$

$$\Big| \frac{\partial g}{\partial \omega_2}(s, \xi, v, \omega_2) \Big| \leqslant 2 \, \mu \, c_2 \, \big(2 \, | \, v \, | + | \, \omega_2 \, | \big) \tag{5.18}$$

are measurable and bounded on bounded subsets of $\mathbb{R}^2 \times \mathbb{R}^4 \times \mathbb{R}$. The derivatives $\partial g(s, \xi, v, \omega_2)/\partial v_{21}$ and $\partial g(s, \xi, v, \omega_2)/\partial v_{22}$ can be estimated in analogous way. Consequently, (4.124) and (4.125) hold with $A_l^{(1)}(s) \equiv 0$, $A^{(2)}(s) \equiv 0$ and $B_l^{(1)}(\xi, v, \omega_2)$, $1 \leqslant l \leqslant 4$ and $B^{(2)}(\xi, v, \omega_2)$ as given through (5.16) – (5.18). Consequently, for a given global minimizer (x^*, u^*) of $(\mathrm{R})_2$, the necessary optimality conditions take the claimed form. ∎

Acknowledgement.

The present work has been supported within the project "Relaxation theorems and necessary optimality conditions for semiconvex multidimensional control problems" by the German Research Council.

References.

1. [ADAMS/FOURNIER 07] Adams, R. A.; Fournier, J. J. F.: *Sobolev Spaces.* Academic Press / Elsevier; Amsterdam etc. 2007, 2nd ed.
2. [ANGELOV 11] Angelov, A.: *Multimodale Bildregistrierung durch elastisches Matching von Kantenskizzen.* Diploma thesis; University of Münster 2011
3. [ANGELOV/WAGNER 12] Angelov, A.; Wagner, M.: *Multimodal image registration by elastic matching of edge sketches via optimal control.* University of Leipzig, Institute of Mathematics, Preprint No. 04/2012 (to appear: J. Ind. Manag. Optim.)
4. [BALZANI/NEFF/SCHRÖDER/HOLZAPFEL 06] Balzani, D.; Neff, P.; Schröder, J.; Holzapfel, G. A.: *A polyconvex framework for soft biological tissues. Adjustment to experimental data.* Int. J. of Solids and Structures **43** (2006), 6052 – 6070
5. [BEVAN 03] Bevan, J.: *An example of a $C^{1,1}$ polyconvex function with no differentiable convex representative.* C. R. Acad. Sci. Paris Sér. I Math. **336** (2003), 11 – 14
6. [BEVAN 06] Bevan, J.: *On convex representatives of polyconvex functions.* Proc. Roy. Soc. Edinburgh Sect. A **126** (2006), 23 – 51
7. [BURGER/MODERSITZKI/RUTHOTTO 13] Burger, M.; Modersitzki, J.; Ruthotto, L.: *A hyperelastic regularization energy for image registration.* SIAM J. Sci. Comput. **35** (2013), B132 – B148
8. [DACOROGNA 08] Dacorogna, B.: *Direct Methods in the Calculus of Variations.* Springer; New York etc. 2008, 2nd ed.
9. [DROSKE/RUMPF 04] Droske, M.; Rumpf, M.: *A variational approach to nonrigid morphological image registration.* SIAM J. Appl. Math. **64** (2004), 668 – 687
10. [DROSKE/RUMPF 07] Droske, M.; Rumpf, M.: *Multiscale joint segmentation and registration of image morphology.* IEEE Trans. Pattern Recognition Machine Intelligence **29** (2007), 2181 – 2194
11. [DUNFORD/SCHWARTZ 88] Dunford, N.; Schwartz, J. T.: *Linear Operators. Part I: General Theory.* Wiley-Interscience; New York etc. 1988

12. [Eneya/Bosse/Griewank 13] Eneya, L.; Bosse, T.; Griewank, A.: *A method for pointwise evaluation of polyconvex envelopes.* Afr. Mat. **24** (2013), 1 – 24

13. [Evans/Gariepy 92] Evans, L. C.; Gariepy, R. F.: *Measure Theory and Fine Properties of Functions.* CRC Press; Boca Raton etc. 1992

14. [Fischer/Modersitzki 03] Fischer, B.; Modersitzki, J.: *Curvature based image registration.* J. Math. Imaging Vision **18** (2003), 81 – 85

15. [Gasser/Holzapfel 02] Gasser, T. C.; Holzapfel, G. H.: *A rate-independent elastoplastic constitutive model for biological fiber-reinforced composites at finite strains: continuum basis, algorithmic formulation and finite element implementation.* Computational Mechanics **29** (2002), 340 – 360

