Trace operators in Besov and Triebel-Lizorkin spaces

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Abstract

We determine the trace of Besov spaces $\mathfrak{B}^s_{p,q}(\mathbb{R}^n)$ and Triebel-Lizorkin spaces $\mathfrak{F}^s_{p,q}(\mathbb{R}^n)$ – characterized via atomic decompositions – on hyperplanes $\mathbb{R}^m$, $n > m \in \mathbb{N}$, for parameters $0 < p, q < \infty$ and $s > \frac{1}{p}$. The limiting case $s = \frac{1}{p}$ is investigated as well. We generalize these assertions to traces on the boundary $\Gamma = \partial\Omega$ of bounded C^k domains Ω. Our results remain valid considering the classical spaces $\mathbf{B}^s_{p,q}$, $\mathbf{F}^s_{p,q}$ – defined via differences. Finally, we include some density assertions, which imply that the trace does not exist when $s < \frac{1}{p}$.

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Introduction

In this article we investigate traces of Besov and Triebel-Lizorkin spaces of positive smoothness – sometimes briefly denoted as B- and F-spaces in the sequel. Besov spaces have been studied for many decades already, resulting, for instance, from the study of partial differential equations, interpolation theory, approximation theory, harmonic analysis. Triebel-Lizorkin spaces were introduced independently by Triebel and Lizorkin in the early 1970s. For a detailed treatment together with historical remarks we refer to Triebel [Tri83, Tri92]. If

$$0 < p, q < \infty \qquad \text{and} \qquad s > \frac{1}{p} + \max\left(0, (n-1)\left(\frac{1}{p} - 1\right)\right)$$

the trace of these spaces has been known to be a Besov space for a long time, cf. [Tri83, Sect. 2.7.2]. Since modern subatomic characterizations admit new insights into the nature of these spaces, we are now able to extend these results to $s > \frac{1}{p}$. We deal with the most recent definition $\mathfrak{B}^s_{p,q}(\mathbb{R}^n)$ and $\mathfrak{F}^s_{p,q}(\mathbb{R}^n)$ relying on *atomic decompositions* containing those $f \in L_p(\mathbb{R}^n)$ which can be represented as

$$f(x) = \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{j,m} a_{j,m}(x), \quad x \in \mathbb{R}^n, \tag{0.1}$$

with coefficients $\lambda = \{\lambda_{j,m} \in \mathbb{C} : j \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$ belonging to some appropriate sequence spaces $b^s_{p,q}$ and $f^s_{p,q}$, respectively. In particular, $s > 0$, $0 < p \le \infty$ ($p < \infty$ for the F-spaces), $0 < q \le \infty$, and $a_{j,m}(x)$ are normalized atoms. Furthermore,

$$\|f|\mathfrak{B}^s_{p,q}(\mathbb{R}^n)\| := \inf \|\lambda|b^s_{p,q}\| \qquad \text{and} \qquad \|f|\mathfrak{F}^s_{p,q}(\mathbb{R}^n)\| := \inf \|\lambda|f^s_{p,q}\|$$

where the infimum is taken over all admissible representations (0.1).

Our results naturally extend the ones previously known, i.e. concerning traces on the hyperplane $\mathbb{R}^{n-1}$ we prove for $s > \frac{1}{p}$,

$$\mathrm{Tr}\, \mathfrak{B}^s_{p,q}(\mathbb{R}^n) = \mathfrak{B}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})$$

and

$$\mathrm{Tr}\, \mathfrak{F}^s_{p,q}(\mathbb{R}^n) = \mathfrak{B}^{s-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1}).$$

In the limiting case $s = \frac{1}{p}$ we obtain

$$\mathrm{Tr}\, \mathfrak{B}^{1/p}_{p,q}(\mathbb{R}^n) = L_p(\mathbb{R}^{n-1}), \qquad 0 < q \le \min(p, 1)$$

and

$$\mathrm{Tr}\,\mathfrak{F}^{1/p}_{p,q}(\mathbb{R}^n) = L_p(\mathbb{R}^{n-1}), \qquad 0 < p < 1.$$

Our results may be extended to more general hyperplanes $\mathbb{R}^m$, $n > m \in \mathbb{N}$. Furthermore, we introduce Besov spaces $\mathfrak{B}^s_{p,q}(\Omega)$ and Triebel-Lizorkin spaces $\mathfrak{F}^s_{p,q}(\Omega)$ on domains $\Omega \subset \mathbb{R}^n$. We investigate traces on the boundary $\Gamma = \partial\Omega$ when Ω is a C^k domain and obtain corresponding results.
In particular, all our trace results for Besov spaces $\mathfrak{B}^s_{p,q}$ remain valid for the classical Besov spaces $\mathbf{B}^s_{p,q}$ as well. With some restrictions on the parameters this is also true for Triebel-Lizorkin spaces $\mathbf{F}^s_{p,q}$.
We conclude with the observation that the spaces $\mathfrak{B}^s_{p,q}(\mathbb{R}^n)$ and $\mathfrak{F}^s_{p,q}(\mathbb{R}^n)$ either have a trace in $L_p(\mathbb{R}^{n-1})$ or the collection of all C^∞ functions in $\mathbb{R}^n$ with compact support in $\mathbb{R}^n \setminus \mathbb{R}^{n-1}$ is dense in them. Related dichotomy numbers are introduced and calculated. Similar for C^k domains Ω with boundary Γ.

The paper is organized as follows. In Section 1 we present two different approaches to Besov and Triebel-Lizorkin spaces of positive smoothness and briefly discuss their connection. In Section 2 we recall the concept of how to understand traces on hyperplanes $\mathbb{R}^m$ in these function spaces defined on $\mathbb{R}^n$. With the help of the atomic approach we derive our main results for B- and F-spaces, when $s > \frac{1}{p}$ as well as for the limiting case $s = \frac{1}{p}$. These results are extended to traces on the boundary $\Gamma = \partial\Omega$ of bounded C^k domains Ω in Section 3. Finally, Section 4 contains an interesting assertion concerning the density of C^∞ functions with compact support in $\mathbb{R}^n \setminus \mathbb{R}^m$ and in $\Omega \setminus \Gamma$ in the spaces defined on $\mathbb{R}^n$ and Ω, respectively, when $s < \frac{1}{p}$.

1 Besov and Triebel-Lizorkin spaces with positive smoothness (on $\mathbb{R}^n$)

We use standard notation. Let $\mathbb{N}$ be the collection of all natural numbers and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathbb{R}^n$ be euclidean n-space, $n \in \mathbb{N}$, $\mathbb{C}$ the complex plane. The set of multi-indices $\beta = (\beta_1, \ldots, \beta_n)$, $\beta_i \in \mathbb{N}_0$, $i = 1, \ldots, n$, is denoted by $\mathbb{N}_0^n$, with $|\beta| = \beta_1 + \cdots + \beta_n$, as usual. Moreover, if $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}_0^n$ we put $x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n}$.
We use the equivalence '$\sim$' in

$$a_k \sim b_k \quad \text{or} \quad \varphi(x) \sim \psi(x)$$

always to mean that there are two positive numbers c_1 and c_2 such that

$$c_1\, a_k \le b_k \le c_2\, a_k \quad \text{or} \quad c_1\, \varphi(x) \le \psi(x) \le c_2\, \varphi(x)$$

for all admitted values of the discrete variable k or the continuous variable x, where $\{a_k\}_k$, $\{b_k\}_k$ are non-negative sequences and φ, ψ are non-negative functions. If $a \in \mathbb{R}$, then $a_+ := \max(a, 0)$ and $[a]$ denotes the integer part of a.
Given two (quasi-) Banach spaces X and Y, we write $X \hookrightarrow Y$ if $X \subset Y$ and the natural embedding of X in Y is continuous. All unimportant positive constants will be denoted by c, occasionally with subscripts. For convenience, let both $\mathrm{d}x$ and $|\cdot|$ stand for the (n-dimensional) Lebesgue measure in the sequel.

Let $Q_{j,m}$ with $j \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$ denote a cube in $\mathbb{R}^n$ with sides parallel to the axes of coordinates, centered at $2^{-j}m$, and with side length 2^{-j+1}. For a cube Q in $\mathbb{R}^n$ and $r > 0$, we denote by rQ the cube in $\mathbb{R}^n$ concentric with Q and with side length r times the side length of Q. Furthermore, $\chi_{j,m}$ stands for the characteristic function of $Q_{j,m}$.

1.1 Definitions and basic properties

We give an atomic characterization of Besov and Triebel-Lizorkin spaces $\mathfrak{B}^s_{p,q}(\mathbb{R}^n)$ and $\mathfrak{F}^s_{p,q}(\mathbb{R}^n)$. This provides a constructive definition expanding functions f via atoms – excluding any moment conditions – and suitable coefficients, where the latter belong to certain sequence spaces denoted by $b^s_{p,q}$ and $f^s_{p,q}$.

According to [Tri06, Prop. 9.14] based on [HN07], it turns out that these spaces essentially coincide with the well-known classical Besov and Triebel-Lizorkin spaces $\mathbf{B}^s_{p,q}(\mathbb{R}^n)$ and $\mathbf{F}^s_{p,q}(\mathbb{R}^n)$ – defined via differences.

First we introduce the relevant sequence spaces.

Definition 1.1 *Let* $0 < p \leq \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$, *and* $\lambda = \{\lambda_{j,m} \in \mathbb{C} : j \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$.

(i) *Then*

$$b^s_{p,q} = \left\{ \lambda : \|\lambda | b^s_{p,q}\| = \left(\sum_{j=0}^{\infty} 2^{j(s-\frac{n}{p})q} \left(\sum_{m \in \mathbb{Z}^n} |\lambda_{j,m}|^p \right)^{q/p} \right)^{1/q} < \infty \right\}$$

(*with the usual modification if* $p = \infty$ *and/or* $q = \infty$).

(ii) *Furthermore*

$$f^s_{p,q} = \left\{ \lambda : \|\lambda | f^s_{p,q}\| = \left\| \left(\sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} 2^{jsq} |\lambda_{j,m}|^q \chi_{j,m}(\cdot) \right)^{1/q} |L_p \right\| < \infty \right\}$$

Now we define the atoms.

Definition 1.2 *Let* $K \in \mathbb{N}_0$ *and* $d > 1$. *A* K*-times differentiable complex-valued function* a *on* $\mathbb{R}^n$ (*continuous if* $K = 0$) *is called a* 1_K*-atom if for some* $j \in \mathbb{N}_0$

$$\operatorname{supp} a \subset dQ_{j,m} \quad \textit{for some} \quad m \in \mathbb{Z}^n, \tag{1.1}$$

and

$$|\mathrm{D}^\alpha a(x)| \leq 2^{|\alpha| j} \quad \textit{for} \quad |\alpha| \leq K. \tag{1.2}$$

It is convenient to write $a_{j,m}(x)$ instead of $a(x)$ if this atom is located at $Q_{j,m}$ according to (1.1). Furthermore, K denotes the smoothness of the atom, cf. (1.2). The atomic characterization of function spaces of type $\mathfrak{B}^s_{p,q}(\mathbb{R}^n)$, $\mathfrak{F}^s_{p,q}(\mathbb{R}^n)$ is given below.

Definition 1.3 *Let* $0 < p \leq \infty$, $0 < q \leq \infty$, *and* $s > 0$. *Let* $d > 1$ *and* $K \in \mathbb{N}_0$ *with*

$$K \geq (1 + [s])$$

be fixed.

(i) *Then* $f \in L_p(\mathbb{R}^n)$ *belongs to* $\mathfrak{B}^s_{p,q}(\mathbb{R}^n)$ *if, and only if, it can be represented as*

$$f = \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{j,m} a_{j,m}(x), \quad \textit{convergence being in} \quad L_p(\mathbb{R}^n), \tag{1.3}$$

where the $a_{j,m}$ *are* 1_K*-atoms* $(j \in \mathbb{N}_0)$ *with*

$$\operatorname{supp} a_{j,m} \subset dQ_{j,m}, \qquad j \in \mathbb{N}_0, \quad m \in \mathbb{Z}^n,$$

and $\lambda \in b^s_{p,q}$. *Furthermore,*

$$\inf \|\lambda | b^s_{p,q}\|,$$

where the infimum is taken over all admissible representations (1.3), *is an equivalent quasi-norm in* $\mathfrak{B}^s_{p,q}(\mathbb{R}^n)$.

