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Counterexamples to Elliptic Regularity and Convex Integration

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Dedicated to Yuriĭ Reshetnyak on the occasion his 75th birthday

Abstract. The purpose of this article is to give a survey of recent results on the construction of elliptic equations and systems with critical regularity properties. The constructions are based on the method of convex integration, combined with a careful analysis of oscillations in the spirit of compensated compactness. Our aim is to emphasize the approach which separates the analysis of oscillations from the actual constructions via convex integration, in order to pinpoint the extent to which the structural assumptions in the equations are responsible for the loss of regularity.

1. Introduction

This article is an extended version of the plenary talk "Convex Integration for Elliptic Systems", presented at the International School-Conference on Analysis and Geometry dedicated to the 75th anniversary of Yu. G. Reshetnyak in August–September 2004. The aim is to present a survey of results on the construction of counterexamples to regularity in elliptic equations and systems. The method of construction is based on an analysis of oscillations combined with convex integration.

In recent years there has been growing interest in oscillation phenomena in nonlinear systems of partial differential equations. The systematic analysis of oscillations compatible with systems of partial differential equations finds it's origin in the pioneering work of F. Murat and L. Tartar [**32**] on *compensated compactness*. In the setting of compensated compactness compatible oscillations are described in terms of Young measures, and the aim is to understand the effect of the geometry (the nonlinearity) of the equations on the presence and nature of oscillations. In particular one can often make use of the explicit nonlinear structures in the equations to deduce compactness of families of solutions, and in turn this can lead to existence and regularity theorems.

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On the other hand oscillations naturally arise in nonlinear systems related to geometric problems, such as isometric immersions. The celebrated work of J. Nash and N. Kuiper [**24, 19**] and the far-reaching generalizations by M. Gromov [**15**] lead to a method of constructing solutions to such systems, called *convex integration*, which makes explicit use of the compatible one-dimensional oscillations. Thus convex integration can be seen as a way to recover existence in situations with lack of compactness. A crucial difference, however, is that the solutions constructed via convex integration have usually very low regularity properties.

One would expect at first sight that systems to which convex integration applies are fundamentally different from those associated with well-posed problems such as elliptic equations and systems. However, if one views a whole class of elliptic problems – given by certain structural hypothesis – as one single under-determined system, this point of view turns out to be surprisingly fruitful in studying regularity questions. In particular questions like "Is a certain structural assumption on the coefficients of the equation sufficient to guarantee certain regularity of the solutions?". If the structural assumption is too weak, one often succeeds in producing counterexamples to the required regularity via the method of convex integration, and this method turns out to be very general.

The first such application of convex integration is due to S. Müller and V. Šverák in [**23**] where they construct Lipschitz but nowhere C^1 weak solutions to Euler-Lagrange systems associated with smooth, strongly quasiconvex functionals. The purpose of this paper is to explain the method of Müller and Šverák on the extension of their result to polyconvex functionals as well as to outline another recent application of convex integration to the L^p theory of elliptic equations in the plane in a reasonably self-contained manner. A more general survey of related results regarding convex integration, including a list of open problems, can be found in [**17**].

2. Differential inclusions and laminates

The starting point in our approach is the differential inclusion

$$Du(x) \in E, \tag{2.1}$$

where $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ and $E \subset \mathbb{R}^{m\times n}$ is a prescribed set of matrices. In [**15**] Gromov developed the method of convex integration for constructing C^1 and Lipschitz solutions to problems of the type (2.1) arising from geometric problems. At the same time, motivated in particular by mathematical models of microstructure [**8, 11**], there was increasing interest in differential inclusions with sets E to which Gromov's original approach does not apply. This lead to extensions of the existence theory of (2.1) by several authors, among others S. Müller and V. Šverák [**23**], B. Dacorogna and P. Marcellini [**12**], B. Kirchheim [**16**] and M. Sychev [**30**]. In this section we sketch the main ideas, following the approach of Müller and Šverák.

A central notion is that of *rank-one convexity*. A function $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ is said to be rank-one convex if $t \mapsto f(A + tB)$ is convex whenever $\operatorname{rank} B = 1$. The rank-one convex hull of bounded sets $E \subset \mathbb{R}^{m\times n}$ is defined by separation with rank-one convex functions, as follows. If E is compact, we define

$$E^{rc} = \big\{A \in \mathbb{R}^{m\times n} : \, f(A) \leqq \sup_E f \quad \text{for all rank-one convex } f\colon \mathbb{R}^{m\times n} \to \mathbb{R}\big\},$$

and for general bounded sets $U \subset \mathbb{R}^{m\times n}$

$$U^{rc} = \bigcup_{E\subset U \text{ compact}} E^{rc}.$$

A general result in the theory of convex integration is that the differential inclusion (2.1) admits many nontrivial solutions if E^{rc} is large. More precisely we have (see Theorem 3.1 in [**23**])

Theorem 2.1. *Let $U \subset \mathbb{R}^{m\times n}$ be a bounded open set and let $A \in U^{rc}$. For any open domain $\Omega \subset \mathbb{R}^n$ there exists a piecewise affine Lipschitz map $u\colon \Omega \to \mathbb{R}^m$ such that $u(x) = Ax$ on $\partial\Omega$ and $Du(x) \in U$ a.e. in Ω.*

We say that a Lipschitz mapping u is *piecewise affine* if there exists a decomposition of Ω into countable pairwise disjoint open sets Ω_i with $|\partial\Omega_i| = 0$ such that u is affine on each subset Ω_i and the union of the Ω_i has full measure.

In order to get solutions with critical regularity properties, one needs information on the gradient distribution of the solutions constructed in Theorem 2.1. This requires the notion of *laminates*. A probability measure ν on the space of $m \times n$ matrices is a laminate if

$$\langle \nu, f\rangle \geqq f(\overline{\nu}) \quad \text{for all rank-one convex } f\colon \mathbb{R}^{m\times n} \to \mathbb{R},$$

where $\overline{\nu}$ denotes the barycenter of the measure ν. It follows directly from this definition that laminates are closed under weak* convergence (in the space of Radon measures) and closed under *splitting*: in other words if ν is a laminate of the form $\nu = \lambda\delta_A + (1-\lambda)\tilde{\nu}$, and μ is a laminate with barycenter $\overline{\mu} = A$, then the probability measure

$$\lambda\mu + (1-\lambda)\tilde{\nu} \tag{2.2}$$

is also a laminate. Also, it follows from the classical Jensen's inequality that probability measures of the form

$$\lambda\delta_B + (1-\lambda)\delta_C \quad \text{with rank}\,(B - C) = 1 \tag{2.3}$$

are laminates. Combining these two observations leads to the class of *laminates of finite order*: measures that can be obtained from Dirac masses by splitting a finite number of times using the laminates of the form (2.3). In fact a Hahn-Banach separation argument shows that laminates of finite order are weak*-dense in the class of laminates (see [**26, 23**]):

Theorem 2.2. *Let ν be a laminate and let $U \subset \mathbb{R}^{m\times n}$ be an open set such that $\operatorname{spt}\nu \subset U$. There exists a sequence of laminates of finite order ν_j with support $\operatorname{spt}\nu_j \subset U$ and barycenter $\overline{\nu_j} = \overline{\nu}$ such that the ν_j converge weakly* to ν.*

Now we come to the main building block in the theory.

Proposition 2.3. *Let ν be a laminate supported on a finite set $\{A_1, \dots, A_N\}$, with barycenter $\overline{\nu} = A$. Moreover let $\alpha \in [0, 1)$, $\delta > 0$ and $0 < r < 1/2 \min|A_i - A_j|$.*

For every bounded domain $\Omega \subset \mathbb{R}^n$ there exists a piecewise affine Lipschitz mapping $u\colon \Omega \to \mathbb{R}^m$ such that $u(x) = Ax$ on $\partial\Omega$, $[u - A]_{C^\alpha(\overline{\Omega})} < \delta$ and

$$|\{x \in \Omega : |Du(x) - A_i| < r\}| = \nu(A_i)|\Omega| \quad \textit{for all } i. \tag{2.4}$$

This result is essentially the content of the remark preceding Section 3.3 in [**23**]. For the convenience of the reader we give the proof in detail in the Appendix.

