Comments on Classical Kannan Maps

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Abstract

Existence and convergence proofs for classical Kannan maps are extended to cases where the right hand side is enlarged in the condition for the map T:

$$\|Tx - Ty\|^n \leq \frac{\alpha^n}{2}\big(\|Tx - x\|^n + \|Ty - y\|^n\big)$$

where n is a positive integer, or $\alpha > 1$.

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The following considerations have been motivated by observations made at studying, at first, the invention 1968 [5], and the beginning of investigations of Kannan maps more from historical (or, perhaps, nostalgical) interest. On this occation, some simple improvements nearly present themselves which are going into another direction, as it seems, than the generalizations to be found in the literature concerning. e.g., more general spaces, or even multivalued maps (see., e.g., [1, 2, 4, 11]).

The classical Kannan map T in Banach space has the property

$$\|Tx - Ty\| \leq \frac{\alpha}{2}\big(\|Tx - x\| + \|Ty - y\|\big) \tag{1}$$

where $\alpha \leq 1$. Our intention is (similar as in [3] for generalized contractions) to check up wether the right hand side may be enlarged: It seems, e.g., rather likely to replace the arithmetic meam in (1) by the quadratic mean, or to consider parameters α a little greater that 1. Thus we will generalize (1) as follows

$$\|Tx - Ty\|^n \leq \frac{\alpha^n}{2}\big(\|Tx - x\|^n + \|Ty - y\|^n\big) \tag{2}$$

with any positive integer n and possibly, $\alpha > 1$, and we ask to what extent classical existence, and convergence results could be maintained.

First we will point out that in case $\alpha = 1$, the existence of a fixed point can be proved, also for $n > 1$, in the same way as former by P. Soardi [9, 10], S. Reich [8], or R. Kannan [7].

Then we pass to uniformly convex Banach spaces and prove the convergence of fixed point iterations in case $\alpha = 1$ and any n, as well as existence of a fixed point in cases $\alpha > 1$. Finally, these results can clearly be improved in Hilbert spaces.

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Remark. For nonnegative $a \neq b$, the sequence $\left\{ \sqrt[n]{\frac{1}{2}(a^n + b^n)} \right\}$ is strictly ascending, and tends to $\max\{a, b\}$.

For convenience later, we still provide a simple general lemma (similar, e.g., in [12] or [3]).

Lemma. *Let T be a selfmap of a complete metric space $\big(M, |\cdot,\cdot|\big)$, with the property*

$$|Tx, Ty|^n \leq \frac{\alpha^n}{2}\big(|Tx, x|^n + |Ty, y|^n\big), \tag{3}$$

n being a positive integer, and $\alpha \in (0, \sqrt[n]{2})$. Then T has a fixed point z if $\inf_{x \in M} |Tx, x| = 0$, and any sequence $\{x_\nu\}$, with $\lim |Tx_\nu, x_\nu| = 0$, tends to z. z is unique, T is continuous at z, and

$$|Tx_\nu, z| \leq \frac{\alpha}{\sqrt[n]{2}} |Tx_\nu, x_\nu|. \tag{4}$$

Proof. If $\inf |Tx, x| = 0$, for a strictly descending null sequence $\{\varepsilon_\nu\}$, the sets $M_\nu = \{x \in M \ : \ |Tx, x| \leq \varepsilon_\nu\}$ are nonempty, $\operatorname{diam} T(M_\nu) \leq \alpha\varepsilon_\nu$ owing to (3), and $T(M_{\nu+1}) \subseteq T(M_\nu)$. Select $z_\nu \in M : \{Tz_\nu\}$ is a Cauchy sequence with limit z, and $\lim z_\nu = z$ too. From

$$\begin{aligned} |Tz, z| &\leq |Tz, Tz_\nu| + |Tz_\nu, z| \\ &\leq \frac{\alpha}{\sqrt[n]{2}} \sqrt[n]{|Tz, z|^n + |Tz_\nu, z_\nu|^n} + |Tz_\nu, z| \end{aligned}$$

for $\nu \to \infty$ we conclude that $|Tz, z|$ cannot be positive, thus $Tz = z$. (4) is an immediate consequence of (3), and yields, considering

$$|Tz_\nu, z_\nu| \leq |Tz_\nu, z| + |z_\nu, z|$$

the continuity of T at z. The uniqueness of z is evident, and the statement of convergence is supplied by (4) too. $\square$

Theorem 1. *Let C be a nonempty weakly compact convex subset of a Banach space with close-to-normal structure, and $T : C \to C$ be an n-Kannan map, i.e. for $x, y \in C$, (2) is valid, with $\alpha = 1$. Then T has a fixed point.*

We will sketch the proof following, essentially, Soardi's version [10].