16. [Ginsburg/Ioffe 96] Ginsburg, B.; Ioffe, A. D.: *The maximum principle in optimal control of systems governed by semilinear equations.* In: Mordukhovich, B. S.; Sussmann, H. J. (Eds.): Nonsmooth Analysis and Geometric Methods in Deterministic Optimal Control. Springer; New York etc. 1996 (IMA Volumes in Mathematics and its Applications 78), 81 – 110

17. [Haber/Modersitzki 04] Haber, E.; Modersitzki, J.: *Numerical methods for volume preserving image registration.* Inverse Problems **20** (2004), 1621 – 1638

18. [Henn/Witsch 00] Henn, S.; Witsch, K.: *A multigrid approach for minimizing a nonlinear functional for digital image matching.* Computing **64** (2000), 339 – 348

19. [Henn/Witsch 01] Henn, S.; Witsch, K.: *Iterative multigrid regularization techniques for image matching.* SIAM J. Sci. Comput. **23** (2001), 1077 – 1093

20. [Hintermüller/Keeling 09] Hintermüller, M.; Keeling, S. L.: *Image registration and segmentation based on energy minimization.* In: Pardalos, P. M.; Romeijn, H. E. (Eds.): Handbook of Optimization in Medicine. Springer; New York 2009, 213 – 252

21. [Ioffe/Tichomirow 79] Ioffe, A. D.; Tichomirow, V. M.: *Theorie der Extremalaufgaben.* VEB Deutscher Verlag der Wissenschaften; Berlin 1979

22. [Ito/Kunisch 08] Ito, K.; Kunisch, K.: *Lagrange Multiplier Approach to Variational Problems and Applications.* SIAM; Philadelphia 2008

23. [Klee 55] Klee, V. L.: *Separation properties of convex cones.* Proc. Am. Math. Soc. **6** (1955), 313 – 318

24. [Kunisch/Vexler 07] Kunisch, K.; Vexler, B.: *Optimal vortex reduction for instationary flows based on translation invariant cost functionals.* SIAM J. Control Optim. **46** (2007), 1368 – 1397

25. [Le Guyader/Vese 09] Le Guyader, C.; Vese, L.: *A combined segmentation and registration framework with a nonlinear elasticity smoother.* In: Tai, X.-C.; Mørken, K.; Lysaker, M.; Lie, K.-A. (Eds.): Scale Space and Variational Methods in Computer Vision, Second International Conference, SSVM 2009, Voss, Norway, June 1-5, 2009. Proceedings. Springer; Berlin - Heidelberg 2009 (LNCS 5567), 600 – 611

26. [Lubkoll/Schiela/Weiser 12] Lubkoll, L.; Schiela, A.; Weiser, M.: *An optimal control problem in polyconvex hyperelasticity.* Konrad-Zuse-Zentrum für Informationstechnik Berlin, ZIB-Report 12–08 (2012)

27. [Modersitzki 04] Modersitzki, J.: *Numerical Methods for Image Registration.* Oxford University Press; Oxford 2004

28. [Modersitzki 09] Modersitzki, J.: *FAIR. Flexible Algorithms for Image Registration.* SIAM; Philadelphia 2009

29. [Ogden 03] Ogden, R. W.: *Nonlinear elasticity, anisotropy, material stability and residual stresses in soft tissue.* In: Holzapfel, G. A.; Ogden, R. W. (Eds.): Biomechanics of Soft Tissue in Cardiovascular Systems. Springer; Wien etc. 2003, 65 – 108

30. [Wagner 06] Wagner, M.: *Mehrdimensionale Steuerungsprobleme mit quasikonvexen Integranden.* Habilitation thesis, BTU Cottbus; Cottbus 2006

31. [Wagner 09] Wagner, M.: *Pontryagin's maximum principle for multidimensional control problems in image processing.* J. Optim. Theory Appl. **140** (2009), 543 – 576

32. [Wagner 10] Wagner, M.: *Elastic image registration in presence of polyconvex constraints.* Karl-Franzens-Universität Graz, SFB-Report No. 2010-033. Status: accepted (Proceedings of the International Workshop on Optimal Control in Image Processing, Heidelberg, Germany, May 31 - June 1, 2010)

33. [Wagner 11] Wagner, M.: *Quasiconvex relaxation of multidimensional control problems with integrands* $f(t, \xi, v)$. ESAIM: Control, Optimisation and Calculus of Variations **17** (2011), 190 – 221

34. [Wagner 12] Wagner, M.: *A direct method for the solution of an optimal control problem arising from image registration.* Numerical Algebra, Control and Optimization **2** (2012), 487 – 510

Last modification: 12. 07. 2013

Author's address. *Marcus Wagner:* University of Leipzig, Department of Mathematics, P. O. B. 10 09 20, D-04009 Leipzig, Germany. Homepage / e-mail: `www.thecitytocome.de` / `marcus.wagner@math.uni-leipzig.de`