Proposition 2.3 can be rephrased as follows. For a (Lipschitz) mapping $u\colon \Omega \to \mathbb{R}^m$ let μ_u be the push-forward of Lebesgue measure under the gradient mapping, i.e.

$$\mu_u(G) = \frac{|\{x \in \Omega : \, Du(x) \in G\}|}{|\Omega|},$$

and let λ_r be the restriction of Lebesgue measure (on $\mathbb{R}^{m\times n}$) to $B_r(0)$, i.e.

$$\lambda_r(G) = |G \cap B_r(0)|$$

for Borel sets $G \subset \mathbb{R}^{m\times n}$. Then Proposition 2.3 says that if ν is a finitely supported laminate, then for any small $r > 0$ and any domain Ω there exists a Lipschitz mapping $u\colon \Omega \to \mathbb{R}^m$ with

$$\mu_u = \lambda_r * \nu.$$

This convolution can be seen as a "spreading" of the measure ν. In this sense laminates ν with

$$\operatorname{spt}\nu \subset E \quad \text{and} \quad \overline{\nu} = A \tag{2.5}$$

can be seen as generalized approximate solutions to the inclusion problem

$$Du(x) \in E \text{ a.e. } \Omega \quad \text{and} \quad u(x) = Ax \text{ on } \partial\Omega. \tag{2.6}$$

Also, since laminates correspond to solutions with linear boundary values, they can be seen as representing oscillations compatible with the differential inclusion.

Notice that although μ_u contains no information about the spatial distribution of values of the gradient Du, it does contain certain information relating to the regularity of u, for example

$$\frac{1}{|\Omega|}\int_\Omega |Du|^p dx = p\int_0^\infty \mu_u\{\xi \in \mathbb{R}^{m\times n} : \, |\xi| > t\} t^{p-1} dt.$$

Therefore one is led to the following general approach to the construction of solutions to (2.1) with critical regularity properties:

Step 1. Find nontrivial laminates - with some specific properties - supported in the set E.

Step 2. Use Proposition 2.3 to construct a sequence of approximate solutions which converges strongly so that the specific property of the corresponding laminate is preserved in the limit.

An important feature of this point of view is that Step 1 allows one to focus on which geometric/combinatorial properties of the set E are relevant in determining the regularity properties of solutions to the differential inclusion (2.1). It should be emphasized that this approach can only pick up effects of the geometry of E on the regularity of solutions that come from *oscillation phenomena*.

3. Quasi-linear elliptic systems

Our first example deals with critical points of functionals of the form

$$I[u] = \int_\Omega f(Du(x))\, dx \tag{3.1}$$

for $u\colon \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$, where $f\colon \mathbb{R}^{2\times 2} \to \mathbb{R}$ is a smooth, strongly polyconvex function with bounded second derivatives.

A function is said to be polyconvex if it is a convex function of the minors. More specifically $f\colon \mathbb{R}^{2\times 2} \to \mathbb{R}$ is said to be *strongly polyconvex* if there exists a convex function $g\colon \mathbb{R}^5 \to \mathbb{R}$ and $\gamma > 0$ so that

$$f(X) = \gamma|X|^2 + g(X, \det X) \quad \text{for all } X \in \mathbb{R}^{2\times 2}.$$

Polyconvexity is a commonly used structural assumption in mathematical models of elasticity [**6, 8, 11**], and is motivated by the weak continuity properties of the Jacobian. In particular polyconvex functions are quasiconvex, i.e. $\int_\Omega f(X + D\psi) - f(X)\,dx \geq 0$ for all $X \in \mathbb{R}^{2\times 2}$ and for all $\psi \in C_0^\infty(\Omega, \mathbb{R}^2)$. It is well known from the work of C. B. Morrey Jr. [**22**] that quasiconvexity is a necessary and sufficient condition for the functional I to be lower-semicontinuous with respect to uniform convergence of uniformly Lipschitz functions. Moreover, under the assumption of strong quasiconvexity L. C. Evans in [**13**] proved that *minimizers* of I (in the sense that $I[w] \leq I[w + \psi]$ whenever $\psi \in C_0^\infty$) are partially regular, that is, smooth except on a closed set of Lebesgue measure zero. This result was extended by J. Kristensen and A. Taheri in [**18**] to the case of strong local minimizers (local with respect to variations in $W^{1,p}$ with $p < \infty$). Thus strong polyconvexity (and more generally strong quasiconvexity) leads to a satisfactory theory for minimizers of I. In contrast, we have the following result for critical points from [**31**]:

Theorem 3.1. *Let Ω be the unit ball in $\mathbb{R}^2$. There exists a smooth, strongly polyconvex function $f\colon \mathbb{R}^{2\times 2} \to \mathbb{R}$ with bounded second derivatives such that the functional $I[u] = \int_\Omega f(Du)\,dx$ admits critical points which are Lipschitz but nowhere C^1 in Ω. Moreover f can be chosen so that these critical points are weak local minimizers, i.e. local with respect to variations in $W^{1,\infty}$.*

In the following we will sketch the proof of this theorem. The proof follows closely the method of S. Müller and V. Šverák, who in [**23**] proved the analogue of this theorem with f quasiconvex instead of polyconvex.

First of all we formulate the problem as a differential inclusion. Let

$$E_f = \left\{ \begin{pmatrix} X \\ Df(X)J \end{pmatrix} : X \in \mathbb{R}^{2\times 2} \right\}, \tag{3.2}$$

where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and suppose that $w = (u, v)\colon \Omega \to \mathbb{R}^2 \times \mathbb{R}^2$ is a Lipschitz mapping satisfying

$$Dw \in E_f \quad \text{a.e. in } \Omega. \tag{3.3}$$

Then in particular $Df(Du)J = Dv$, so that $\operatorname{div} Df(Du) = 0$, in other words $u\colon \Omega \to \mathbb{R}^2$ is a critical point of I. Thus it suffices to construct Lipschitz solutions to the inclusion (3.3) which are nowhere C^1.

Secondly, we need to discuss the constraints on the geometry of E_f coming from the assumption of polyconvexity. Note that E_f is a smooth 4-dimensional manifold in $\mathbb{R}^{4\times 2}$. The tangent space is given by

$$T_X E_f = \left\{ \begin{pmatrix} Y \\ D^2 f(X) Y J \end{pmatrix} : Y \in \mathbb{R}^{2\times 2} \right\}.$$

Hence $T_X E_f$ contains rank-one matrices if and only if there exist $a, b, n \in \mathbb{R}^2$ such that

$$D^2 F(X)(a \otimes n)J = b \otimes n.$$

Using that $(a \otimes n)J = a \otimes n^\perp$ we find

$$\langle D^2F(X)a \otimes n^\perp, a \otimes n^\perp\rangle = \langle b \otimes n, a \otimes n^\perp\rangle = 0.$$

On the other hand it is easy to see that strong polyconvexity of f implies the strong Legendre–Hadamard condition, which can be written in the form

$$\langle D^2F(X)\xi \otimes \eta, \xi \otimes \eta\rangle \geq \gamma|\xi|^2|\eta|^2 \quad \text{for all } \xi, \eta \in \mathbb{R}^2.$$

So if f is strongly polyconvex, the tangent space T_XE_f cannot contain matrices of rank one. In fact, by an observation of J. M. Ball in [**7**] more is true: $\operatorname{rank}(A-B) > 1$ for any two matrices $A, B \in E_f$. These properties of E_f reflect the ellipticity of the Euler–Lagrange system corresponding to I.