Proof. If $\inf_{x \in C} \|Tx - x\| = \rho$, for a strictly descending sequence $\{\rho_\nu\}$ tending to ρ the sets $C_\nu = \big\{x \in C \ : \ \|Tx - x\| \leq \rho_\nu\big\}$ are nonempty, and $C_{\nu+1} \subseteq C_\nu$. Apparently, $\|T^2x - Tx\| \leq \|Tx - x\|$, i.e., $T(C_\nu) \subseteq C_\nu$. By virtue of (2), $\operatorname{diam} T(C_\nu) \leq \rho_n u$. The convex hull

$$\operatorname{coh} T(C_\nu) = \Big\{z = \sum \lambda_i Tx_i \ : \ x_i \in C_\nu,\, \lambda_i \geq 0,\, \sum \lambda_i = 1\Big\}$$

has diameter ρ_ν too ($\sum$ stands for a finite sum). The closed convex hull $D_\nu = \overline{\operatorname{coh}}\, T(C_\nu)$ is contained in C_ν: For any $y \in D_\nu$, and any $\varepsilon > 0$, there is a $z \in \operatorname{coh} T(C_\nu)$ with

$\|z-y\| \le \varepsilon$, and

$$\begin{aligned}\|y-Ty\| &\le \|y-z\| + \left\|\sum \lambda_i Tx_i - Ty\right\| \\ &\le \varepsilon + \sum \lambda_i \|Tx_i - Ty\| \\ &\le \varepsilon + \sum \lambda_i \sqrt[n]{\frac{1}{2}\bigl(\|Tx_i - x_i\|^n + \|Ty-y\|^n\bigr)} \\ &\le \varepsilon + \sqrt[n]{\frac{1}{2}\bigl(\rho_\nu^n + \|Ty-y\|^n\bigr)},\end{aligned}$$

independent of the choice of the x_i. Therefore $\|Ty-y\| > \rho_\nu$ is impossible if ε is sufficiently small, thus $y \in C_\nu$, $D_\nu \subseteq C_\nu$, and $T(D_\nu) \subseteq T(C_\nu) \subseteq D_\nu$. $D := \cap_{\nu+1}^{\infty} D_\nu \neq 0$ because of the weak compactness, $T(D) \subseteq D$, and $\|Ty-y\| = \rho$ for all $y \in D$. On the other hand, $\operatorname{diam} D \le \rho$. Now, in any closed convex subset D with positive diameter ρ, close-to-normal structure claims a $z \in D$ such that $\|x-z\| < \rho$ for all $x \in D$. But the last two statements contradict this condition if $\rho > 0$. Therefore $\rho = 0$ and $D = \{z\}$: $Tz = z$. □

Incidentally, from a part of this proof we can deduce:

Supplement. *If C is a closed convex set of diameter δ in a Banach space which is minimal with respect to an n-Kannan selfmap T as above, then $\|Tx-x\| = \delta$ for all $x \in C$.*

Theorem 2. *Let C be a nonempty closed convex set in a uniformly convex Banach space, and $T : C \to C$ an n-Kannan map. Then there exists $\alpha_0 > 1$ such that T has a fixed point if $\alpha < \alpha_0$ in (2).*

Proof. For $y = Tx$ in (2) we obtain

$$\|T^2x - Tx\|^n \le \frac{\alpha^n}{2-\alpha^n}\|Tx-x\|^n = \beta^n \|Tx-x\|^n. \tag{5}$$

Since in case $\alpha < 1$ clearly $\lim_{\nu\to\infty} \|T^{\nu+1}x - T^\nu x\| = 0$ such a fixed point will exist owing to the lemma, we can assume $\alpha \ge 1$ in the following.