(ii) *In addition,* $f \in L_p(\mathbb{R}^n)$ *belongs to* $\mathfrak{F}^s_{p,q}(\mathbb{R}^n)$ *if, and only if, it can be represented as*

$$f = \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{j,m} a_{j,m}(x), \quad \textit{convergence being in} \quad L_p(\mathbb{R}^n), \tag{1.4}$$

where the $a_{j,m}$ *are* 1_K*-atoms* $(j \in \mathbb{N}_0)$ *with*

$$\operatorname{supp} a_{j,m} \subset dQ_{j,m}, \qquad j \in \mathbb{N}_0, \quad m \in \mathbb{Z}^n,$$

and $\lambda \in f^s_{p,q}$. *Furthermore,*

$$\inf \|\lambda | f^s_{p,q}\|,$$

where the infimum is taken over all admissible representations (1.4), *is an equivalent quasi-norm in* $\mathfrak{F}^s_{p,q}(\mathbb{R}^n)$.

Remark 1.4 Moreover, the atomic approaches for B- and F-spaces are strongly linked with the *classical approaches* which introduce $\mathbf{B}^s_{p,q}(\mathbb{R}^n)$ and $\mathbf{F}^s_{p,q}$ as those subspaces of $L_p(\mathbb{R}^n)$ such that

$$\|f|\mathbf{B}^s_{p,q}(\mathbb{R}^n)\|_r = \|f|L_p(\mathbb{R}^n)\| + \left(\int_0^1 t^{-sq}\omega_r(f,t)_p^q \,\frac{\mathrm{d}t}{t}\right)^{1/q}$$

and

$$\|f|\mathbf{F}^s_{p,q}(\mathbb{R}^n)\|_r = \|f|L_p(\mathbb{R}^n)\| + \left\|\left(\int_0^1 t^{-sq} d^r_{t,p}f(\cdot)^q \frac{\mathrm{d}t}{t}\right)^{1/q} |L_p(\mathbb{R}^n)\right\|$$

are finite, respectively, where $0 < p \leq \infty$, ($p < \infty$ for F-spaces), $0 < q \leq \infty$ (with the usual modification if $q = \infty$), $s > 0$, $r \in \mathbb{N}$ with $r > s$. Here $\omega_r(f,t)_p$ stands for the usual r-th modulus of smoothness of a function $f \in L_p(\mathbb{R}^n)$,

$$\omega_r(f,t)_p = \sup_{|h|\leq t} \|\Delta^r_h f \mid L_p(\mathbb{R}^n)\|, \quad t > 0,$$

and $d^r_{t,p}f(\cdot)$ denotes the ball means of $f \in L_p(\mathbb{R}^n)$,

$$d^r_{t,p}f(x) = \left(t^{-n}\int_{|h|\leq t} |(\Delta^r_h f)(x)|^p \mathrm{d}h\right)^{1/p}, \quad x \in \mathbb{R}^n, \quad t > 0,$$

where

$$(\Delta^1_h f)(x) = f(x+h) - f(x) \quad \text{and} \quad (\Delta^{r+1}_h f)(x) = \Delta^1_h(\Delta^r_h f)(x), \qquad h \in \mathbb{R}^n.$$

Recent results by Hedberg, Netrusov [HN07] on atomic decompositions and by Triebel [Tri06, Sect. 9.2] on the reproducing formula prove coincidences

$$\mathbf{B}^s_{p,q}(\mathbb{R}^n) = \mathfrak{B}^s_{p,q}(\mathbb{R}^n), \qquad s > 0, \quad 0 < p, q \leq \infty,$$

and

$$\mathbf{F}^s_{p,q}(\mathbb{R}^n) = \mathfrak{F}^s_{p,q}(\mathbb{R}^n), \qquad s > n\left(\frac{1}{\min(p,q)} - \frac{1}{p}\right), \quad 0 < p < \infty, \quad 0 < q \leq \infty,$$

(in terms of equivalent quasi-norms).
In particular, this implies that all our results for Besov spaces $\mathfrak{B}^s_{p,q}(\mathbb{R}^n)$ could as well be stated in terms of the classical spaces $\mathbf{B}^s_{p,q}(\mathbb{R}^n)$. The same is true for the F-spaces with the above restriction on the parameter s.

1.2 Embeddings

We recall some embeddings for the spaces $\mathfrak{B}^s_{p,q}(\mathbb{R}^n)$, $\mathfrak{F}^s_{p,q}(\mathbb{R}^n)$ that were proven in [HS08, Sch08] and which will subsequently be needed. Let $\mathfrak{A} \in \{\mathfrak{B}, \mathfrak{F}\}$.

Proposition 1.5 *Let* $s > 0$, $0 < p \leq \infty$ ($p < \infty$ *for F-spaces*), $0 < q \leq \infty$.

(i) *Let* $\varepsilon > 0$, $0 < u \leq \infty$, *and* $q \leq v \leq \infty$. *Then*

$$\mathfrak{A}^{s+\varepsilon}_{p,u}(\mathbb{R}^n) \hookrightarrow \mathfrak{A}^s_{p,q}(\mathbb{R}^n) \qquad \textit{and} \qquad \mathfrak{A}^s_{p,q}(\mathbb{R}^n) \hookrightarrow \mathfrak{A}^s_{p,v}(\mathbb{R}^n). \tag{1.5}$$

(ii) *Let* $0 < p_0 < p < p_1 \leq \infty$, $s_0, s_1 > 0$ *such that*

$$s_0 - \frac{n}{p_0} = s - \frac{n}{p} = s_1 - \frac{n}{p_1}, \tag{1.6}$$

and $0 < q, u, v \leq \infty$. *If*

$$0 < u \leq p \leq v \leq \infty, \tag{1.7}$$

then

$$\mathfrak{B}^{s_0}_{p_0,u}(\mathbb{R}^n) \hookrightarrow \mathfrak{F}^s_{p,q}(\mathbb{R}^n) \hookrightarrow \mathfrak{B}^{s_1}_{p_1,v}(\mathbb{R}^n). \tag{1.8}$$

In terms of boundedness we have the following results which may be also be found in [HS08, Sch08].

Proposition 1.6 *Let* $0 < p \leq \infty$ (*with* $p < \infty$ *for F-spaces*), *and* $0 < q \leq \infty$. *Then*

$$\mathfrak{F}^{n/p}(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n) \qquad \text{if, and only if,} \qquad 0 < p \leq 1, \quad 0 < q \leq \infty, \tag{1.9}$$

and

$$\mathfrak{B}^{n/p}_{p,q}(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n) \qquad \text{if, and only if,} \qquad 0 < p < \infty, \quad 0 < q \leq 1, \tag{1.10}$$

where L_∞ *in* (1.9) *and* (1.10) *can be replaced by* C.

Moreover, by (1.5) and (1.8) we obtain

$$\mathfrak{A}^s_{p,q}(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n), \qquad s > \frac{n}{p}, \quad 0 < p, q \leq \infty, \tag{1.11}$$

(with $p < \infty$ if $\mathfrak{A} = \mathfrak{F}$), where L_∞ can be replaced by C, too.

2 Traces on hyperplanes in $\mathbb{R}^n$

Let $\mathfrak{A}^s_{p,q}$ denote one of the spaces $\mathfrak{B}^s_{p,q}$ or $\mathfrak{F}^s_{p,q}$. We briefly explain our understanding of the trace operator on hyperplanes, since when dealing with L_p functions the pointwise trace has no obvious meaning. If $x = (x_1, \ldots, x_n)$ put $x' = (x_1, \ldots, x_{n-1})$. We ask for the trace of $f \in \mathfrak{A}^s_{p,q}(\mathbb{R}^n)$ on the hyperplane

$$\mathbb{R}^{n-1} = \{x \in \mathbb{R}^n : x = (x', 0)\}.$$

A clarification of this problem is of crucial interest for boundary value problems of elliptic differential operators.
Obviously, any $f \in S(\mathbb{R}^n)$ has a pointwise trace

$$(\mathrm{Tr}\, f)(x) := (\mathrm{Tr}\,_{\{x_n=0\}} f)(x) = f(x', 0) \quad \text{on} \quad \mathbb{R}^{n-1}.$$

Nevertheless, if $f \in \mathfrak{A}^s_{p,q}(\mathbb{R}^n)$ the trace operator

$$\mathrm{Tr}\, : \mathfrak{A}^s_{p,q}(\mathbb{R}^n) \hookrightarrow \mathfrak{A}^s_{p,q}(\mathbb{R}^{n-1})$$

is to be understood in the following sense. One asks, whether there is a constant $c > 0$ such that

$$\|\varphi(\cdot, 0)|\mathfrak{A}^s_{p,q}(\mathbb{R}^{n-1})\| \leq c\|\varphi|\mathfrak{A}^s_{p,q}(\mathbb{R}^n)\|, \qquad \text{for all} \quad \varphi \in S(\mathbb{R}^n). \tag{2.1}$$

Since the embedding $S(\mathbb{R}^n) \hookrightarrow \mathfrak{A}^s_{p,q}(\mathbb{R}^n)$ is dense for $0 < p, q < \infty$ one approximates $f \in \mathfrak{A}^s_{p,q}(\mathbb{R}^n)$ by $\varphi_j \in S(\mathbb{R}^n)$, where $j \in \mathbb{N}$. If one has (2.1), then $\{\varphi_j(x', 0)\}_{j=1}^\infty$ is a Cauchy sequence in $\mathfrak{A}^s_{p,q}(\mathbb{R}^{n-1})$. Its limit element – which by (2.1) is independent of the approximating sequence $\{\varphi_j\}_{j=1}^\infty \subset S(\mathbb{R}^n)$ – is called the *trace* of $f \in \mathfrak{A}^s_{p,q}(\mathbb{R}^n)$ and denoted by $\mathrm{Tr}\, f$. Completion implies

$$\|\mathrm{Tr}\, f|\mathfrak{A}^s_{p,q}(\mathbb{R}^{n-1})\| \leq c\|f|\mathfrak{A}^s_{p,q}(\mathbb{R}^n)\|, \qquad f \in \mathfrak{A}^s_{p,q}(\mathbb{R}^n),$$

and

$$\mathrm{Tr}\, : \mathfrak{A}^s_{p,q}(\mathbb{R}^n) \hookrightarrow \mathfrak{A}^s_{p,q}(\mathbb{R}^{n-1})$$

is a linear and bounded operator.

Remark 2.1 We can extend (2.1) to spaces $\mathfrak{A}^s_{p,q}(\mathbb{R}^n)$ with $p = \infty$ and/ or $q = \infty$ in the following way. If $p = \infty$, then by (1.11), $\mathfrak{B}^s_{\infty,q}(\mathbb{R}^n)$ with $s > 0$ is embedded in the space of continuous functions and Tr makes sense pointwise. If $q = \infty$, then one has by (1.5)

$$\mathfrak{A}^s_{p,\infty}(\mathbb{R}^n) \hookrightarrow \mathfrak{A}^{s-\varepsilon}_{p,1}(\mathbb{R}^n) \qquad \text{for any} \quad \varepsilon > 0.$$

Let $s > \frac{1}{p}$ and $\varepsilon > 0$ be small enough such that one has

$$s > s - \varepsilon > \frac{1}{p}.$$

Since by [Tri08a, Rem. 13] traces are independent of the source spaces and of the target spaces one can now define Tr for $\mathfrak{A}^s_{p,\infty}(\mathbb{R}^n)$ by restriction of Tr for $\mathfrak{A}^{s-\varepsilon}_{p,1}(\mathbb{R}^n)$ to $\mathfrak{A}^s_{p,\infty}(\mathbb{R}^n)$. Hence (2.1) is always meaningful.

2.1 The trace problem in $\mathfrak{B}^s_{p,q}(\mathbb{R}^n)$

Our main result concerning traces in Besov spaces on hyperplanes in $\mathbb{R}^n$ is stated below.