Therefore in order to proceed with the approach outlined in Section 2 we need to have examples of nontrivial laminates which are supported on finite sets $\{A_1, \ldots, A_k\}$ with $\operatorname{rank}(A_i - A_j) > 1$. One class of such examples is given by the T_k configurations ($k \geq 4$), which we will explain below. Theorem 3.1 can then be deduced from the following two propositions:

Proposition 3.2. *There exists a smooth, strongly polyconvex $f\colon \mathbb{R}^{2\times 2} \to \mathbb{R}$ with bounded second derivatives such that E_f contains a T_5 configuration.*

Proposition 3.3. *If $f_0 \in C^2(\mathbb{R}^{2\times 2})$ is such that E_{f_0} contains a T_k configuration, then for any $\delta > 0$ there exists $f \in C^2(\mathbb{R}^{2\times 2})$ with $\sup|D^2f - D^2f_0| \leq \delta$ such that the inclusion $Dw \in E_f$ admits a Lipschitz solution which is nowhere C^1.*

Notice that in Proposition 3.3 there is no structural assumption on f_0. It can be seen as a general method of passing from the existence of a nontrivial laminate to the existence of Lipschitz but nowhere C^1 solutions. In fact the proof of Proposition 3.3 is the same as in [**23**]. Thus in some sense all the work in showing that strong polyconvexity is not enough to rule out pathological solutions is done in Proposition 3.2.

3.1. T_k **configurations.** It is of fundamental importance, in view of applications to elliptic partial differential equations, that there exist laminates supported on sets $\{A_1, \ldots, A_k\}$ with no rank-one connections, i.e. such that $\operatorname{rank}(A_i - A_j) > 1$ for all $i \neq j$. This fact has been observed independently by a number of authors in different contexts (e.g. [**5, 10, 25, 29, 33**]). The simplest example consists of four diagonal 2×2 matrices:

$$A_1 = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \quad A_3 = \begin{pmatrix} -3 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_4 = \begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix}.$$

In fact this set of matrices played a crucial role in the construction in [**23**]. The important property is the following cyclic structure (see Figure 1):

Definition 3.4. An ordered set of $k \geq 4$ matrices $\{A_i\}_{i=1}^k \subset \mathbb{R}^{m\times n}$ without rank-one connections is said to form a T_k configuration if there exist matrices $P, C_i \in \mathbb{R}^{m\times n}$ and real numbers $\kappa_i > 1$ such that

$$\begin{aligned} A_1 &= P + \kappa_1 C_1, \\ A_2 &= P + C_1 + \kappa_2 C_2, \\ &\vdots \\ A_k &= P + C_1 + \ldots + C_{k-1} + \kappa_k C_k, \end{aligned}$$

and moreover $\operatorname{rank}(C_i) = 1$ and $\sum_{i=1}^k C_i = 0$.

Lemma 3.5. *Let $\{A_1, \dots, A_k\}$ be a T_k configuration, and for $i = 1, \dots, k$ let $P_i = P + C_1 + \cdots + C_{i-1}$ (so that $P_1 = P$). Then*

$$\{P_1, \dots, P_k\} \subset \{ X_1, \dots, X_k\}^{rc}.$$

In particular for each $j = 1, \dots, k$ there exist numbers $\nu_i^{(j)} \in (0, 1)$ so that the probability measures $\nu^{(j)} = \sum_{i=1}^k \nu_i^{(j)} \delta_{A_i}$ are laminates with barycenters $\bar{\nu}^{(j)} = P_j$.

Proof. Let $f \colon \mathbb{R}^{m\times n} \to \mathbb{R}$ be a rank-one convex function vanishing at the points $A_1, \dots, A_k$. We have that for each i the inequality

$$f(P_{i+1}) \le \frac{1}{\kappa_i} f(A_i) + \Big(1 - \frac{1}{\kappa_i}\Big) f(P_i) = \Big(1 - \frac{1}{\kappa_i}\Big) f(P_i)$$

holds. Combining these inequalities yields $f(P_i) \le 0$ for all i. □

A simple application (see [**31**]) of the implicit function theorem yields:

Lemma 3.6. *Suppose $(A_1, \dots, A_k) \in (\mathbb{R}^{m\times n})^k$ is a T_k configuration. Then locally near $(A_1, \dots, A_k)$ there exists a smooth manifold $\mathcal{M}_k \subset (\mathbb{R}^{m\times n})^k$ of dimension $(m + n)k$ such that all k-tuples $(B_1, \dots, B_k) \in \mathcal{M}_k$ are T_k configurations.*

3.2. Construction of Lipschitz but nowhere C^1 solutions. In this section we sketch the proof of Proposition 3.3. There are two basic ingredients.

First of all recall that E_f is a smooth 4-dimensional manifold in $\mathbb{R}^{4\times 2}$ and assume that $A_1^0, \dots, A_k^0 \in E_f$ is a T_k configuration. Let $\mathcal{M}_k$ be the manifold of dimension $6k$ locally near $(A_1^0, \dots, A_k^0)$ consisting of T_k configurations (c.f. Lemma 3.6), and let $\mathcal{E}_f = E_f^{\times k}$ the set of (ordered) k-tuples of matrices in E_f ($\dim \mathcal{E}_f = 4k$). We know that $\mathcal{M}_k \cap \mathcal{E}_f$ is nonempty since it contains $(A_1^0, \dots, A_k^0)$. The first ingredient is to show that a small perturbation of f (in C^2) can achieve the following generic situation: the intersection $\mathcal{M}_k \cap \mathcal{E}_f$ is *transversal*, hence it is a manifold (in $(\mathbb{R}^{4\times 2})^k$) of dimension $2k$, and moreover the map

$$\pi_j \colon \mathcal{M}_k \cap \mathcal{E}_f \to \mathbb{R}^{4\times 2} \text{ defined by } (A_1, \dots, A_k) \mapsto P_j$$

(c.f. Definition 3.4) is open. Note that the tangent space $T\mathcal{E}_f$ depends on $D^2 f$, so the idea is to show that a generic perturbation of $D^2 f$ at the points A_i^0 can perturb $T\mathcal{E}_f$ to a sufficiently general position. Note that if the intersection of $\mathcal{M}_k$ and $\mathcal{E}_f$ is transversal, then $\dim \mathcal{M}_k \cap \mathcal{E}_f = 2k \ge 8$ since $k \ge 4$, so it makes sense to ask for openness of π_j. The details can be found in Section 5 of [**31**].

Using the openness of π_j we define a sequence of open sets $U_i \subset \mathbb{R}^{4\times 2}$ such that $U_i \subset U_{i+1}^{rc}$ and $U_i \to E_f$ in the sense that if $B_i \in U_i$ with $B_i \to B$, then $B \in E_f$. In Gromov's terminology such a sequence of sets is called an *in-approximation*. First, let $\phi_j \colon \mathcal{M}_k \cap \mathcal{E}_f \to \mathbb{R}^{4\times 2}$ be defined by $(A_1, \dots, A_k) \mapsto A_j$ and let $z_0 = (A_1^0, \dots, A_k^0)$. By our assumptions $D\pi_j$ restricted to the tangent space $T_{z_0}(\mathcal{M}_k \cap \mathcal{E}_f)$ has full rank, and so for all but finitely many values of λ the linear map

$$(1 - \lambda) D\pi_j + \lambda D\phi_j \tag{3.4}$$

has full rank. Let $\lambda_n \in (0, 1)$ be an increasing sequence with $\lambda_n \to 1$ so that the maps in (3.4) have full rank for all $n \in \mathbb{N}$ and j. Let

$$\Phi_n^j \overset{\text{def}}{=} (1 - \lambda_n)\pi_j + \lambda_n \phi_j.$$

Then $\Phi_n^j\colon \mathcal{M}_k \cap \mathcal{E}_f \to \mathbb{R}^{4\times 2}$ are local submersions. In order to ensure that in addition $U_n \subset U_{n+1}^{\mathrm{rc}}$, we choose an increasing sequence of relatively open sets

$$\mathcal{O}_{n-1} \subset \mathcal{O}_n \subset \mathcal{M}_k \cap \mathcal{E}_f \cap \big(B_\delta(A_1^0) \times \cdots \times B_\delta(A_k^0)\big)$$

and let $U_{n,j} = \Phi_n^j(\mathcal{O}_n)$, $U_n = \bigcup_{j=1}^k U_{n,j}$. By adjusting the sequence λ_n if necessary, we may assume that $P^0 \in U_1^{rc}$. The key point is that for each $A \in U_n$ there exists $A_j \in U_{n+1,j}$ for $j = 1, \ldots, k$ forming a T_k configuration such that $A \in \{A_1, \ldots, A_k\}^{rc}$, and moreover $U_{n,j} \to B_\delta(A_j^0)$.