For $z := \frac{1}{2}(Tx + T^2x)$, $x \in C$

$$\|Tz - z\| \le \frac{1}{2}\|Tz - Tx\| + \frac{1}{2}\|Tz - T^2x\|,$$

we obtain, by (2) and (5),

$$\begin{aligned}\|Tz - T^2x\| &\le \alpha \sqrt[n]{\frac{1}{2}\bigl(\|Tz-z\|^n + \|T^2x - Tx\|^n\bigr)} \\ &\le \alpha \sqrt[n]{\frac{1}{2}\bigl(\|Tz-z\|^n + \beta^n\|Tx-x\|^n\bigr)}\end{aligned} \tag{6}$$

and the same holds, all the more, for $\|Tz - Tx\|$. Therefore

$$\|Tz-z\|^n \le \beta^{2n}\|Tx-x\|^n, \quad \beta^n = \frac{\alpha^n}{2-\alpha^n} \tag{7}$$

Inserting this into the former inequalities (6), we get (note $\alpha^n = 2\beta^n/(1+\beta^n)$)

$$\|Tz - T^2x\|^n \leq \frac{\alpha^n}{2}(\beta^{2n} + \beta^n)\|Tx - x\|^n = \beta^{2n}\|Tx - x\|^n \tag{8}$$

which holds also for $\|Tz - Tx\|$.

Let, now, $\inf_{x\in C}\|Tx - x\| = \sigma$. We have to show $\sigma = 0$. Assume, on the contrary, $\sigma > 0$. By uniform convexity, and (6), we have

$$\begin{aligned}\|Tz - z\| &= \frac{1}{2}\|(Tz - Tx) + (Tz - T^2x)\| \\ &\leq \beta^2\|Tx - x\|\Big(1 - \delta\big(\frac{\sigma}{\beta^2\|Tx - x\|}\big)\Big)\end{aligned} \tag{9}$$

where $\delta(\cdot)$ is the modulus of convexity. The equation

$$\varphi(\beta) := \beta^2\Big(1 - \delta\big(\frac{1}{\beta^2}\big)\Big) = 1 \tag{10}$$

has a unique solution $\beta_0 > 1$, because the function φ is strictly increasing and continuous in $[1, \infty)$, $\varphi(1) < 1$ and $\lim_{\beta\to\infty}\varphi(\beta) = \infty$.

For $\beta < \beta_0$, and $\varepsilon > 0$ such that $\beta^2(1+\varepsilon) < \beta_0^2$ too, select $x \in C$ with $\|Tx - x\| \leq (1+\varepsilon)\sigma$. With it, (9) reads

$$\|Tx - z\| \leq \beta^2(1+\varepsilon)\sigma\Big(1 - \delta\big(\frac{1}{(1+\varepsilon)\beta^2}\big)\Big) < \sigma$$

which contradicts the meaning of σ, and therefore $\sigma = 0$. As

$$1 < \alpha_0^n := \frac{2\beta_o^n}{1+\beta_0^n} < 2 \tag{11}$$

for the solution β_0 of (10), a fixed point will exist if $\alpha < \alpha_0$ thanks to the lemma. □

Remark. In Hilbert space, α_0 can be calculated explicitely; for $n = 1$, e.g., $\alpha_0 = 1.0278\ldots$ (for a better value cf. theorem 4 below).

Some considerations in this proof can be used for a simple convergence proof at least in the case $\alpha = 1$, generalizing, e.g., Soardi's result for $n = 1$ concerning the fixed point iteration

$$z_{\nu+1} = \frac{1}{2}\big(Tz_\nu + T^2z_\nu\big). \tag{12}$$

If we set, above, $x = z_\nu$, and $z = z_{\nu+1}$, then (7) states that $\sigma_{\nu+1} = \|Tz_{\nu+1} - z_{\nu+1}\|$ are members of a nonascending sequence, and $\sigma = \lim\sigma_{\nu+1}$ will exist. On the other hand, (8) yields

$$\|Tz_{\nu+1} - T^2z_\nu\| \leq \sigma_\nu, \quad \|Tz_{\nu+1} - Tz_\nu\| \leq \sigma_\nu$$

and considering

$$\lim\|Tz_{\nu+1} - z_{\nu+1}\| = \lim\frac{1}{2}\|(Tz_{\nu+1} - Tz_\nu) + (Tz_{\nu+1} - T^2z_\nu)\| = \sigma$$

we see that $\sigma > 0$ would entail $\lim\|Tz_\nu - T^2z_\nu\| = 0$ owing to uniform convexity. But then (6) reads