Theorem 2.2 *Let $n \geq 2$, $0 < p, q \leq \infty$, and $s - \frac{1}{p} > 0$. Then $\operatorname{Tr}_{x_n=0} = \operatorname{Tr}$ is a linear and bounded operator from $\mathfrak{B}^s_{p,q}(\mathbb{R}^n)$ onto $\mathfrak{B}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})$,*

$$\operatorname{Tr} \mathfrak{B}^s_{p,q}(\mathbb{R}^n) = \mathfrak{B}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1}).$$

Proof : Our constructions follow closely [FJ85, Sect. 5].
Step 1. By Definition 1.3 every $f \in \mathfrak{B}^s_{p,q}$ has an optimal atomic decomposition of the form

$$f(x) = \sum_{j \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda_{j,m} a_{j,m}(x), \qquad x \in \mathbb{R}^n,$$

with

$$\|f|\mathfrak{B}^s_{p,q}\| \sim \|\lambda|b^s_{p,q}\|.$$

In this step we wish to prove that

$$\operatorname{Tr} \mathfrak{B}^s_{p,q}(\mathbb{R}^n) \subset \mathfrak{B}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1}). \tag{2.2}$$

According to (2.1) and the explanations given thereafter we may restrict ourselves to smooth functions f. For $f \in S(\mathbb{R}^n)$ and the trace operator

$$\operatorname{Tr} f(x) = f(x', 0),$$

assumption (2.2) is equivalent to

$$\|f(\cdot, 0)|\mathfrak{B}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})\| \leq c\|f|\mathfrak{B}^s_{p,q}(\mathbb{R}^n)\|, \tag{2.3}$$

for some $c > 0$ independent of $f \in \mathfrak{B}^s_{p,q}$.
We put

$$\mathbb{R}^{n-1} := \{x = (x', x_n) \in \mathbb{R}^n : x_n = 0\}.$$

Considering the trace operator we see that for $m = (m', m_n) \in \mathbb{Z}^n$

$$\operatorname{Tr} f(x) = f(x', 0) = \sum_j \sum_{m'} \sum_{m_n \in I} \lambda_{j,(m',m_n)} a_{j,(m',m_n)}(x', 0),$$

where for fixed j we only sum over a finite index set $I = I(j, m')$ (depending on $d > 1$) with

$$\operatorname{supp} a_{j,(m',m_n)} \cap \mathbb{R}^{n-1} \neq \emptyset \qquad \text{if} \qquad m_n \in I.$$

We define new atoms via

$$b_{j,m'}(x') := \begin{cases} \dfrac{\sum_{m_n \in I} \lambda_{j,(m',m_n)} a_{j,(m',m_n)}(x', 0)}{\sum_{m_n \in I} |\lambda_{j,(m',m_n)}|} & \text{, if } \displaystyle\sum_{m_n \in I} |\lambda_{j,(m',m_n)}| \neq 0, \\ 0 & \text{, otherwise.} \end{cases}$$

For our construction only atoms $a_{j,m}$ are of interest with supports in cubes $dQ_{j,m}$, for which $dQ_{j,m}$ has a non-empty intersection with the hyperplane $\mathbb{R}^{n-1}$.
Let $Q = Q_{j,m}$ be one of these cubes and let Q' be the projection of Q on that hyperplane (now being identified with $\mathbb{R}^{n-1}$), i.e. $Q' = Q'_{j,m'}$. Furthermore

$$\eta_{j,m'} := \sum_{m_n \in I} |\lambda_{j,(m',m_n)}|, \qquad j \in \mathbb{N}_0, \quad m' \in \mathbb{Z}^{n-1}.$$

The restriction (or trace) of f to $\mathbb{R}^{n-1}$ is now

$$\operatorname{Tr} f(x) = f(x', 0) = \sum_j \sum_{m'} \eta_{j,m'} b_{j,m'}(x') \tag{2.4}$$

whenever the sum converges. In fact, we have convergence in $L_p(\mathbb{R}^{n-1})$ for all $f \in \mathfrak{B}^s_{p,q}(\mathbb{R}^n)$, cf. Step 2 below.
We show that $b_{j,m'}$ represent suitable atoms according to Definition 1.2. Observe that $b_{j,m'}$ are again $C^K(\mathbb{R}^{n-1})$ functions that additionally satisfy

$$\operatorname{supp} b_{j,m'} \subset \left(\bigcup_{m_n \in I} dQ_{j,(m',m_n)} \right) \cap \mathbb{R}^{n-1} = dQ'_{j,m'}, \tag{2.5}$$

and for $\alpha' \in \mathbb{N}_0^{n-1}$

$$\begin{aligned} |\mathrm{D}^{\alpha'} b_{j,m'}(x')| &\leq \frac{|\sum_{m_n} \lambda_{j,m} \mathrm{D}^{(\alpha',0)} a_{j,m}(x',0)|}{\sum_{m_n} |\lambda_{j,m}|} \\ &\leq \frac{2^{j|\alpha'|} \sum_{m_n} |\lambda_{j,m}|}{\sum_{m_n} |\lambda_{j,m}|} \\ &= 2^{j|\alpha'|}, \qquad |\alpha'| \leq K, \end{aligned} \tag{2.6}$$

which establishes that $b_{j,m'}$ is a suitable atom for our representation of $\operatorname{Tr} f$ in $\mathfrak{B}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})$.
Using our new coefficients $\eta = \{\eta_{j,m'}\}_{j,m'}$ we calculate for the norm

$$\begin{aligned} \|\operatorname{Tr} f | \mathfrak{B}^{s-\frac{1}{p}}_{pq}(\mathbb{R}^{n-1})\| &\leq \|\eta | b^{s-\frac{1}{p}}_{p,q}\| \\ &= \left(\sum_j 2^{j\left((s-\frac{1}{p})-\frac{n-1}{p}\right)q} \left(\sum_{m'} |\eta_{j,m'}|^p \right)^{q/p} \right)^{1/q} \\ &= \left(\sum_j 2^{j(s-\frac{n}{p})q} \left(\sum_{m'} \Big| \sum_{m_n \in I} |\lambda_{j,m}| \Big|^p \right)^{q/p} \right)^{1/q} \\ &\leq c \left(\sum_j 2^{j(s-\frac{n}{p})q} \left(\sum_{m'} \sum_{m_n \in I} |\lambda_{j,m}|^p \right)^{q/p} \right)^{1/q} \\ &\leq c' \left(\sum_j 2^{j(s-\frac{n}{p})q} \left(\sum_{m} |\lambda_{j,m}|^p \right)^{q/p} \right)^{1/q} \\ &\sim \|f | \mathfrak{B}^s_{p,q}(\mathbb{R}^n)\| \end{aligned}$$

(with obvious modifications if $p = \infty$ and/or $q = \infty$), where the sequence spaces $b^{s-\frac{1}{p}}_{p,q}$ are defined according to Definition 1.1(i) with index set in $\mathbb{N}_0 \times \mathbb{Z}^{n-1}$. We used in the 4th line, that the cardinality of the index set $I = I(j,m')$ is actually independent of j, m'. This proves (2.3).

Step 2. The existence, or non-existence, of the trace of $\mathfrak{B}^s_{p,q}(\mathbb{R}^n)$ is equivalent to the question whether we can make sense of the sums in (2.4) whenever (2.5) and (2.6) hold, since any such expression can arise from a suitable $f \in \mathfrak{B}^s_{p,q}(\mathbb{R}^n)$.
If $p \geq 1$ it is known from older results that for $s - \frac{1}{p} > 0$ the sums in (2.4) always converge in L_p (in particular in S') and therefore the trace exists, cf. [Tri83, Sect. 2.7.2].
Suppose now $0 < p < 1$. Then (2.4) does converge in L_p (but not necessarily in S'), which may be seen calculating

$$\begin{aligned} \left\| \sum_{j=0}^{\infty} \sum_{m'} \eta_{j,m'} b_{j,m'}(x') | L_p(\mathbb{R}^{n-1}) \right\|^p &\leq \sum_j \sum_{m'} |\eta_{j,m'}|^p \|b_{j,m'} | L_p(\mathbb{R}^{n-1})\|^p \\ &\leq \sum_j \sum_{m'} |\eta_{j,m'}|^p \int_{dQ'_{j,m'}} |b_{j,m'}(x')|^p \mathrm{d}x' \\ &\leq \sum_j \sum_{m'} |\eta_{j,m'}|^p |dQ'_{j,m'}| \end{aligned}$$

$$
\begin{aligned}
&\leq c\sum_j\sum_{m'}|\eta_{j,m'}|^p 2^{-j(n-1)}\\
&= c\sum_j 2^{-j(n-1)}\sum_{m'}\Big|\sum_{m_n\in I}|\lambda_{j,m}|\Big|^p\\
&\leq c'\sum_j 2^{-j(n-1)}\sum_m|\lambda_{j,m}|^p\\
&= c'\sum_j 2^{-j(sp-1)}2^{j(sp-1)}2^{-j(n-1)}\sum_m|\lambda_{j,m}|^p\\
&\leq c''\left(\sum_j 2^{j((sp-1)-(n-1))\frac{q}{p}}\left(\sum_m|\lambda_{j,m}|^p\right)^{q/p}\right)^{p/q}\\
&\sim \|f|\mathfrak{B}^s_{p,q}(\mathbb{R}^n)\|^p,
\end{aligned}
$$

where in the second but last line when $\frac{q}{p}\leq 1$ we can use the embedding $\ell_{\frac{q}{p}}\hookrightarrow \ell_1$, and in the case $\frac{q}{p}>1$ an application of Hölder's inequality gives the desired result, since $sp-1>0$. This proves the convergence in L_p.

Step 3. It is fairly easy to see that the trace map Tr is onto $\mathfrak{B}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})$, since any 1_K-atom $b_{j,m'}\in C^K(\mathbb{R}^{n-1})$ satisfying (1.1), (1.2) can be obtained as the restriction of a 1_K-atom $a_{j,m}\in C^K(\mathbb{R}^n)$ (simply construct $a_{j,m}$ by multiplying $b_{j,m'}$ with a suitable 1_K-atom b_{j,m_n} defined on $\mathbb{R}$ with $b_{j,m_n}(0)=1$).

To establish the extension property we show that for given $g\in\mathfrak{B}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})$ there exists a function $f\in\mathfrak{B}^s_{p,q}(\mathbb{R}^n)$ with

$$
f(x',0)=g(x') \qquad \text{and} \qquad \|f|\mathfrak{B}^s_{p,q}(\mathbb{R}^n)\|\leq c\|g|\mathfrak{B}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})\|.
$$

In order to obtain a bounded extension operator

$$
\mathrm{Ex}:\mathfrak{B}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})\to\mathfrak{B}^s_{p,q}(\mathbb{R}^n), \qquad (\mathrm{Ex}\,g)(x)=f(x),
$$

with

$$
\mathrm{Tr}\circ\mathrm{Ex}=\mathrm{id}_{\mathfrak{B}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})},
$$

let $g\in\mathfrak{B}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})$ with optimal atomic decomposition, i.e.,

$$
g(x')=\sum_j\sum_{m'}\lambda_{j,m'}b_{j,m'}(x') \qquad \text{and} \qquad \|g|\mathfrak{B}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})\|\sim\|\lambda|b^{s-\frac{1}{p}}_{p,q}\|. \tag{2.7}
$$

We set

$$
f(x',x_n)=\sum_j\sum_{m'}\sum_{m_n}\lambda_{j,m}a_{j,m}(x',x_n),
$$

with coefficients

$$
\lambda_{j,m}=\begin{cases}\lambda_{j,m'}, & m_n=0\\ 0, & m_n\neq 0\end{cases},
$$

and

$$
a_{j,m}(x',x_n)=b_{j,m'}(x')b_{j,m_n}(x_n),
$$

where b_{j,m_n} are 1_K-atoms according to Definition 1.2 satisfying

$$
b_{j,m_n}(0)=1 \qquad \text{and} \qquad \operatorname{supp} b_{j,m_n}\subset[2^{-j}(m_n-1),2^{-j}(m_n+1)].
$$

Therefore $a_{j,m}$ are 1_K-atoms and we see that

$$
f(x',0)=g(x'), \qquad x'\in\mathbb{R}^{n-1}.
$$

Furthermore, we estimate

$$\|f|\mathfrak{B}^s_{p,q}(\mathbb{R}^n)\| \le \left(\sum_j 2^{j(s-\frac{n}{p})q}\left(\sum_m |\lambda_{j,m}|^p\right)^{q/p}\right)^{1/q}$$
$$= \left(\sum_j 2^{j\left[(s-\frac{1}{p})-\frac{n-1}{p}\right]q}\left(\sum_{m'} |\lambda_{j,m'}|^p\right)^{q/p}\right)^{1/q} \sim \|g|\mathfrak{B}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})\|.$$

Hence we have established the existence of a bounded (but not linear – cf. Remark 2.3) extension operator Ex from the trace space into the original space, which finally completes the proof.