The second ingredient in the proof of Proposition 3.3 is to apply Proposition 2.3 iteratively to obtain a sequence of piecewise affine Lipschitz mappings $w^{(n)}\colon \Omega \to \mathbb{R}^4$ with the following properties:

(a) $w^{(n)}(x) = P^0 x$ on $\partial\Omega$, and $Dw^{(n)}(x) \in U_n$ for a.e. $x \in \Omega$,
(b) $\sup_\Omega |w^{(n+1)} - w^{(n)}| \le 2^{-n}$,
(c) for all $\tilde\Omega \subset \Omega$ there exists $n_0 \in \mathbb{N}$, $c_{\tilde\Omega} > 0$ such that for all $n \ge n_0$

$$|\{x \in \tilde\Omega :\ Dw^{(n)}(x) \in U_{n,j}\}| \ge c_{\tilde\Omega} \quad \text{for all } j = 1, \ldots, k,$$

(d) $\int_\Omega |Dw^{(n+1)} - Dw^{(n)}|\, dx \le C(\lambda_{n+1} - \lambda_n)$.

First we define $w^{(0)}(x) = P^0 x$. To obtain $w^{(n+1)}$ from $w^{(n)}$ we decompose Ω into a union of pairwise disjoint open sets of diameter no more than $\frac{1}{n}$,

$$\Big|\Omega \setminus \bigcup_\alpha \Omega_\alpha^n\Big| = 0,$$

so that $w^{(n)}$ is affine in each open set. In each Ω_α^n we replace $w^{(n)}$ with the piecewise affine mapping obtained from the following one-step construction:

Lemma 3.7. *Let $A \in U_{n,j}$. For any domain $\omega \subset \Omega$ there exists a piecewise affine Lipschitz mapping $w\colon \omega \to \mathbb{R}^4$ with the following properties*:

(i) *$w(x) = Ax$ on $\partial\omega$, and $Dw(x) \in U_{n+1}$ in ω,*
(ii) *$|w(x) - Ax| < 2^{-(n+1)}$ in ω,*
(iii) *$|\{x \in \omega :\ Dw(x) \in U_{n+1,j}\}| > \frac{\lambda_n}{\lambda_{n+1}} |\omega|$,*
(iv) *$\int_\omega |Dw - A|\, dx \le C(\lambda_{n+1} - \lambda_n)|\omega|$.*

Proof. Assume for simplicity that $j = 1$. By our construction of the in-approximation, there exists $(A_1, \ldots, A_k) \in \mathcal{O}_n$ forming a T_k configuration so that A is contained in the segment $[P_1, A_1]$. In Figure 1, solid lines show the original T_k contained in E_f, and dashed lines the perturbed T_k with $A \in [P_1, A_1]$. As $(A_1, \ldots, A_k) \in \mathcal{O}_{n+1}$ also, there exist new points $\tilde A_j \in U_{n+1,j}$ on the segments $[P_j, A_j]$. But then, since $\tilde A_j$ themselves form a T_k with $\pi_1(\tilde A_1, \ldots, \tilde A_k) = P_1$, there exist coefficients $\nu_j \in (0,1)$ such that the probability measure $\nu = \sum_{j=1}^k \nu_j \delta_{\tilde A_j}$ is a laminate with barycenter P_1. Consequently

$$\mu \overset{\text{def}}{=} \frac{\lambda_n}{\lambda_{n+1}} \delta_{\tilde A_1} + \Big(1 - \frac{\lambda_n}{\lambda_{n+1}}\Big)\nu$$

is a laminate supported in U_{n+1} with barycenter A. Moreover

$$\mu(U_{n+1,1}) > \frac{\lambda_n}{\lambda_{n+1}}. \tag{3.5}$$

We now apply Proposition 2.3 with the laminate μ to obtain the mapping $w\colon \omega \to \mathbb{R}^4$. Indeed, (i) and (ii) follow directly from Proposition 2.3 and since U_n are open

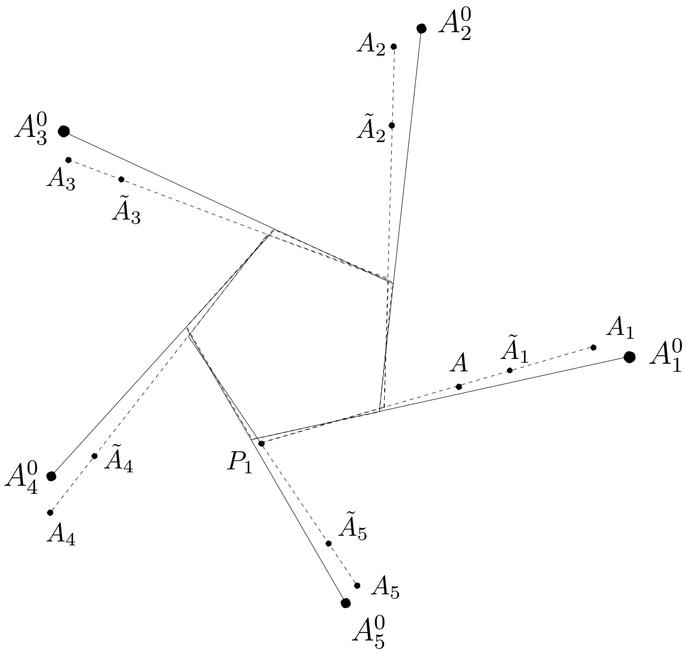

FIGURE 1. Original and perturbed T_k's.

sets, and (iii) follows from the estimate (3.5) together with (2.4). To prove (iv) note that by (iii) the gradient Dw takes values near $\tilde{A}_1$ in a large portion of the domain ω, and $|A - \tilde{A}_1| = (\lambda_{n+1} - \lambda_n)|P_1 - A_1|$. Hence

$$\int_\omega |Dw - A|\,dx = \int_{\{Dw \in U_{n+1,1}\}} |Dw - A|\,dx + \int_{\{Dw \notin U_{n+1,1}\}} |Dw - A|\,dx$$

$$\leq C|\omega|(\lambda_{n+1} - \lambda_n) + C|\omega|\Big(1 - \frac{\lambda_n}{\lambda_{n+1}}\Big) \leq C\Big(1 + \frac{1}{\lambda_1}\Big)|\omega|(\lambda_{n+1} - \lambda_i). \qquad \square$$

It is clear that with this construction the sequence $w^{(n)}$ satisfies (a), (b) and (d). To see that (c) is also satisfied, let $\tilde{\Omega} \subset \Omega$ be any subset. For large enough $n_0 \in \mathbb{N}$ there exists α such that $\Omega_\alpha^{n_0} \subset \tilde{\Omega}$. Hence from the proof of Lemma 3.7 we see that there exists $\varepsilon > 0$ so that for each j

$$|\{x \in \Omega_\alpha^{n_0} : Dw^{(n_0+1)}(x) \in U_{n_0+1,j}\}| > \varepsilon|\Omega_\alpha^{n_0}|.$$

But then, from (iii) it follows that for each $n > n_0$ and each j

$$|\{x \in \tilde{\Omega} : Dw^{(n)}(x) \in U_{n,j}\}| > \frac{\lambda_{n-1}}{\lambda_n}\frac{\lambda_{n-2}}{\lambda_{n-1}} \cdots \frac{\lambda_{n_0}}{\lambda_{n_0+1}}\varepsilon|\Omega_\alpha^{n_0}| \geq \lambda_{n_0}\varepsilon|\Omega_\alpha^{n_0}|.$$

From (b) and (d) it follows that our sequence $w^{(n)}$ converges to some limit w uniformly and in $W^{1,1}$. Moreover, w is Lipschitz with $w(x) = P^0 x$ on $\partial\Omega$ and

$$Dw(x) \in E_f \quad \text{a.e. in } \Omega.$$

Finally, (c) implies that Dw has essential oscillation of order 1 in any open subset of Ω, hence w is nowhere C^1. This proves Proposition 3.3.