$$\|Tz_{\nu+1} - T^2z_\nu\| \leq \frac{\alpha}{\sqrt[n]{2}}\|Tz_{\nu+1} - z_{\nu+1}\| + \varepsilon_\nu, \quad \varepsilon_\nu \to 0$$

and

$$\sigma_{\nu+1} = \|Tz_{\nu+1} - z_{\nu+1}\| \leq \frac{1}{2}\|Tz_{\nu+1} - Tz_\nu\| + \frac{1}{2}\|Tz_{\nu+1} - T^2 z\|$$
$$\leq \frac{1}{2}\sqrt[n]{\frac{1}{2}(\sigma_{\nu+1}^n + \sigma_\nu^n)} + \frac{1}{2}\frac{1}{\sqrt[n]{2}}\sigma_{\nu+1} + \frac{1}{2}\varepsilon_\nu.$$

For $\nu \to \infty$, we would obtain

$$\sigma \leq \frac{1}{2}\sigma + \frac{1}{2}\frac{1}{\sqrt[n]{2}}\sigma = \frac{1}{2}\left(1 + \frac{1}{\sqrt[n]{2}}\right)\sigma$$

being impossible if $\sigma > 0$, and $\{z_\nu\}$ converges as per lemma, such the result of Soardi [10] for $n = 1$ is extended any n:

Theorem 3. *Let C, T be as in theorem 2, with $\alpha = 1$. Then every sequence in C, governed by rule* (12) *above converges to the fixed point.*

Addendum. *In the classical case $n = 1$, $\alpha = 1$ even the more plain iteration rule*

$$x_{\nu+1} = \frac{1}{2}(x_\nu + Tx_\nu) \tag{13}$$

will provide a sequence $\{x_\nu\}$ converging to the fixed point.

Proof. Assume, without loss of generality, $z = Tz = 0$. Using (4)

$$\|Tx_\nu\| \leq \frac{1}{2}\|Tx_\nu - x_\nu\| \leq \frac{1}{2}\big(\|Tx_\nu\| + \|x_\nu\|\big),$$

thus $\|Tx_\nu\| \leq \|x_\nu\|$, and therefore

$$\|x_{\nu+1}\| \leq \frac{1}{2}\big(\|x_\nu\| + \|Tx_\nu\|\big) \leq \|x_\nu\|.$$

Let $\lim \|x_\nu\| = \rho$. Select a subsequence, designated as $\{x_\nu\}$ again, with

$$\lim \|x_\nu\| = \rho, \quad \lim \|Tx_\nu\| = \sigma \leq \rho,$$

and

$$\lim \|x_{\nu+1}\| = \lim \Big\| \frac{1}{2}(x_\nu + Tx_\nu) \Big\| = \rho.$$

By uniform convexity, if $\rho > 0$, we would obtain $\lim \|Tx_\nu - x_\nu\| = 0$, and with it,

$$\lim \|Tx_\nu\| = 0, \quad \lim \|x_\nu\| = 0.$$

Therefore $\rho = 0$: (13) is fixed point iteration. □

Finally, we will pass to n-Kannan maps in Hilbert space where we will confine ourselves to the case $n = 2$ (including $n = 1$) which maybe earliest of interest; besides, it seems to be especially fitting here. This becomes evident in the proof of the following assertion:

Proposition. *Let C be a nonempty closed subset of a Hilbert space, and $T : C \to C$ a 2-Kannan map with $\alpha = 1$, i.e.*

$$\|Tx - Ty\|^2 \leq \frac{1}{2}\left(\|Tx - x\|^2 + \|Ty - y\|^2\right)$$

for $x, y \in C$. Then every sequence $\{x_\nu\}$ formed by the rule (13) *will converge to the fixed point z, and*

$$\|x_{\nu+1} - z\| \le \frac{1}{\sqrt{2}}\|x_\nu - z\|.$$

Proof. The existence of a fixed point z is given by theorem 2. To simplify matters, we will assume $z = 0$. Then for $x = x_\nu$, $y = z = Tz = 0$ (or, from(4)) we have

$$\|Tx_\nu\|^2 \le \frac{1}{2}\|Tx_\nu - x_\nu\|^2.$$

By virtue of the parallelogramm identity we obtain

$$\begin{aligned}\|Tx_\nu - x_\nu\|^2 + \|Tx_\nu + x_\nu\|^2 &= 2\|x_\nu\|^2 + 2\|Tx_\nu\|^2 \\ &\le 2\|x_\nu\|^2 + \|Tx_\nu - x_\nu\|^2\end{aligned}$$

and, with it, already the assertion:

$$\|x_{\nu+1}\|^2 = \left\|\frac{1}{2}(Tx_\nu + x_\nu)\right\|^2 \le \frac{1}{2}\|x_\nu\|^2.$$

□

After this, we may expect results also in case $\alpha > 1$.