□

Remark 2.3 Note that the constructed extension operator in Step 3 is bounded but not linear when $s-\frac{1}{p}<(n-1)\left(\frac{1}{p}-1\right)$. In [Tri08b, Ch. 6.2] it is shown that in this case

$$\left(\mathfrak{B}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})\right)' = \{0\}, \qquad s-\frac{1}{p}<(n-1)\left(\frac{1}{p}-1\right),$$

which implies the impossibility of frame representations in these spaces and therefore the optimal coefficients $\lambda_{j,m'}$ as well as the atoms $a_{j,m}$ in (2.7) are not linear with respect to g in this case.
Alternatively we could use the subatomic approach as described in [Tri06, Sect. 9.2] instead of atomic decompositions in Step 3 of Theorem 2.2. Then again the constructed extension operator turns out to be bounded but not linear – but in this case linearity fails only in terms of the coefficients but not for the building blocks. We sketch the proof.
For a given $g\in\mathfrak{B}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})$ we need to construct a function $f\in\mathfrak{B}^s_{p,q}(\mathbb{R}^n)$ such that

$$f(x',0)=g(x') \qquad \text{and} \qquad \|f|\mathfrak{B}^s_{p,q}(\mathbb{R}^n)\| \le c\|g|\mathfrak{B}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})\|.$$

Let $g\in\mathfrak{B}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})$ with optimal subatomic decomposition, i.e.,

$$g(x') = \sum_{\beta'}\sum_j\sum_{m'} \lambda^{\beta'}_{j,m'} k^{\beta'}_{j,m'}(x') \qquad \text{and} \qquad \|g|\mathfrak{B}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})\| \sim \|\lambda|b^{s-\frac{1}{p},\varrho}_{p,q}\|, \tag{2.8}$$

where $\varrho>0$ is chosen later on. Put

$$(\mathrm{Ex}\, g)(x) = f(x) = \sum_{\beta',j,m'}\sum_{m_n=-1}^{-2^J} \lambda^{\beta'}_{j,m'} k^{\beta'}_{j,m'}(x') k^0_{j,m_n}(x_n),$$

where $k^0_{j,m_n}(x_n)$ are 1-dimensional (standardized) building blocks. It is easy to see that $f(x',0)=g(x')$, since $\sum_{m_n=-1}^{-2^J} k^0(0-m_n)=1$. The following calculation for $\alpha=(\alpha',\alpha_n)\in\mathbb{N}^n_0$

$$\left|\mathrm{D}^\alpha k^{\beta'}_{j,m'}(x')k^0_{j,m_n}(x_n)\right| = \left|\sum_{\gamma'+\delta'=\alpha'} \mathrm{D}^{\gamma'}(2^{-J}(2^jx'-m'))^{\beta'}\mathrm{D}^{\delta'}k(2^jx'-m')\frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}k(2^jx_n-m_n)\right|$$
$$\le \sum_{\gamma'+\delta'=\alpha'} 2^{j|\gamma'|}2^{(c-\varepsilon)|\beta'|}2^{j|\delta'|}\sup_{z'\in\mathbb{R}^{n-1}}\left|\mathrm{D}^{\delta'}k(z')\right| 2^{j\alpha_n}\sup_{z\in\mathbb{R}}\left|\frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}k(z)\right|$$
$$\le c_{k,K}2^{r|\beta'|}2^{j|\alpha|}, \qquad \text{since} \quad |\gamma'|+|\delta'|+\alpha_n=|\alpha|$$

together with

$$\mathrm{supp}\, k^{\beta'}_{j,m'}(x')k^0_{j,m_n}(x_n)\subset dQ_{j,m}$$

shows that

$$\frac{k^{\beta'}_{j,m'}(x')k^0_{j,m_n}(x_n)}{c_{k,K}2^{r|\beta'|}}$$

represent suitable atoms according to Definition (1.2). Furthermore we estimate for $\eta \le 1$,

$$\begin{aligned}
\|f|\mathfrak{B}^s_{p,q}(\mathbb{R}^n)\|^\eta &= \|\sum_{\beta'} f^{\beta'}|\mathfrak{B}^s_{p,q}(\mathbb{R}^n)\|^\eta \\
&\le \sum_{\beta'} \|f^{\beta'}|\mathfrak{B}^s_{p,q}(\mathbb{R}^n)\|^\eta \\
&\le \sum_{\beta'} \left(\sum_j 2^{j(s-\frac{n}{p})q} \left(\sum_{m'} \sum_{m_n=-1}^{-2^{-J}} \left|\lambda^{\beta'}_{j,m'} c_{k,K} 2^{r|\beta'|}\right|^p \right)^{q/p} \right)^{\eta/q} \\
&\le c \sum_{\beta'} 2^{\eta r|\beta'|} \left(\sum_j 2^{j\left[(s-\frac{1}{p})-\frac{n-1}{p}\right]q} \left(\sum_{m'} |\lambda^{\beta'}_{j,m'}|^p \right)^{q/p} \right)^{\eta/q} \\
&\le c \left(\sum_{\beta'} 2^{-\delta|\beta'|} \right) \left(\sup_{\beta'} 2^{\varrho|\beta'|} \left(\sum_j 2^{j\left[(s-\frac{1}{p})-\frac{n-1}{p}\right]q} \left(\sum_{m'} |\lambda^{\beta'}_{j,m'}|^p \right)^{q/p} \right)^{\eta/q} \right) \\
&\le c' \|g|\mathfrak{B}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})\|^\eta,
\end{aligned}$$

where we set $\varrho = \eta r + \delta$ in the second but last line.

Remark 2.4 So far we only considered $\mathrm{Tr}\, f$=$\mathrm{Tr}_{\{x_n=0\}} f$. But it is obvious that traces on hyperplanes of dimension $1, 2, \ldots, n-2$ can be obtained by iteration of Theorem 2.2. Let $n > m \in \mathbb{N}$ and $\mathrm{Tr}_{\{x_{m+1}=\cdots=x_n=0\}} = \mathrm{Tr}\, f$. Then

$$\mathrm{Tr}\, \mathfrak{B}^s_{p,q}(\mathbb{R}^n) = \mathfrak{B}^{s-\frac{n-m}{p}}_{p,q}(\mathbb{R}^m) \qquad \text{when} \qquad s > \frac{n-m}{p}, \quad 0 < p, q \le \infty.$$

We now discuss what happens in the limiting case $s = \frac{1}{p}$.

Corollary 2.5 *Let* $0 < p < \infty$, $0 < q \le \min(1, p)$. *Then*

$$\mathrm{Tr}\, \mathfrak{B}^{1/p}_{p,q}(\mathbb{R}^n) = L_p(\mathbb{R}^{n-1}).$$

P r o o f : Step 1. Using the same construction as in Theorem 2.2, we need to show that

$$\mathrm{Tr}\, \mathfrak{B}^{1/p}_{p,q}(\mathbb{R}^n) \subset L_p(\mathbb{R}^{n-1}),$$

i.e. the sums in (2.4) converge in $L_p(\mathbb{R}^{n-1})$ if $f \in \mathfrak{B}^{1/p}_{p,q}(\mathbb{R}^n)$, $0 < q \le \min(1, p)$. If $0 < p < 1$ and $q \le p$ this is observed by the following calculation

$$\begin{aligned}
\|\mathrm{Tr}\, f|L_p(\mathbb{R}^{n-1})\|^p &\le \sum_j \sum_{m'} |\eta_{j,m'}|^p \int_{dQ'_{j,m'}} |b_{j,m'}(x')|^p \mathrm{d}x' \\
&\le c \sum_j 2^{-j(n-1)} \sum_m |\lambda_{j,m}|^p \\
&\le c' \left(\sum_j 2^{-j(n-1)\frac{q}{p}} \left(\sum_m |\lambda_{j,m}|^p \right)^{q/p} \right)^{p/q} \\
&\sim \|f|\mathfrak{B}^{1/p}_{p,q}(\mathbb{R}^n)\|, \qquad (2.9)
\end{aligned}$$

where in the second but last line we used the embedding $\ell_{\frac{q}{p}} \hookrightarrow \ell_1$. When $p \geq 1$ and $0 < q \leq 1$ we obtain

$$
\begin{aligned}
\|\mathrm{Tr}\, f|L_p(\mathbb{R}^{n-1})\| &\leq \sum_j \left(\int_{dQ'_{j,m'}} \Big| \sum_{m'} \eta_{j,m'} b_{j,m'}(x') \Big|^p \mathrm{d}x' \right)^{1/p} \\
&\sim \sum_j \left(\sum_{m'} |\eta_{j,m'}|^p \int_{dQ'_{j,m'}} |b_{j,m'}(x')|^p \mathrm{d}x' \right)^{1/p} \\
&\leq c \sum_j 2^{-j\frac{(n-1)}{p}} \left(\sum_m |\lambda_{j,m}|^p \right)^{1/p} \\
&\leq c' \left(\sum_j 2^{-j\frac{(n-1)q}{p}} \left(\sum_m |\lambda_{j,m}|^p \right)^{q/p} \right)^{1/q} \\
&\sim \|f|\mathfrak{B}^{1/p}_{p,q}(\mathbb{R}^n)\|, \qquad (2.10)
\end{aligned}
$$

using the fact that we only have a controlled overlap of the atoms $b_{j,m'}$ for fixed j, and $\ell_q \hookrightarrow \ell_1$ in the second but last line. Now (2.9) and (2.10) prove that Tr is a bounded (and linear) operator from $\mathfrak{B}^{1/p}_{p,q}(\mathbb{R}^n)$ into $L_p(\mathbb{R}^{n-1})$.

Step 2. In order to see that Tr is onto $L_p(\mathbb{R}^{n-1})$, it is sufficient to show that each $h \in L_p(\mathbb{R}^{n-1})$ has a decomposition

$$
h(x') = \sum_j \sum_{m'} \eta_{j,m'} b_{j,m'}, \qquad (2.11)
$$

where the $b_{j,m'}$'s satisfy

$$
|\mathrm{D}^{\alpha'} b_{j,m'}(x')| \leq 2^{j|\alpha'|}, \quad |\alpha'| \leq K, \quad \alpha' \in \mathbb{N}_0^{n-1}, \quad \text{and} \qquad \mathrm{supp}\, b_{j,m'} \subset Q'_{j,m'}
$$

– since any such representation can be obtained as the restriction of the trace operator applied to an $f \in \mathfrak{B}^{1/p}_{p,q}(\mathbb{R}^n)$. Additionally we require that

$$
\left(\sum_j 2^{-j\frac{(n-1)}{p}q} \left(\sum_{m'} |\eta_{j,m'}|^p \right)^{q/p} \right)^{1/q} \leq c\|h|L_p(\mathbb{R}^{n-1})\|, \qquad (2.12)
$$

since for $f(x',0) = h(x')$ this leads to – cf. Step 3 of Theorem 2.2 –

$$
\|f|\mathfrak{B}^{1/p}_{p,q}(\mathbb{R}^n)\| \leq c\|h|L_p(\mathbb{R}^{n-1})\|.
$$

Out proof follows closely [FJ85, Th. 5.1]. To establish such a decomposition, start by picking a $\kappa \in C_0^\infty(\mathbb{R}^{n-1})$ satisfying

$$
\mathrm{supp}\,\kappa \subset [0,1]^{n-1} =: \Omega, \qquad 0 \leq \kappa(\cdot) \leq 1, \quad \text{and} \quad \|1-\kappa|L_p(\Omega)\| \leq \min\left(\frac{1}{5}, \left(\frac{1}{5}\right)^{1/p} \right).
$$