3.3. Polyconvex counterexamples. The purpose of this section is to show how Proposition 3.2 can be reduced to a problem in *linear programming*. We will repeatedly write 4×2 matrices in block form

$$A=\begin{pmatrix}X\\Y\end{pmatrix},$$

where $X,Y\in\mathbb{R}^{2\times 2}$. By the definition of the set E_f a T_k configuration $(A_1,\dots,A_k)$ lies in E_f if and only if

$$Df(X_i)J=Y_i\quad\text{for } i=1,\dots,k. \tag{3.6}$$

One approach to Proposition 3.2 could be to fix a (strongly) polyconvex function f and solve (3.6) using the parametrization in Definition 3.4. However, this leads to a large system of nonlinear algebraic equations. The twist is to fix instead a T_k configuration and solve (3.6) for f.

LEMMA 3.8. *Let $(A_1,\dots,A_k)$ be a T_k configuration. There exists a smooth, strongly polyconvex function f with bounded second derivatives satisfying* (3.6) *if and only if the linear system of* (*strict*) *inequalities*

$$c_i-c_j+d_i\det(X_i-X_j)+\langle X_i-X_j,Y_iJ\rangle<0 \tag{3.7}$$

is solvable for $c,d\in\mathbb{R}^k$.

PROOF. Recall that $f\colon\mathbb{R}^{2\times 2}\to\mathbb{R}$ is strongly polyconvex if there exists a convex function $g\colon\mathbb{R}^5\to\mathbb{R}$ and $\gamma>0$ such that $f(X)=\gamma|X|^2+g(X,\det X)$. Therefore there exists a strongly polyconvex function f for which $A_i\in E_f$ for all $i=1,\dots,k$ if and only if there exists $\gamma>0$ and a convex function g satisfying

$$\partial_X g(\tilde X_i)+\partial_d g(\tilde X_i)\mathrm{cof}\,X_i=-Y_iJ-2\gamma X_i\qquad\text{for } i=1,\dots,k. \tag{3.8}$$

Here ∂_d means derivative with respect to the determinant term, and for $X\in\mathbb{R}^{2\times 2}$ we write $\tilde X=(X,\det X)\in\mathbb{R}^5$. Suppose we are given real numbers c_i and vectors $B_i,\tilde X_i\in\mathbb{R}^5$ for $i=1,\dots,k$. It is well known that there exists a (smooth) convex function g with the property that $g(\tilde X_i)=c_i$ and $Dg(\tilde X_i)=B_i$ if the data satisfies the system of $k(k-1)$ inequalities

$$c_j>c_i+\big\langle B_i,\tilde X_j-\tilde X_i\big\rangle_{\mathbb{R}^5}\qquad\text{for all } i\neq j. \tag{3.9}$$

Indeed, let $g_0(\tilde X)=\max_i\big(c_i+\big\langle B_i,\tilde X-\tilde X_i\big\rangle\big)$. Take a smooth mollifier ϕ on $\mathbb{R}^5$ supported in a small ball around the origin and satisfying $\int\phi(\tilde Y)\,d\tilde Y=1$ and $\int\tilde Y\phi(\tilde Y)\,d\tilde Y=0$. Since the inequalities (3.9) are strict, taking the support of ϕ sufficiently small we ensure that in a neighbourhood of each $\tilde X_i$

$$\phi*g_0(\tilde X)=\int\big(c_i+\big\langle B_i,(\tilde X-\tilde Y)-\tilde X_i\big\rangle\big)\phi(\tilde Y)\,d\tilde Y=c_i+\big\langle B_i,\tilde X-\tilde X_i\big\rangle=g_0(\tilde X).$$

Therefore $g=\phi*g_0$ gives the required smooth and convex function.

Substituting (3.8) into (3.9) gives

$$\begin{aligned}c_j&>c_i+\big\langle B_i,\tilde X_j-\tilde X_i\big\rangle_{\mathbb{R}^5}\\&=c_i+\big\langle\partial_X g(\tilde X_i),X_j-X_i\big\rangle+\partial_d g(\tilde X_i)(\det X_j-\det X_i)\\&=c_i-\big\langle Y_iJ+2\gamma X_i+\partial_d g(\tilde X_i)\mathrm{cof}\,X_i,X_j-X_i\big\rangle+\partial_d g(\tilde X_i)(\det X_j-\det X_i).\end{aligned}$$

Writing $d_i = \partial_d g(\tilde{X}_i)$ we deduce that a convex function g satisfying (3.8) exists if there exist real numbers c_i, d_i satisfying the system

$$c_i - c_j + d_i \det(X_i - X_j) + \langle X_i - X_j, Y_i J\rangle < -2\gamma\langle X_i, X_i - X_j\rangle. \tag{3.10}$$

But if (3.7) is solvable, then also (3.10) is solvable for sufficiently small $\gamma > 0$. □

Unfortunately the system (3.7) is not feasible for a generic choice of T_k configuration. In fact in [**17**] it is proved that (3.7) cannot be feasible for $k = 4$ for any choice of T_4 configuration. However, we note that Y_i also appears linearly, so if after fixing the "base" $\{X_i\}$ we treat $\{Y_i\}$ as extra variables, for $k = 5$ we obtain a system of 20 inequalities in 16 variables. This enables one to "guess" $\{X_i\}$ for which the system will be feasible. In [**31**] a T_5 configuration is exhibited for which (3.7) is feasible. Such an example concludes the proof of Proposition 3.2, and therefore the proof of Theorem 3.1.

4. L^p theory of elliptic equations

We turn to our second example of the approach outlined in Section 2, which concerns the L^p regularity of solutions to second order linear equations in the plane with measurable coefficients. The results in this section were obtained in joint work with K. Astala and D. Faraco in [**2**]. We consider equations of the form

$$\operatorname{div} \sigma(x)\nabla u = 0 \quad \text{in } \Omega \subset \mathbb{R}^2, \tag{4.1}$$

where σ is measurable and uniformly elliptic in the sense that

$$\frac{1}{K}|\xi|^2 \le \sigma_{ij}(x)\xi_i\xi_j \le K|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^2 \tag{4.2}$$

for some $K > 1$. It is well known (see [**9, 21**]) that there exist exponents $q_K < 2 < p_K$ with the property that any weak solution $u \in W^{1,q}_{\mathrm{loc}}$ to (4.1) for some $q > q_K$ is automatically in $W^{1,p}_{\mathrm{loc}}$ for all $p < p_K$. Recent developments in the theory of planar quasiconformal mappings, in particular the area distortion theorem of K. Astala [**1**] and the invertibility of Beltrami operators [**4**] lead to the precise identification of these exponents in [**20**], namely

$$q_K = \frac{2K}{K+1}, \quad p_K = \frac{2K}{K-1}. \tag{4.3}$$

This higher integrability property was extended recently to the lower critical exponent $q = q_K$ in [**27**]. There are classical examples built on radial stretchings which show that for general σ (subject to (4.2)) the range of exponents cannot be improved. Using convex integration we give another class of examples, which shows that no restriction on the range of σ can improve on the range of exponents $q_K < 2 < p_K$.

Theorem 4.1. *Let Ω be the unit ball in $\mathbb{R}^2$ and let $K > 1$.*

i) *There exists a measurable function $\sigma\colon \Omega \to \{\frac{1}{K}, K\}$ such that the solution $u \in W^{1,2}(\Omega)$ to the Dirichlet problem*

$$\begin{cases} \operatorname{div} \sigma(x)\nabla u(x) = 0 & \text{in } \Omega,\\ u(x) = x_1 & \text{on } \partial\Omega \end{cases} \tag{4.4}$$

satisfies for every ball $B(x,r) \subset \Omega$

$$\int_{B(x,r)} |\nabla u|^{\frac{2K}{K-1}} = \infty. \tag{4.5}$$

ii) *For every* $\alpha \in (0,1)$ *there exists a measurable function* $\sigma\colon \Omega \to \{\frac{1}{K}, K\}$ *and a function* $u \in W^{1,q}(\Omega) \cap C^\alpha(\overline{\Omega})$ *for all* $q < \frac{2K}{K+1}$ *such that* $u(x) = x_1$ *on* $\partial\Omega$ *and*

$$\operatorname{div} \sigma(x)\nabla u(x) = 0$$

in the sense of distributions, but for every ball $B(x,r) \subset \Omega$

$$\int_{B(x,r)} |\nabla u|^{\frac{2K}{K+1}} = \infty. \tag{4.6}$$

This theorem should be compared with the result of L. C. Piccinini and S. Spagnolo in [**28**] which shows that for equations of the type (4.4) with σ scalar valued, the Hölder regularity of solutions improves beyond the general case when σ is matrix-valued.