Theorem 4. *Let $T : C \to C$ be a selfmap of a nonempty closed convex subset of a Hilbert space, with the property*

$$\|Tx - Ty\|^2 \le \frac{\alpha^2}{2}\left(\|Tx - x\|^2 + \|Ty - y\|^2\right) \tag{14}$$

for $x, y \in C$. If $1 \le \alpha^2 < \alpha_0^2 := \frac{1}{4}(9 - \sqrt{17})$, every sequence $\{z_\nu\}$ set up according rule (12)

$$z_{\nu+1} = \frac{1}{2}(Tz_\nu + T^2 z_\nu)$$

will converge to a fixed point z, and at this the displacements tend to zero as

$$\|Tz_{\nu+1} - z_{\nu+1}\| \le \frac{1}{\sqrt{2}}\frac{\alpha}{2 - \alpha^2}\|Tz_\nu - z_\nu\|. \tag{15}$$

Accordingly,

$$\|z_\nu - z\| \le \left(1 + \frac{\alpha}{\sqrt{2}}\right)\|Tz_\nu - z_\nu\|.$$

Proof. We apply the parallelogram identity to the parallelogramm with vertices Tz_ν, $T^2 z_\nu$, $Tz_{\nu+1}$:

$$\begin{aligned}&\|(Tz_{\nu+1} - Tz_\nu) + (Tz_{\nu+1} - T^2 z_\nu)\|^2 + \|T^2 z_\nu - Tz_\nu\|^2 \\ &\qquad = 2\|Tz_{\nu+1} - Tz_\nu\|^2 + 2\|Tz_{\nu+1} - T^2 z_\nu\|^2 \\ &\qquad \le \alpha^2\left(\|Tz_{\nu+1} - z_{\nu+1}\|^2 + \|Tz_\nu - z_\nu\|^2\right) \\ &\qquad\qquad + \alpha^2\left(\|Tz_{\nu+1} - z_{\nu+1}\|^2 + \|T^2 z_\nu - Tz_\nu\|^2\right)\end{aligned}$$

by (14), therefore

$$\begin{aligned}\|Tz_{\nu+1} - z_{\nu+1}\|^2 &= \|Tz_{\nu+1} - \frac{1}{2}(Tz_\nu + T^2 z_\nu)\|^2 \\ &= \frac{1}{4}\|(Tz_{\nu+1} - Tz_\nu) + (Tz_{\nu+1} - T^2 z_\nu)\|^2 \\ &\le \frac{\alpha^2}{2}\|Tz_{\nu+1} - z_{\nu+1}\|^2 + \frac{\alpha^2}{4}\|Tz_\nu - z_\nu\|^2 + \frac{1}{4}(\alpha^2 - 1)\|T^2 z_\nu - Tz_\nu\|^2 \\ &\le \frac{\alpha^2}{2}\|Tz_{\nu+1} - z_{\nu+1}\|^2 + \frac{1}{4}\frac{\alpha^2}{2-\alpha^2}\|Tz_\nu - z_\nu\|^2,\end{aligned}$$

considering (5), and out of this (15) follows easily. The last inequality in the theorem is given using (4):

$$\begin{aligned}\|z_\nu - z\| &\le \|z_\nu - Tz_\nu\| + \|Tz_\nu - Tz\| \\ &\le \Big(1 + \frac{\alpha}{\sqrt{2}}\Big)\|Tz_\nu - z_\nu\|.\end{aligned}$$

α_0^2 is the solution of $\frac{1}{2}\big(\frac{\alpha}{2-\alpha^2}\big)^2 = 1$; clearly $\alpha_0^2 < 2$. According to the lemma, T has a fixed point. $\square$

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