If

$$
Q'_{j,m'} = \{x' : m_i 2^{-j} \leq x_i < (m_i+1)2^{-j}, i = 1, \ldots, n-1\},
$$

put

$$
b_{j,m'}(x') := C \cdot \kappa(2^j x' - m'), \qquad (2.13)
$$

such that $\mathrm{supp}\, b_{j,m'} \subset Q'_{j,m'}$, where $m' = (m_1, \ldots, m_{n-1})$ and C is chosen small enough for $b_{j,m'}$ to satisfy

$$
|\mathrm{D}^{\alpha'} b_{j,m'}(x')| \leq 2^{j|\alpha'|}, \qquad |\alpha'| \leq K, \quad \alpha' \in \mathbb{N}_0^{n-1}.
$$

Fix a non-negative $h \in L_p(\mathbb{R}^{n-1})$. (It suffices to prove the assumption for such functions, since an arbitrary $h \in L_p(\mathbb{R}^{n-1})$ can be reduced to the sum of two real-valued functions

$$
h = \mathfrak{Re}\, h + i\mathfrak{Im}\, h
$$

and any real-valued function $h \in L_p(\mathbb{R}^{n-1})$ can be decomposed into two non-negative functions h_+, h_- such that

$$h(x) = h_+(x) - h_-(x), \qquad \text{where} \quad h_+ := \max(h, 0), \quad h_- := \max(-h, 0).$$

The full generality of (2.11), (2.12) for arbitrary $h \in L_p(\mathbb{R}^{n-1})$ then follows by standard arguments.)
By choosing the side length 2^{-j_1} small enough, it is possible to find a simple function

$$e_1(x') = \sum_{m'} r_{j_1,m'} \chi_{j_1,m'}(x')$$

such that

$$e_1 \geq 0 \qquad \text{and} \qquad \|h - e_1|L_p(\mathbb{R}^{n-1})\| \leq \min\left(\frac{1}{4}, \left(\frac{1}{4}\right)^{1/p}\right) \|h|L_p(\mathbb{R}^{n-1})\|.$$

We define the smooth version

$$\tilde{e_1}(x') = \sum_{m'} \eta_{j_1,m'} b_{j_1,m'}(x'),$$

where the $b_{j_1,m'}$'s are given by (2.13) and $\eta_{j_1,m'} = \frac{r_{j_1,m'}}{C}$ with the same constant C. Setting

$$D = \frac{C}{\min\left(\frac{5}{4}, \left(\frac{5}{4}\right)^{1/p}\right)}$$

for $p \geq 1$ we see that

$$\begin{aligned}\left(\sum_{m'} 2^{-j_1(n-1)} |\eta_{j_1,m'}|^p\right)^{1/p} &= \frac{\|e_1|L_p(\mathbb{R}^{n-1})\|}{C} \\ &\leq \frac{\|h - e_1|L_p(\mathbb{R}^{n-1})\| + \|h|L_p(\mathbb{R}^{n-1})\|}{C} \\ &\leq \frac{\|h|L_p(\mathbb{R}^{n-1})\|}{D},\end{aligned}$$

and when $0 < p < 1$ we obtain the same estimate via

$$\begin{aligned}\sum_{m'} 2^{-j_1(n-1)} |\eta_{j_1,m'}|^p &= \frac{\|e_1|L_p(\mathbb{R}^{n-1})\|^p}{C^p} \\ &\leq \frac{\|h - e_1|L_p(\mathbb{R}^{n-1})\|^p + \|h|L_p(\mathbb{R}^{n-1})\|^p}{C^p} \\ &\leq \frac{\|h|L_p(\mathbb{R}^{n-1})\|^p}{D^p}.\end{aligned}$$

We picked κ so that $\|e_1 - \tilde{e_1}|L_p(\mathbb{R}^{n-1})\| \leq \min\left(\frac{1}{5}, \left(\frac{1}{5}\right)^{1/p}\right) \|e_1|L_p(\mathbb{R}^{n-1})\|$. Hence for $p \geq 1$,

$$\begin{aligned}\|h - \tilde{e_1}|L_p(\mathbb{R}^{n-1})\| &\leq \|h - e_1|L_p(\mathbb{R}^{n-1})\| + \|e_1 - \tilde{e_1}|L_p(\mathbb{R}^{n-1})\| \\ &\leq \left\{\min\left(\frac{1}{4}, \left(\frac{1}{4}\right)^{1/p}\right) + \min\left(\frac{1}{5}, \left(\frac{1}{5}\right)^{1/p}\right)\frac{C}{D}\right\} \|h|L_p(\mathbb{R}^{n-1})\| \\ &\leq \frac{1}{2} \|h|L_p(\mathbb{R}^{n-1})\|. \end{aligned} \tag{2.14}$$

Similar for $0 < p < 1$. If this process is repeated with h replaced by $h - \tilde{e_1}$, we obtain $\tilde{e_2} = \sum_{m'} \eta_{j_2,m'} b_{j_2,m'}$ such that

$$\left(\sum_{m'} 2^{-j_2(n-1)} |\eta_{j_2,m'}|^p\right)^{1/p} \leq \frac{\|h - \tilde{e_1}|L_p(\mathbb{R}^{n-1})\|}{D} \leq \frac{\|h|L_p(\mathbb{R}^{n-1})\|}{2D}$$

and

$$\|h - \tilde{e_1} - \tilde{e_2}|L_p(\mathbb{R}^{n-1})\| \leq \frac{\|h - \tilde{e_1}|L_p(\mathbb{R}^{n-1})\|}{2} \leq \frac{\|h|L_p(\mathbb{R}^{n-1})\|}{4},$$

where we used (2.14). We can also arrange that $j_2 > j_1$. Continuing this process inductively we obtain the functions $\tilde{e}_i = \sum_{m'} \eta_{j_i,m'} b_{j_i,m'}$, $i = 1, 2, \ldots$, satisfying

$$\left(\sum_{m'} 2^{-j_i(n-1)} |\eta_{j_i,m'}|^p \right)^{1/p} \le \frac{\|h|L_p(\mathbb{R}^{n-1})\|}{2^{i-1}D}, \tag{2.15}$$

$$\left\| h - \sum_{i=1}^{m} \tilde{e}_i | L_p(\mathbb{R}^{n-1}) \right\| \le 2^{-m} \|h|L_p(\mathbb{R}^{n-1})\|, \qquad m = 1, 2, \ldots, \tag{2.16}$$

and $j_{i+1} > j_i$ for every i. The required decomposition of h is $h(x') = \sum_{i=1}^{\infty} \tilde{e}_i(x')$. By (2.16) this sum converges in $L_p(\mathbb{R}^{n-1})$ and from (2.15) we see that

$$\left(\sum_{i=1}^{\infty} \left(\sum_{m'} 2^{-j_i(n-1)} |\eta_{j_i,m'}|^p \right)^{q/p} \right)^{1/q} \le c \|h|L_p(\mathbb{R}^{n-1})\|.$$

This completes the proof. □

Remark 2.6 We actually proved a bit more than stated. Note that Step 3 in the proof of Theorem 2.2 together with Step 2 of Corollary 2.5 establish the existence of a bounded extension operator, i.e. for given $g \in L_p(\mathbb{R}^{n-1})$ there exists a function $f \in \mathfrak{B}^{1/p}_{p,q}(\mathbb{R}^n)$ with

$$f(x', 0) = g(x') \qquad \text{and} \qquad \|f|\mathfrak{B}^{1/p}_{p,q}(\mathbb{R}^n)\| \le c\|g|L_p(\mathbb{R}^{n-1})\|.$$

In particular, we have

$$\mathrm{Ex} : L_p(\mathbb{R}^{n-1}) \to \mathfrak{B}^{1/p}_{p,q}(\mathbb{R}^n), \qquad (\mathrm{Ex}\, g)(x) = f(x),$$

with

$$\mathrm{Tr} \circ \mathrm{Ex} = \mathrm{id}_{L_p(\mathbb{R}^{n-1})}.$$

Remark 2.7 As in Remark 2.4 we obtain similar results for the limiting case when dealing with hyperplanes $\mathbb{R}^m$, $n > m \in \mathbb{N}$. Using Theorem 2.2 and Corollary 2.5, by iteration we obtain

$$\mathrm{Tr}\, \mathfrak{B}^{\frac{n-m}{p}}_{p,q}(\mathbb{R}^n) = L_p(\mathbb{R}^m), \qquad 0 < p < \infty, \quad 0 < q \le \min(1, p).$$

Remark 2.8 Our results are best possible in the sense that the sums in (2.4) do not necessarily converge in

$$L_p + L_\infty := \{f : f = g_p + g_\infty, \quad g_i \in L_i(\mathbb{R}^n)\},$$

normed by

$$\|f|L_p + L_\infty\| := \inf_{\substack{f = g_p + g_\infty \\ g_i \in L_i}} (\|g_p|L_p(\mathbb{R}^n)\| + \|g_\infty|L_\infty(\mathbb{R}^n)\|),$$

if $s = \frac{1}{p}$ and $q > p$. Therefore the trace does not exist. (Note that Proposition 1.5(i) establishes

$$\mathfrak{B}^{\frac{1}{p}}_{p,q}(\mathbb{R}^n) \hookrightarrow \mathfrak{B}^s_{p,u}(\mathbb{R}^n), \qquad s < \frac{1}{p}, \quad 0 < u \le \infty,$$

from which then also follows that the trace in general does not exist if $s < \frac{1}{p}$.)
This can be seen in the following way. Let $s = \frac{1}{p}$, $q > p$, and pick a sequence

$$\{\eta_j\}_{j=2}^{\infty} \in \ell_q \setminus \ell_p$$

(or in $c_0 \setminus \ell_p$ if $q = \infty$).

Furthermore, we choose a collection of dyadic cubes $\{E_j\}_{j=2}^{\infty}$ with

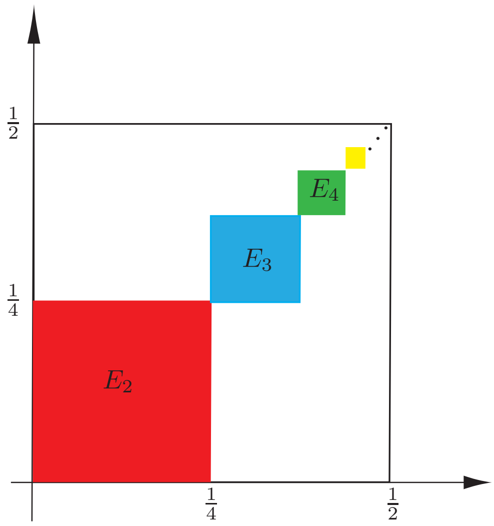
$$E_j \subset [0,1]^{n-1}$$

and length

$$l(E_j) = 2^{-j},$$

that additionally satisfy

$$E_j \cap E_k \neq \emptyset \quad \text{if} \quad j \neq k.$$



Put

$$\eta_{j,m'} := \begin{cases} 2^{j\frac{n-1}{p}}\eta_j, & Q_{j,m'} = E_j, \\ 0, & \text{otherwise}, \end{cases}$$

and let $b_{j,m'}$ be 1_K-atoms in $\mathbb{R}^{n-1}$, i.e.

$$\operatorname{supp} b_{j,m'} \subset dQ_{j,m'},$$

$$\left|\mathrm{D}^{\alpha'} b_{j,m'}(x')\right| \leq 2^{j|\alpha'|}, \qquad |\alpha'| \leq K,$$

for which additionally

$$b_{j,m'}(x') \geq c \quad \text{if} \quad x' \in Q'_{j,m'}, \quad c > 0.$$

Then

$$\|\eta|b^{1/p}_{p,q}\| = \left(\sum_{j=0}^{\infty} 2^{-j\frac{n-1}{p}q}\left(\sum_{m'\in\mathbb{Z}^{n-1}} |\eta_{j,m'}|^p\right)^{q/p}\right)^{1/q} = \left(\sum_{j=2}^{\infty}|\eta_j|^q\right)^{1/q} < \infty,$$

and it is clear that

$$\sum_{j,m'} \eta_{j,m'} b_{j,m'}$$

would arise as the trace of a suitable $f \in \mathfrak{B}^{1/p}_{p,q}(\mathbb{R}^n)$ if the trace operator was continuous. But if we let

$$g_N(x') := \sum_{j=2}^{N}\sum_{m'} \eta_{j,m'} b_{j,m'}(x'), \qquad N \text{ large},$$

then $\operatorname{supp} g_N \subset [0,1]^{n-1}$. Since $L_\infty([0,1]^{n-1}) \hookrightarrow L_p([0,1]^{n-1})$ we estimate

$$\begin{aligned} \|g_N|L_p + L_\infty\| \geq c\|g_N|L_p\| &= \left\|\sum_{j=2}^{N}\sum_{m'} \eta_{j,m'} b_{j,m'}|L_p\right\| \\ &\sim \left(\sum_{j=2}^{N} 2^{j(n-1)}|\eta_j|^p \int\limits_{\substack{dE_j,\\ E_j = Q'_{j,m'}}} |b_{j,m'}(x')|^p \mathrm{d}x'\right)^{1/p} \\ &\geq c'\left(\sum_{j=2}^{N}|\eta_j|^p\right)^{1/p} \longrightarrow \infty \quad \text{as} \quad N \to \infty. \end{aligned}$$

Therefore the sum $\sum_{j,m'} \eta_{j,m'} b_{j,m'}$ cannot converge in $L_p + L_\infty$.