Before discussing the proof of Theorem 4.1 we briefly state the analogous results for equations of the form

$$a_{ij}(x)\,\partial_i\partial_j u = 0 \quad \text{in } \Omega \subset \mathbb{R}^2, \tag{4.7}$$

where a_{ij} is measurable and uniformly elliptic in the sense that

$$\frac{1}{\sqrt{K}}|\xi|^2 \le a_{ij}\xi_i\xi_j \le \sqrt{K}|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^2$$

for some $K > 1$. The precise L^p theory follows from recent work of K. Astala, T. Iwaniec and G. Martin in [**3**], showing that if $u \in W^{2,q}_{\mathrm{loc}}$ is a solution to (4.7) for some $q > q_K$, then $u \in W^{2,p}_{\mathrm{loc}}$ for all $p < p_K$, with q_K, p_K again defined by (4.3). Here we obtain the following analogue of Theorem 4.1:

Theorem 4.2. *Let* $\Omega \subset \mathbb{R}^2$ *be the unit ball and let* $K > 1$.

i) *There exists measurable* $a\colon \Omega \to \left\{ \begin{pmatrix} \frac{1}{\sqrt{K}} & 0 \\ 0 & \sqrt{K} \end{pmatrix}, \begin{pmatrix} \sqrt{K} & 0 \\ 0 & \frac{1}{\sqrt{K}} \end{pmatrix} \right\}$ *such that the solution* $u \in W^{2,2}(\Omega)$ *to the Dirichlet problem*

$$\begin{cases} a_{ij}(x)\partial_i\partial_j u = 0 & \text{in } \Omega, \\ u(x) = \frac{1}{2}(x_1^2 - x_2^2) & \text{on } \partial\Omega \end{cases} \tag{4.8}$$

satisfies for every ball $B(x,r) \subset \Omega$

$$\int_{B(x,r)} |D^2 u|^{\frac{2K}{K-1}} = \infty.$$

ii) *For every* $\alpha \in (0,1)$ *there exists* $a\colon \Omega \to \left\{ \begin{pmatrix} \frac{1}{\sqrt{K}} & 0 \\ 0 & \sqrt{K} \end{pmatrix}, \begin{pmatrix} \sqrt{K} & 0 \\ 0 & \frac{1}{\sqrt{K}} \end{pmatrix} \right\}$ *and* $u \in W^{2,q}(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$ *for all* $q < \frac{2K}{K+1}$ *such that*

$$a_{ij}(x)\,\partial_i\partial_j u = 0 \quad \text{a.e. in } \Omega, \tag{4.9}$$

but for every ball $B(x,r) \subset \Omega$

$$\int_{B(x,r)} |D^2 u|^{\frac{2K}{K+1}} = \infty.$$

We will show the main steps in the proof of Theorem 4.1, the proof of Theorem 4.2 is similar. The approach is again the one outlined in Section 2. The reformulation of the problem as a first order differential inclusion follows directly from the connection with planar quasiregular mappings. Indeed, $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ is a weak solution of (4.1) with σ scalar valued if and only if $u = \mathrm{Re}\,(w)$ for a map $w\colon \Omega \to \mathbb{C}$ satisfying

$$w_{\bar z} = \mu \overline{w_z},$$

where μ is related to σ via $\mu = \frac{1-\sigma}{1+\sigma}$. Thus equations of the form (4.1) with $\sigma(x) \in \{K, \frac{1}{K}\}$ correspond to the differential inclusion

$$Dw \in E_K, \tag{4.10}$$

where

$$E_K = \{A = (a_+, a_-) : a_- = \pm k \bar a_+\} \quad \text{with} \quad k = \frac{K-1}{K+1}.$$

Here we write 2×2 matrices in conformal coordinates $A = (a_+, a_-)$ using the identification $\mathbb{R}^{2\times 2} \cong \mathbb{C} \times \overline{\mathbb{C}}$. Note that E_K is the union of two 2-dimensional subspaces in $\mathbb{R}^{2\times 2}$ which contain no rank-one matrices. On the other hand, there are of course rank-one connections *between* the two subspaces, in contrast with the situation in Section 3.

Following the general philosophy in this paper that laminates satisfying (2.5) can be viewed as generalized solutions to the differential inclusion (2.6), where in this case the specific inclusion problem is given in (4.10), we first need examples of sequences of laminates $\nu_n \overset{*}{\rightharpoonup} \nu$, whose weak limit is a probability measure with unbounded support satisfying for some $c > 1$ and $p > 1$

$$\frac{1}{c} t^{-p} < \nu\big(\{|A| > t\}\big) < c t^{-p} \quad \forall t > 0. \tag{4.11}$$

Such laminates, called staircase laminates, were first introduced by D. Faraco in [**14**] where the author used them to prove a result slightly weaker than part (i) of Theorem 4.1. We will discuss staircase laminates in Section 4.1. It should come as no surprise that the two optimal exponents p in (4.11) for which laminates supported in E_K exist are precisely q_K and p_K given by (4.3). In the next section we show how to construct such staircase laminates in E_K. Finally in Section 4.2 we show the main steps in the construction of solutions to the inclusion (4.10) in the weak Lebesgue space L^p_{weak} corresponding to (4.11).

4.1. Staircase laminates. The construction of staircase laminates for E_K can be best illustrated in the diagonal plane $\mathbb{D} = \left\{\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} : x, y \in \mathbb{R}\right\}$. The intersection $E_K \cap \mathbb{D}$ consists of two lines E_K^+ and E_K^- through the origin as shown in Figure 2, and the rank-one lines in $\mathbb{D}$ are precisely the coordinate directions. In the following we identify coordinates (x, y) with diagonal matrices $\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$. For simplicity we assume that $K > 2$. Let

$$I_n = (n, n),\ A_n = \Big(n, \frac{n}{K}\Big),\ B_n = \Big(\frac{n+1}{K}, n+1\Big),\ P_n = (n, n+1).$$

It can be easily verified (see Figure 2), that the probability measure

$$\mu_n = \frac{K}{(n+1)K - n} A_n + \frac{n}{n+1} \frac{K}{(n+1)K - n} B_n + \frac{n}{n+1} \frac{n(K-1) - 1}{n(K-1) + K} I_{n+1}$$

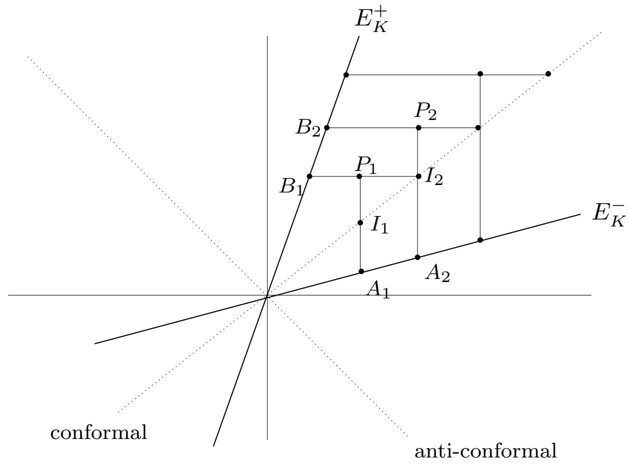

FIGURE 2. Construction of the upper staircase.

is a (second order) laminate with support $\operatorname{spt}\mu_n = \{A_n, B_n, I_{n+1}\}$ and barycenter $\overline{\mu_n} = I_n$. Now we define the laminates ν_n by setting $\nu_1 = \mu_1$ and writing $\nu_n = \lambda\delta_{I_{n+1}} + (1-\lambda)\tilde{\nu}$ we define (c.f. (2.2))

$$\nu_{n+1} = \lambda\mu_{n+1} + (1-\lambda)\tilde{\nu}. \tag{4.12}$$

In this way we obtain $\overline{\nu_n} = I_1$, $\operatorname{spt}\nu_n = \{A_1, \ldots, A_n, B_1, \ldots, B_n, I_{n+1}\}$, and

$$\nu_n(I_{n+1}) = \prod_{j=1}^{n} \frac{j}{j+1}\frac{j(K-1)-1}{j(K-1)+K} = \frac{1}{n+1}\prod_{j=1}^{n}\frac{j(K-1)-1}{j(K-1)+K}. \tag{4.13}$$

By (4.12) and since $\nu_n(I_{n+1}) \leq \frac{1}{n+1}$, we find that $\nu_n \overset{*}{\rightharpoonup} \nu$ for some probability measure ν with $\bar{\nu} = I_1$ and $\operatorname{spt}\nu = \{A_j, B_j : j \in \mathbb{N}\}$. Moreover, we have

LEMMA 4.3. *The measure ν satisfies*

$$\frac{1}{c}t^{-\frac{2K}{K-1}} < \nu\big(\{|A| > t\}\big) < ct^{-\frac{2K}{K-1}} \quad \forall t > 0. \tag{4.14}$$

PROOF. First of all notice that by the construction (4.12) for all $n \in \mathbb{N}$ we have

$$\nu\big(\{|A| > n\}\big) = \nu_n(I_{n+1}) = \frac{1}{n+1}\prod_{j=1}^{n}\frac{j(K-1)-1}{j(K-1)+K}.$$

Thus it suffices to show that for some fixed constant c

$$\left|\prod_{j=1}^{n}\frac{j(K-1)-1}{j(K-1)+K} - n^{-\frac{K+1}{K-1}}\right| \leq c. \tag{4.15}$$

Taking logarithms we have

$$\log\prod_{j=1}^{n}\frac{j(K-1)-1}{j(K-1)+K} \approx \sum_{j=1}^{n} -\frac{K+1}{j(K-1)+K} \approx -\frac{K+1}{K-1}\log n,$$

from which (4.15) follows. □

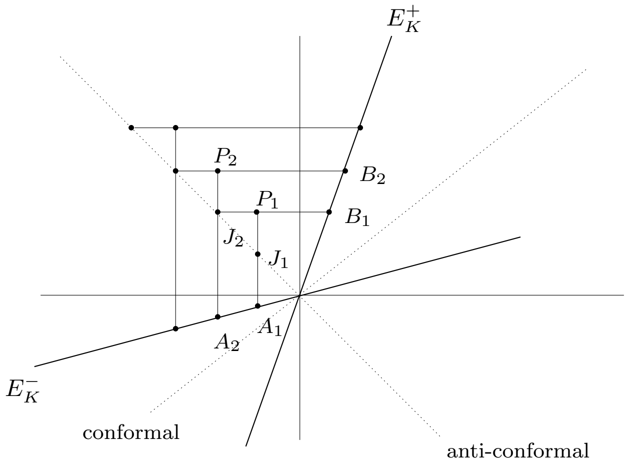

FIGURE 3. Construction of the lower staircase.

The staircase laminate for part (ii) of the theorem is constructed in an analogous manner, shown in Figure 3. The key difference is that the mass is pushed out to infinity along the anti-conformal plane instead of the conformal plane. We refer the interested reader to Section 3.2 in [**2**].

4.2. Construction of solutions in weak Lebesgue spaces. Suppose that ν_n is a sequence of laminates defined as in (4.12) and such that the weak* limit ν satisfies the estimate (4.11) for some $p > 1$ and is supported in E_K. In this section we outline how the construction of the corresponding solutions to the inclusion $Dw \in E_K$ is carried out. It should be pointed out that in the case $p \geq p_K$ using elliptic estimates it is possible to prove the existence of solutions corresponding to the laminate ν via an elegant Baire category approach (see [**2, 34**]). However, as this method is not applicable for the case $p \leq q_K$, and in order to present a unified approach, we sketch the construction using a method similar to the one in Section 3.2.

The first step is to construct approximate solutions. The in-approximations take the form (c.f. Figure 2)

$$\begin{aligned} U_n = \{A \in \mathbb{R}^{2\times 2} :\, & 2^{-(n+1)}\tau(|A|) < \operatorname{dist}(A, E_K) < 2^{-n}\tau(|A|) \text{ and there exists} \\ & P \in E_1,\, Q \in E_K \text{ such that } \operatorname{rank}(P - Q) = 1 \text{ and } A \in [P, Q]\}, \end{aligned}$$

where $\tau\colon [0,\infty) \to (0, 1/2]$ is a continuous non-increasing function with $\tau(0) > 0$ and $\int_1^\infty \frac{\tau(t)}{t}\, dt < \infty$, to be chosen later. Note that in our notation E_1 is the set of conformal matrices.

LEMMA 4.4. *For any bounded domain $\Omega \subset \mathbb{R}^2$, any $\delta > 0$ and $\alpha \in [0, 1)$ there exists a piecewise affine mapping $w \in W^{1,1}(\Omega, \mathbb{R}^2) \cap C^\alpha(\overline{\Omega}, \mathbb{R}^2)$ such that $w(x) = x$ on $\partial\Omega$, $[w(x) - x]_{C^\alpha} < \delta$, $Dw \in U_n$ a.e. in Ω and furthermore*

$$\frac{1}{c} t^{-p} < |\{x \in \Omega :\, |Dw(x)| > t\}| < c t^{-p}. \tag{4.16}$$

This can be done by applying Proposition 2.3 iteratively and using the defining sequence of laminates ν_n from Section 4.1. It is important to note that the map obtained in Lemma 4.4 is piecewise affine.

Now we construct a sequence of piecewise affine mappings $w^{(n)}$ in a similar way as in Section 3.2. The analogue of Lemma 3.7 is the following:

Lemma 4.5. *Let $A \in U_n$. For any bounded subdomain $\omega \subset \Omega$ there exists a piecewise affine map $w \in W^{1,1}(\omega, \mathbb{R}^2) \cap C^\alpha(\overline{\omega}, \mathbb{R}^2)$ with the properties*:

(i) *$w(x) = Ax$ on $\partial\omega$, and $Dw(x) \in U_{n+1}$ in ω,*
(ii) *$[w(x) - Ax]_{C^\alpha} < 2^{-(n+1)}$,*
(iii) *$\int_\omega |Dw - A|\, dx \le C 2^{-n} |\omega|$,*
(iv) *for all $t > |A|$*

$$\frac{1}{c} 2^{-n} t^{-p} |\omega| < |\{x \in \omega : |Dw(x)| > t\}| < c 2^{-n} t^{-p} |\omega|.$$

Sketch of proof. The proof is very similar to the proof of Lemma 3.7. We use the definition of U_n to find $P \in E_K$ to $Q \in E_1$ with rank$\,(P - Q) = 1$ so that for some $\lambda \in (0, 1)$ we have $\lambda P + (1 - \lambda) Q = A$. Then we use the conformal invariance of E_K and E_1 to pass from the boundary value $w(x) = x$ in Lemma 4.4 to any conformal affine boundary value $w(x) = Qx$. This would yield the estimate

$$|\{x \in \omega : |Dw(x)| > t\}| \sim |Q|^p t^{-p} (1 - \lambda) |\omega|.$$

The key point is to obtain the estimate (iv), which is independent of $|Q|$. This is where the definition of U_n is important: it ensures, with $\tau(t) = \min(1, t^{1-p})$, that $(1 - \lambda) \sim 2^{-n} |A|^{-p}$ (note also that $|Q| \sim| A|$). □

The construction of the sequence $w^{(n)}$ now proceeds precisely as in Section 3.2. To obtain $w^{(n+1)}$ from $w^{(n)}$ we decompose Ω into a union of pairwise disjoint open sets of diameter no more than $\frac{1}{n}$ with $|\Omega \setminus \bigcup_j \Omega_j^n| = 0$ so that $w^{(n)}$ is affine in each open set, and in each Ω_j^n we replace $w^{(n)}$ with the piecewise affine mapping obtained from Lemma 4.5.