2.2 The trace problem in $\mathfrak{F}^s_{p,q}(\mathbb{R}^n)$

With the help of our previous results on traces in $\mathfrak{B}^s_{p,q}(\mathbb{R}^n)$ we are now able to investigate the trace problem for the spaces $\mathfrak{F}^s_{p,q}(\mathbb{R}^n)$. It turns out that the trace is actually independent of the parameter q. We make use of the following Proposition. A proof may be found in [FJ90, Prop. 2.7].

Proposition 2.9 *Let $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, and $E_{j,m} \subset Q_{j,m}$ measurable sets with $|E_{j,m}| \sim |Q_{j,m}|$. Then*

$$\|\lambda|f^s_{p,q}\| \sim \left\| \left(\sum_{j,m} 2^{jsq} |\lambda_{j,m}\chi_{E_{j,m}}(\cdot)|^q \right)^{1/q} |L_p(\mathbb{R}^n) \right\|.$$

The next Theorem states our main result.

Theorem 2.10 *Let $n \geq 2$, $0 < p < \infty$, $0 < q \leq \infty$, and $s - \frac{1}{p} > 0$.*

$$\mathrm{Tr}\, \mathfrak{F}^s_{p,q}(\mathbb{R}^n) = \mathfrak{B}^{s-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1}).$$

Proof : It is sufficient to show that the trace of $\mathfrak{F}^s_{p,q}$ is independent of q, i.e.

$$\mathrm{Tr}\, \mathfrak{F}^s_{p,q}(\mathbb{R}^n) = \mathrm{Tr}\, \mathfrak{F}^s_{p,p}(\mathbb{R}^n) = \mathrm{Tr}\, \mathfrak{B}^s_{p,p}(\mathbb{R}^n),$$

since then the rest follows immediately from Theorem 2.2.
If $0 < q < r \leq \infty$, we have the embedding $\mathfrak{F}^s_{p,q} \hookrightarrow \mathfrak{F}^s_{p,r}$ yielding

$$\mathrm{Tr}\, \mathfrak{F}^s_{p,q} \hookrightarrow \mathrm{Tr}\, \mathfrak{F}^s_{p,r}.$$

In order to prove the other direction let $f \in \mathfrak{F}^s_{p,r}$ with optimal atomic decomposition

$$f(x) = \sum_{j,m} \lambda_{j,m} a_{j,m}(x), \qquad x \in \mathbb{R}^n,$$

i.e.,

$$\|f|\mathfrak{F}^s_{p,r}\| \sim \|\lambda|f^s_{p,r}\|.$$

In particular, by Definition 1.3

$$\mathrm{supp}\, a_{j,m} \subset dQ_{j,m}.$$

We need to show that there exists an $\tilde{f} \in \mathfrak{F}^s_{p,q}$ such that

$$\mathrm{Tr}\, f = \mathrm{Tr}\, \tilde{f}.$$

Let $\mathbb{R}^{n-1} := \{x = (x', x_n) \in \mathbb{R}^n : x_n = 0\}$ and set

$$\tilde{\lambda}_{j,m} := \begin{cases} \lambda_{j,m}, & dQ_{j,m} \cap \mathbb{R}^{n-1} \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, put $\tilde{a}_{j,m}(x) = a_{j,m}(x)$ and consider

$$\tilde{f}(x) = \sum_{j,m} \tilde{\lambda}_{j,m} a_{j,m}(x). \tag{2.17}$$

From the construction we immediately see that

$$\mathrm{Tr}\, f = \mathrm{Tr}\, \tilde{f}.$$

Note that in (2.17) we only sum over finitely many $m_n \in I(j, m')$, where the index set is actually independent of j, m'. This can be seen by observing that

$$m_n \in I \quad \text{if, and only if,} \quad dQ_{j,m} \cap \mathbb{R}^{n-1} \neq \emptyset$$

which is equivalent to

$$m_n \in I \quad \text{if, and only if,} \quad 0 \in \left(2^{-j}m_n - d2^{-j-1}, 2^{-j}m_n + d2^{-j-1}\right).$$

But this yields

$$m_n \in I \quad \text{if, and only if,} \quad 0 \in \left(m_n - \frac{d}{2}, m_n + \frac{d}{2}\right),$$

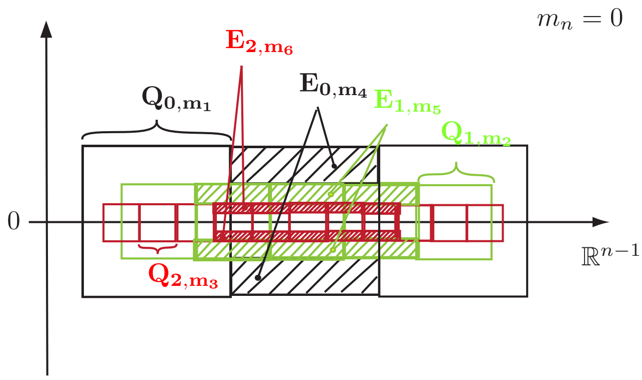
establishing the independence of the index set I on j and m'.
We want to apply Proposition 2.9. Therefore we wish to construct suitable sets $E_{j,m}$ such that

$$E_{j,(m',m_n)} \subset Q_{j,(m',m_n)} \qquad \text{and} \qquad |E_{j,(m',m_n)}| \sim |Q_{j,(m',m_n)}|, \tag{2.18}$$

which do not intersect for fixed $m_n \in I$.

If $m_n = 0$ put

$$E_{j,(m',0)} \quad := \{x \in Q_{j,(m',0)} : 2^{-j-1} < |x_n| < 2^{-j}\},$$

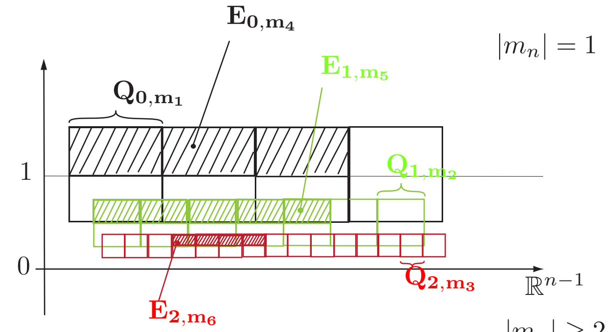


whereas for $|m_n| = 1$ we set

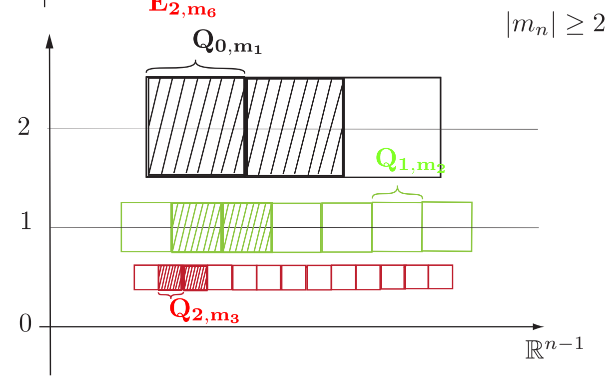
$$E_{j,(m',1)} := \{x \in Q_{j,(m',1)} : 0 < x_n - 2^{-j} < 2^{-j-1}\},$$

and

$$E_{j,(m',-1)} := \{x \in Q_{j,(m',-1)} : -2^{-j-1} < x_n + 2^{-j} < 0\},$$



and if $|m_n| \geq 2$ we can simply choose $E_{j,m} := Q_{j,m}$.



Clearly we have (2.18). In particular, for fixed m_n the sets $E_{j,(m',m_n)}$ have pairwise disjoint support for all $j \in \mathbb{N}_0$, $m' \in \mathbb{Z}^{n-1}$. Hence, if $q < \infty$ we calculate

$$\begin{aligned} \|\tilde{f}|\mathfrak{F}^s_{p,q}(\mathbb{R}^n)\| &\leq \|\tilde{\lambda}|f^s_{p,q}\| \\ &\sim \left\| \left(\sum_{j=0}^{\infty} \sum_{m\in\mathbb{Z}^n} 2^{jsq} |\tilde{\lambda}_{j,m}|^q \chi_{j,m}(\cdot) \right)^{1/q} |L_p(\mathbb{R}^n) \right\| \\ &\sim \left\| \left(\sum_{j=0}^{\infty} \sum_{m\in\mathbb{Z}^n} 2^{jsq} |\tilde{\lambda}_{j,m}|^q \chi_{E_{j,m}}(\cdot) \right)^{1/q} |L_p(\mathbb{R}^n) \right\| \end{aligned}$$

$$\sim \sum_{m_n\in I} \left\| \left(\sum_{j=0}^{\infty} \sum_{m'} 2^{jsq} |\lambda_{j,(m',m_n)}|^q \chi_{E_{j,(m',m_n)}}(\cdot) \right)^{1/q} |L_p(\mathbb{R}^n) \right\|$$

$$\sim \sum_{m_n\in I} \left\| \sum_{j=0}^{\infty} \sum_{m'} 2^{js} |\lambda_{j,(m',m_n)}| \chi_{E_{j,(m',m_n)}}(\cdot) |L_p(\mathbb{R}^n) \right\|$$

$$\sim \sum_{m_n\in I} \left\| \left(\sum_{j=0}^{\infty} \sum_{m'} 2^{jsr} |\lambda_{j,(m',m_n)}|^r \chi_{E_{j,(m',m_n)}}(\cdot) \right)^{1/r} |L_p(\mathbb{R}^n) \right\|$$

$$\sim \left\| \left(\sum_{j=0}^{\infty} \sum_{m\in\mathbb{Z}^n} 2^{jsr} |\tilde{\lambda}_{j,m}|^r \chi_{E_{j,m}}(\cdot) \right)^{1/r} |L_p(\mathbb{R}^n) \right\|$$

$$\sim \|\tilde{\lambda}|f^s_{p,r}\| \leq \|f|\mathfrak{F}^s_{p,r}(\mathbb{R}^n)\| < \infty,$$

where in the third and 8th step we made use of Proposition 2.9. The q and $1/q$ in line 5 cancel and can be replaced by r and $1/r$, since the sets $E_{j,(m',m_n)}$ have disjoint supports for fixed $m_n \in I$. In particular, $\tilde{f} \in \mathfrak{F}^s_{p,q}$ and therefore

$$\mathrm{Tr}\,\mathfrak{F}^s_{p,r} \subset \mathrm{Tr}\,\mathfrak{F}^s_{p,q},$$

which completes the proof. □

We investigate the limiting case when $s = \frac{1}{p}$ as well.