The convergence of the sequence to a limit $w \in W^{1,1}(\Omega, \mathbb{R}^2) \cap C^\alpha(\overline{\Omega}, \mathbb{R}^2)$ with $Dw \in E_K$ follows directly from (i)-(iii). Furthermore, using (iv), for all $n \in \mathbb{N}$ and $t > 1$ we have

$$|\{x \in \Omega : |Dw^{(n+1)}| > t\}| \le |\{x \in \Omega : |Dw^{(n)}| > t\}| + c 2^{-n} t^{-p} |\Omega|,$$

which shows that $w \in L^p_{\text{weak}}$. To obtain a bound from below, let $B \subset \Omega$ be a ball. For large enough $n_0 \in \mathbb{N}$ there exists j such that $\Omega_j^{n_0} \subset B$. From (iv) in Lemma 4.5 we obtain for $t > t_0$

$$|\{x \in \Omega_j^{n_0} : |Dw^{(n_0+1)}(x)| > t\}| \ge \frac{1}{c} 2^{-n_0} t^{-p} |\Omega_j^{n_0}|.$$

On the other hand from the proof of Lemma 4.5 we see that in each step $|Dw^{(n+1)} - Dw^{(n)}| < 2^{-n}$ on a subset of volume fraction at least $1 - 2^{-n}$ (more precisely λ), hence for all $n > n_0$ and $t > t_0$ we have

$$\begin{aligned} |\{x \in B : |Dw^{(n)}(x)| > t\}| &\ge |\{x \in \Omega_j^{n_0} : |Dw^{(n)}(x)| > t\}| \\ &\ge \frac{1}{c} 2^{-n_0} t^{-p} |\Omega_j^{n_0}| \prod_{j=1}^\infty (1 - 2^{-j}) \ge \frac{1}{c_B} t^{-p}. \end{aligned}$$

This finishes the construction of the map w with $Dw \in E_K$ and such that for any ball $B \subset \Omega$ and any $t > 1$

$$\frac{1}{c_B} t^{-p} \leq |\{x \in \Omega : |Dw(x)| > t\}| \leq c_B t^{-p}.$$

Appendix A. Proof of Proposition 2.3

We may assume without loss of generality that $\nu(A_i) > 0$ for all i. Let $U = \bigcup_{i=1}^N B_r(A_i)$.

Step 1. In the case when ν is a laminate of finite order, the result is precisely Lemma 3.2 in [**23**]. It relies on a simple construction for the case when $\nu = \lambda\delta_B + (1-\lambda)\delta_C$, which is then iterated a finite number of times in a way naturally suggested by the definition of a laminate of finite order. It should be pointed out that in [**23**] the result is proved for $\alpha = 0$. For general $\alpha < 1$ the proof is exactly the same for obtaining the estimate $[u - A]_{C^\alpha(\overline{\Omega})} < \delta$, namely by the standard technique of decomposing Ω into a disjoint union of rescaled copies of itself upto measure zero, and placing rescaled mappings of the form

$$u_{x_i,r_i}(x) = r_i u\left(\frac{x - x_i}{r_i}\right) + Ax_i$$

in each copy. With such a rescaling the Hölder norms decrease.

Step 2. The main difficulty in obtaining the result for a general laminate is that when passing to the limit we need to keep the precise volume fraction (2.4) as well as keep the limit mapping piecewise affine. Both of these properties are crucial to the applicability of the proposition. Therefore we proceed in the following way, suggested by V. Šverák.

Let $\mathcal{L}_A$ be the set of vectors $\mathbf{v} \in \mathbb{R}^N$ such that there exists a laminate μ with barycenter $\bar{\mu} = A$ and support $\operatorname{spt}\mu \subset U$ so that

$$\mathbf{v}_i = \nu\big(B_r(A_i)\big) \quad \text{for all } i. \tag{A.1}$$

Similarly, let $\mathcal{F}_A$ be the set of vectors $\mathbf{v} \in \mathbb{R}^N$ such that there exists a piecewise affine Lipschitz mapping $u \colon \Omega \to \mathbb{R}^m$ with $u(x) = Ax$ on $\partial\Omega$, $[u - A]_{C^\alpha} < \delta$ and $Du(x) \in U$ a.e. $x \in \Omega$ with

$$\mathbf{v}_i = \frac{|\{x \in \Omega : Du(x) \in B_r(A_i)\}|}{|\Omega|} \quad \text{for all } i.$$

It is clear that $\mathcal{L}_A$ and $\mathcal{F}_A$ are convex and both lie in the set

$$\Delta = \Big\{\mathbf{v} \in \mathbb{R}^N : \sum_{i=1}^N \mathbf{v}_i = 1 \text{ and } \mathbf{v}_i \geq 0 \text{ for all } i\Big\}.$$

Let $\mathbf{v}$ denote the vector corresponding (in the sense of (A.1)) to the laminate ν. We claim that $\mathbf{v}$ lies in the interior of $\mathcal{L}_A$ (relative to Δ).

To prove our claim, we perturb ν in the following way: Choose $0 < \eta < r/2$. By standard results on rank-one convex hulls (see Theorem 4.9 in [**16**]) $\{A_1, \ldots, A_N\}^{rc}$ is connected, hence there exists $P \in \mathbb{R}^{m\times n}$ with $|P| = \eta$ so that $A_1 - P \in \{A_1, \ldots, A_N\}^{rc}$. Thus

$$A_1 \in \{A_1 + P, A_2 + P, \ldots, A_N + P\}^{rc},$$

and in particular there exists a laminate μ with barycenter $\overline{\mu} = A_1$ supported on the points $A_i + P$, $i = 1, \ldots, N$. Note that because $\overline{\mu} = A_1$,

$$1 - C\eta < \mu(A_1 + P) < 1$$

for some constant C independent of η. But then

$$\nu^1 = \nu(A_1)\mu + \sum_{i=2}^{N} \nu(A_i)\delta_{A_i}$$

is a laminate with $\overline{\nu^1} = A$ and $\operatorname{spt}\nu^1 \subset U$. Moreover, the corresponding vector $\mathbf{v}^1$ satisfies

$$\mathbf{v}_1^1 < \mathbf{v}_1^0 - C_1\eta \quad \text{and} \quad \mathbf{v}_i^1 > \mathbf{v}_i^0 + C_i\eta \quad \text{for } i > 1$$

with the constants C_i independent of η. Since we can perform such a perturbation with any A_i in place of A_1, and since $\mathcal{L}_A$ is convex, we conclude that $\mathbf{v} \in \operatorname{int}\mathcal{L}_A$.

Step 3. Using Theorem 2.2 for any $\varepsilon > 0$ we find a laminate of finite order μ with support $\operatorname{spt}\mu \subset U^{rc}$ for which $|\mu(B_r(A_i)) - \nu(B_r(A_i))| < \varepsilon$ for all i. Then Step 1 provides a piecewise affine Lipschitz mapping u_1 corresponding to μ. Now in each subdomain of Ω where u_1 is affine with gradient in $U^{rc} \setminus U$ we can replace it with a piecewise affine Lipschitz map whose gradient lies in U using Theorem 2.1. Thus we obtain a piecewise affine Lipschitz map u_2 such that $Du_2 \in U$ and

$$\big||\{x \in \Omega : Du_2(x) \in B_r(A_i)\}| - \nu(B_r(A_i))\big| < N\varepsilon.$$

This shows that $\mathbf{v} \in \overline{\mathcal{F}_A}$. On the other hand Step 2 shows that $\mathbf{v}$ cannot lie on the boundary, hence necessarily $\mathbf{v} \in \mathcal{F}_A$, i.e. there exists a piecewise affine Lipschitz mapping $u : \Omega \to \mathbb{R}^m$ with $u(x) = Ax$ on $\partial\Omega$, $[u - A]_{C^\alpha} < \delta$ and

$$|\{x \in \Omega : Du \in B_r(A_i)\}| = \nu(A_i)|\Omega| \quad \text{for all } i$$

as required.

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