Corollary 2.11 *Let $0 < p \leq 1$ and $0 < q \leq \infty$. Then*

$$\mathrm{Tr}\,\mathfrak{F}^{1/p}_{p,q}(\mathbb{R}^n) = L_p(\mathbb{R}^{n-1}).$$

Proof: In Theorem 2.10 we established the independence of the trace of $\mathfrak{F}^s_{p,q}$ on q. Therefore Corollary 2.5 yields

$$\mathrm{Tr}\,\mathfrak{F}^{1/p}_{p,q} = \mathrm{Tr}\,\mathfrak{F}^{1/p}_{p,p} = \mathrm{Tr}\,\mathfrak{B}^{1/p}_{p,p}(\mathbb{R}^n) = L_p(\mathbb{R}^{n-1}), \qquad 0 < p \leq 1.$$

□

Remark 2.12 Again by iteration of Theorem 2.10 and Corollary 2.11 we obtain results for traces on hyperplanes of dimension $1, 2, \ldots, n-2$. Let $n > m \in \mathbb{N}$ and $\mathrm{Tr}_{\{x_{m+1}=\cdots=x_n=0\}} = \mathrm{Tr}\,f$. Then for $s > \frac{n-m}{p}$

$$\mathrm{Tr}\,\mathfrak{F}^s_{p,q}(\mathbb{R}^n) = \mathfrak{B}^{s-\frac{n-m}{p}}_{p,q}(\mathbb{R}^m), \qquad 0 < p < \infty, \quad 0 < q \leq \infty$$

and in the limiting case when $s = \frac{n-m}{p}$ we have

$$\mathrm{Tr}\,\mathfrak{F}^{\frac{n-m}{p}}_{p,q}(\mathbb{R}^n) = L_p(\mathbb{R}^m), \qquad 0 < p \leq 1, \quad 0 < q \leq \infty.$$

3 Traces on the boundary of C^k domains Ω

3.1 Definitions and basic notation

We consider mainly three domains in the sequel: the Euclidean n-space $\mathbb{R}^n$, the half-space

$$\mathbb{R}^n_+ = \{x : x = (x', x_n) \in \mathbb{R}^n, x' \in \mathbb{R}^{n-1}, x_n > 0\},$$

and bounded C^k domains $\Omega \subset \mathbb{R}^n$. All these domains can be regarded as special cases of so-called (ε, δ)-domains. Recall that domain always stands for open set. The boundary of Ω is denoted by $\Gamma = \partial\Omega$.

Definition 3.1 **(i)** *Let Ω be a domain in $\mathbb{R}^n$ with $\Omega \neq \mathbb{R}^n$. Then Ω is said to be an (ε, δ)-domain, where $0 < \varepsilon < \infty$ and $0 < \delta < \infty$, if it is connected and if for any $x \in \Omega$, $y \in \Omega$ with $|x - y| < \delta$ there is a curve $L \subset \Omega$, connecting x and y such that $|L| \leq \varepsilon^{-1}|x - y|$ and*

$$\operatorname{dist}(z, \partial\Omega) \geq \varepsilon \min(|x - z|, |y - z|), \qquad z \in L. \tag{3.1}$$

(ii) *Let Ω be a bounded domain in $\mathbb{R}^n$. Then Ω is said to be a C^∞ domain if there exist N open balls $K_1, \ldots, K_N$ such that*

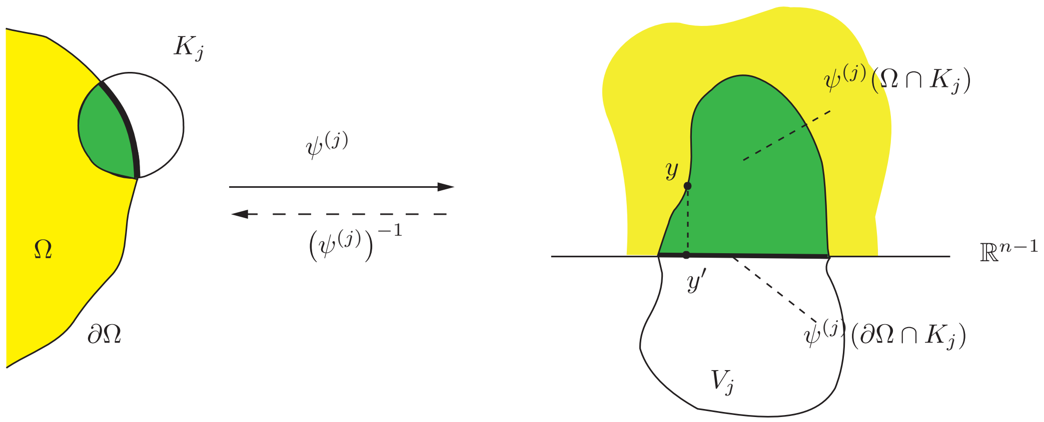
$$\bigcup_{j=1}^{N} K_j \supset \partial\Omega \qquad \text{and} \qquad K_j \cap \partial\Omega \neq \emptyset \quad \text{if} \quad j = 1, \ldots, N,$$

with the following property: for every ball K_j there are diffeomorphic C^∞-maps $\psi^{(j)}$ (curvilinear coordinates) such that

$$y = \psi^{(j)}(x) : K_j \iff V_j = \psi^{(j)}(K_j), \quad j = 1, \ldots, N,$$

where

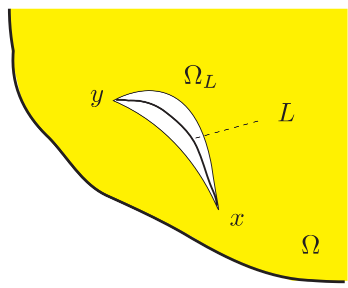
$$\psi^{(j)}(K_j \cap \Omega) \subset \mathbb{R}^n_+, \qquad \psi^{(j)}(K_j \cap \partial\Omega) \subset \mathbb{R}^{n-1}.$$



One may assume that V_j is simply connected and that the upper boundary of $\psi^{(j)}(K_j \cap \Omega)$ can be described by $y_n = \tau^{(j)}(y')$, where $\tau^{(j)}$ is a C^∞ function.

Remark 3.2

(i) It is well-known that (ε, δ)-domains play a crucial role concerning questions of extendability. What is meant by (ε, δ)-domain can be seen in the figure aside. In particular, (3.1) ensures that with L there is also a surrounding croissant-like subdomain $\Omega_L \subset \Omega$. They include minimally smooth domains and hence C^∞ domains. $\mathbb{R}^n_+$ is a further example of an (ε, δ)-domain.
The two domains with the cusp as indicated below are not (ε, δ)-domains (let locally $\Omega \subset \mathbb{R}^2$ be below or above the curve $|x_1|^\alpha$, $0 < \alpha < 1$, respectively) since the indicated connecting curves are either too long (cf. Figure 1) or possible croissants are not fat enough (cf. Figure 2).



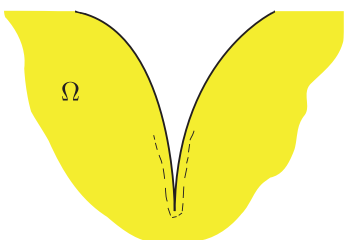


Figure 1

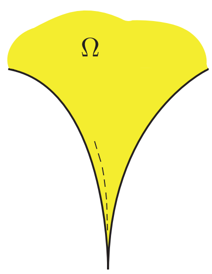
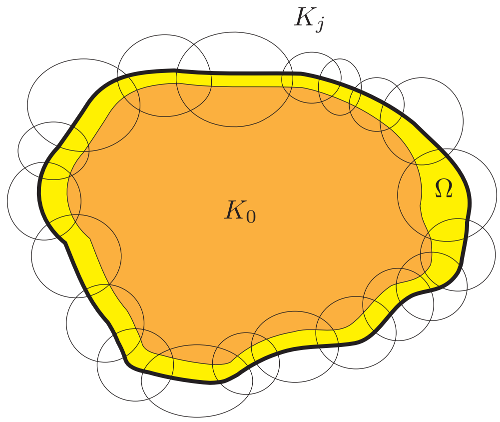


Figure 2

(ii) If we replace C^∞ by C^k with $k = 1, 2, 3, \ldots$, we have the definition of a bounded C^k domain. Furthermore, the C^k-maps $\psi^{(j)}$ can be extended outside K_j in such a way that the extended vector-functions (denoted by $\psi^{(j)}$ as well) yield diffeomorphic mappings from $\mathbb{R}^n$ onto itself (k-diffeomorphisms).

(iii) Note that (ε, δ)-domains include *minimally smooth* domains in the sense of Stein, cf. [Ste70, p. 189] and therefore as special cases bounded C^k domains.

Remark 3.3 *Resolution of unity.* Let K_j with $j = 1, \ldots, N$ be the same balls as in Definition 3.1. Let K_0 be an inner domain with $\bar{K}_0 \subset \Omega$, as indicated in the figure aside; hence



$$\partial\Omega \subset \bigcup_{j=1}^{N} K_j$$

and

$$\Omega \subset K_0 \cup \left(\bigcup_{j=1}^{N} K_j \right).$$

Let $\{\varphi_j\}_{j=0}^N$ be a related resolution of unity of $\bar{\Omega}$, i.e. φ_j are non-negative functions with

$$\varphi_0 \in D(K_0), \qquad \varphi_j \in D(K_j), \quad j = 1, \ldots, N, \tag{3.2}$$

and

$$\sum_{j=0}^{N} \varphi_j(x) = 1 \quad \text{if } x \in \bar{\Omega}. \tag{3.3}$$

Obviously, the restriction of φ_j to $\partial\Omega$ is a resolution of unity with respect to $\partial\Omega$.

Decomposition. Now we can decompose $f \in L_p(\Omega)$ such that

$$f(x) = \varphi_0(x) f(x) + \sum_{j=1}^{N} \varphi_j(x) f(x), \qquad x \in \Omega,$$

where the term $\varphi_0 f$ can be extended outside of Ω by zero.

3.2 Besov and Triebel-Lizorkin spaces on domains

We are now going to define the spaces $\mathbf{B}^s_{p,q}(\Omega)$. Let us introduce an adapted notation of generalized differences

$$\Delta^r_h f(x, \Omega) := \begin{cases} \Delta^r_h f(x), & x, x+h, \ldots, x+rh \in \Omega \\ 0, & \text{otherwise.} \end{cases}$$

The modulus of smoothness of order r of a function $f \in L_p(\Omega)$ is then

$$\omega_r(f, t, \Omega)_p := \sup_{|h| \le t} \|\Delta^r_h f(x, \Omega) | L_p(\Omega)\|.$$

Definition 3.4 *Let $0 < p, q \le \infty$, $s > 0$, and $r \in \mathbb{N}$ such that $r > s$. Then the Besov space $\mathbf{B}^s_{p,q}(\Omega)$ contains all $f \in L_p(\Omega)$ such that*

$$\|f | \mathbf{B}^s_{p,q}(\Omega)\| = \|f | L_p(\Omega)\| + \left(\int_0^1 t^{-sq} \omega_r(f, t, \Omega)^q_p \, \frac{\mathrm{d}t}{t} \right)^{1/q} \tag{3.4}$$

(with the usual modification if $q = \infty$) is finite.

The following theorem tells us that it is possible to define an extension operator, Ex, which extends functions in $\mathbf{B}^s_{p,q}(\Omega)$ to all of $\mathbb{R}^n$. A proof can be found in [DS93, Th. 6.1].

Theorem 3.5 *Let $0 < p, q \le \infty$, $s > 0$, and let $\Omega \subset \mathbb{R}^n$ be an (ε, δ)-domain. Then there exists a bounded extension operator*

$$\mathrm{Ex} : \mathbf{B}^s_{p,q}(\Omega) \longrightarrow \mathbf{B}^s_{p,q}(\mathbb{R}^n), \qquad \mathrm{Ex}\, f\big|_{\Omega} = f,$$

satisfying

$$\|\mathrm{Ex}\, f|\mathbf{B}^s_{p,q}(\mathbb{R}^n)\| \le c\|f|\mathbf{B}^s_{p,q}(\Omega)\|,$$

where the constant c is independent of $f \in \mathbf{B}^s_{p,q}(\Omega)$.

Remark 3.6 In particular, for any (ε, δ)-domain Ω we have

$$\|f|\mathbf{B}^s_{p,q}(\Omega)\| \le \|\mathrm{Ex}\, f|\mathbf{B}^s_{p,q}(\mathbb{R}^n)\| \le c\|f|\mathbf{B}^s_{p,q}(\Omega)\|.$$

Furthermore, it is possible to give a direct definition of the spaces $\mathfrak{B}^s_{p,q}(\Omega)$, $\mathfrak{F}^s_{p,q}(\Omega)$. We use the following abbreviation

$$\sum_m{}^{\Omega,j} = \sum_{m\in\mathbb{Z}^n, Q_{jm}\cap\Omega\neq\emptyset}, \qquad \text{where} \quad j \in \mathbb{N}_0.$$

We define the relevant sequence spaces.

Definition 3.7 *Let $0 < p \le \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$. Furthermore, let $\Omega \subset \mathbb{R}^n$ and $\lambda = \{\lambda_{j,m} \in \mathbb{C} : j \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$. Then*

$$b^s_{p,q}(\Omega) = \left\{ \lambda : \|\lambda|b^s_{p,q}(\Omega)\| = \left(\sum_{j=0}^{\infty} 2^{j(s-\frac{n}{p})q} \left(\sum_{m\in\mathbb{Z}^n}^{\Omega,j} |\lambda_{j,m}|^p \right)^{q/p} \right)^{1/q} < \infty \right\}$$

and

$$f^s_{p,q}(\Omega) = \left\{ \lambda : \|\lambda|f^s_{p,q}(\Omega)\| = \left\| \left(\sum_{j=0}^{\infty} \sum_{m\in\mathbb{Z}^n}^{\Omega,j} 2^{jsq}|\lambda_{j,m}|^q \chi_{j,m}(\cdot) \right)^{1/q} |L_p(\mathbb{R}^n) \right\| < \infty \right\}$$

(*with the usual modification if $p = \infty$ and/or $q = \infty$*).

Now we come to the atomic approach.

Definition 3.8 *Let $0 < p \le \infty$ ($p < \infty$ for the F-spaces), $0 < q \le \infty$, and $s > 0$. Let $d > 1$ and $K \in \mathbb{N}_0$ with*

$$K \ge (1 + [s])$$

be fixed.

(i) *Then $f \in L_p(\Omega)$ belongs to $\mathfrak{B}^s_{p,q}(\Omega)$ if, and only if, it can be represented as*

$$f(x) = \sum_{j=0}^{\infty} \sum_{m\in\mathbb{Z}^n}^{\Omega,j} \lambda_{j,m} a_{j,m}(x), \quad \textit{convergence being in} \quad L_p(\Omega), \tag{3.5}$$

where the $a_{j,m}$ are 1_K-atoms ($j \in \mathbb{N}_0$) with

$$\operatorname{supp} a_{j,m} \subset dQ_{j,m}, \qquad j \in \mathbb{N}_0, \quad m \in \mathbb{Z}^n,$$

and $\lambda \in b^s_{p,q}$. Furthermore,

$$\|f|\mathfrak{B}^s_{p,q}(\Omega)\| := \inf \|\lambda|b^s_{p,q}(\Omega)\|,$$

where the infimum is taken over all admissible representations (3.5).

(ii) *Similarly $\mathfrak{F}^s_{p,q}(\Omega)$ is the collection of all $f \in L_p(\Omega)$ which can be represented as*

$$f(x) = \sum_{j=0}^{\infty} \sum_{m\in\mathbb{Z}^n}^{\Omega,j} \lambda_{j,m} a_{j,m}(x), \quad \textit{convergence being in} \quad L_p(\Omega), \tag{3.6}$$

where the $a_{j,m}$ are 1_K-atoms $(j \in \mathbb{N}_0)$ with

$$\operatorname{supp} a_{j,m} \subset dQ_{j,m}, \qquad j \in \mathbb{N}_0, \quad m \in \mathbb{Z}^n,$$

and $\lambda \in f^s_{p,q}$. Furthermore,

$$\|f|\mathfrak{F}^s_{p,q}(\Omega)\| := \inf \|\lambda|f^s_{p,q}(\Omega)\|,$$

where the infimum is taken over all admissible representations (3.6).

Remark 3.9 Let $\mathfrak{A} \in \{\mathfrak{B}, \mathfrak{F}\}$. According to their definition, the spaces $\mathfrak{A}^s_{p,q}(\Omega)$ can as well be regarded as restrictions of the corresponding spaces on $\mathbb{R}^n$ in the usual interpretation, i.e.,

$$\mathfrak{A}^s_{p,q}(\Omega) = \{f \in L_p(\Omega) : \quad \text{there exists} \quad g \in \mathfrak{A}^s_{p,q}(\mathbb{R}^n) \quad \text{with} \quad g\big|_\Omega = f\},$$

furnished with the norm

$$\|f|\mathfrak{A}^s_{p,q}(\Omega)\| = \inf \|g|\mathfrak{A}^s_{p,q}(\mathbb{R}^n)\| \quad \text{with} \quad g\big|_\Omega = f.$$

Here $g\big|_\Omega = f$ denotes the restriction of g to Ω considered as an element of $D'(\Omega)$ meaning

$$g(\varphi) = f(\varphi) \qquad \text{for all } \varphi \in D(\Omega).$$

It turns out that the results from [HN07] as stated in Remark 1.4 can be extended to domains Ω.

Theorem 3.10 *Let $\Omega \subset \mathbb{R}^n$ be an (ε, δ)-domain, $s > 0$, and $0 < p, q \le \infty$. Then*

$$\mathbf{B}^s_{p,q}(\Omega) = \mathfrak{B}^s_{p,q}(\Omega) \tag{3.7}$$

(*in terms of equivalent quasi-norms*).

Proof : First let $f \in \mathfrak{B}^s_{p,q}(\Omega)$ and let $g \in \mathfrak{B}^s_{p,q}(\mathbb{R}^n)$ be such that

$$\|f|\mathfrak{B}^s_{p,q}(\Omega)\| \sim \|g|\mathfrak{B}^s_{p,q}(\mathbb{R}^n)\| \qquad \text{with} \qquad g\big|_\Omega = f.$$

Then we see that

$$\|f|\mathbf{B}^s_{p,q}(\Omega)\| \le \|g|\mathbf{B}^s_{p,q}(\mathbb{R}^n)\| \sim \|g|\mathfrak{B}^s_{p,q}(\mathbb{R}^n)\| \sim \|f|\mathfrak{B}^s_{p,q}(\Omega)\|,$$

where the first step follows from Definition 3.4.
Conversely if $f \in \mathbf{B}^s_{p,q}(\Omega)$, we make use of the extension operator constructed in [DS93] for $\mathfrak{B}^s_{p,q}(\Omega)$. For $s > 0$, $0 < p, q \le \infty$, there is a bounded mapping

$$\mathrm{Ex} : \mathbf{B}^s_{p,q}(\Omega) \longrightarrow \mathbf{B}^s_{p,q}(\mathbb{R}^n), \quad \text{with} \quad \mathrm{Ex}\, f\big|_\Omega = f$$

such that for $f \in \mathbf{B}^s_{p,q}(\Omega)$

$$\|\mathrm{Ex}\, f|\mathbf{B}^s_{p,q}(\mathbb{R}^n)\| \le c\|f|\mathbf{B}^s_{p,q}(\Omega)\|.$$

Putting $g := \mathrm{Ex}\, f$ we obtain

$$\|f|\mathfrak{B}^s_{p,q}(\Omega)\| \le \|g|\mathfrak{B}^s_{p,q}(\mathbb{R}^n)\| \sim \|g|\mathbf{B}^s_{p,q}(\mathbb{R}^n)\| \le c\|f|\mathbf{B}^s_{p,q}(\Omega)\|,$$

where the first step is obvious from Remark 3.9. This completes the proof. □

Remark 3.11 Remark 3.9 together with Theorem 3.10 yields an extrinsic characterization for the spaces $\mathbf{B}^s_{p,q}(\Omega)$. Let $f \in \mathbf{B}^s_{p,q}(\Omega)$, then

$$\|f|\mathbf{B}^s_{p,q}(\Omega)\| = \inf \|g|\mathbf{B}^s_{p,q}(\mathbb{R}^n)\|,$$

where the infimum is taken over all $g \in \mathbf{B}^s_{p,q}(\mathbb{R}^n)$ such that $g\big|_\Omega = f$.

3.3 The trace problem in $\mathfrak{B}^s_{p,q}(\Omega)$ and $\mathfrak{F}^s_{p,q}(\Omega)$

The boundary $\Gamma = \partial\Omega$ of a bounded C^k domain Ω will be furnished in the usual way with a surface measure $\mathrm{d}\sigma$. The corresponding complex-valued Lebesgue spaces $L_p(\Gamma)$, $0 < p \le \infty$, are normed by

$$\|g|L_p(\Gamma)\| = \left(\int_\Gamma |g(\gamma)|^p \mathrm{d}\sigma(\gamma)\right)^{1/p}$$

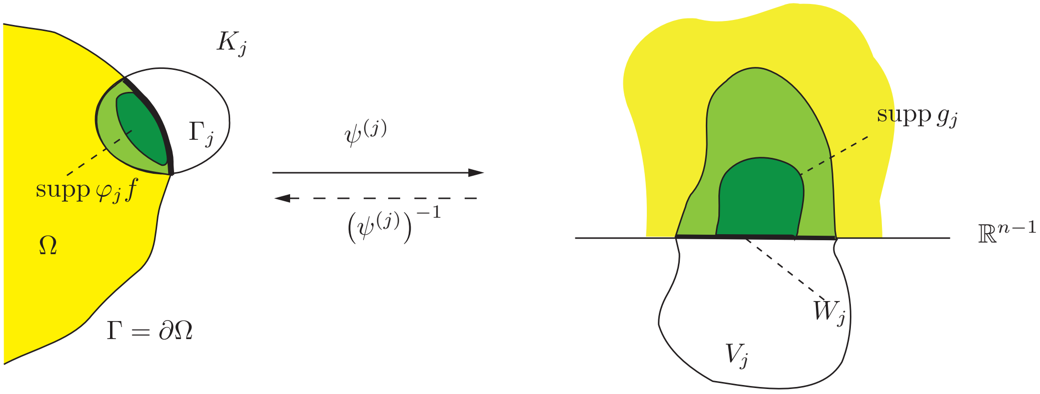
(with obvious modifications if $p = \infty$). We furthermore require the introduction of Besov spaces on Γ. We rely on the resolution of unity according to (3.2), (3.3) and the local diffeomorphisms $\psi^{(j)}$ mapping $\Gamma_j = \Gamma \cap K_j$ onto $W_j = \psi^{(j)}(\Gamma_j)$. Define

$$g_j(y) := (\varphi_j f) \circ (\psi^{(j)})^{-1}(y), \qquad j = 1, \dots, N,$$

which restricted to $y = (y', 0) \in W_j$,

$$g_j(y') = (\varphi_j f) \circ (\psi^{(j)})^{-1}(y'), \qquad j = 1, \dots, N, \quad f \in L_p(\Gamma),$$

makes sense. This results in functions $g_j \in L_p(W_j)$ with compact supports in the $(n-1)$-dimensional C^k domain in W_j – we do not distinguish notationally between g_j and $(\psi^{(j)})^{-1}$ as functions of $(y', 0)$ and of y'.



Definition 3.12 *Let $n \ge 2$, and let Ω be a bounded C^k domain in $\mathbb{R}^n$ with $\Gamma = \partial\Omega$, and φ_j, $\psi^{(j)}$, W_j be as above. Assume $s > 0$ and $0 < p, q \le \infty$. Then we introduce*

$$\mathfrak{B}^s_{p,q}(\Gamma) = \{f \in L_p(\Gamma) : g_j \in \mathfrak{B}^s_{p,q}(W_j),\ j = 1, \dots, N\},$$

equipped with

$$\|f|\mathfrak{B}^s_{p,q}(\Gamma)\| := \sum_{j=1}^N \|g_j|\mathfrak{B}^s_{p,q}(W_j)\|.$$

Remark 3.13 We furnish $\mathfrak{B}^s_{p,q}(W_j)$ with the intrinsic $(n-1)$-dimensional norms according to Definition 3.8.
Note that we could furthermore replace W_j in the definition of the norm above by $\mathbb{R}^{n-1}$, if we extend g_j outside W_j with zero, hence

$$\|f|\mathfrak{B}^s_{p,q}(\Gamma)\| \sim \sum_{j=1}^N \|g_j|\mathfrak{B}^s_{p,q}(\mathbb{R}^{n-1})\|.$$

In analogy to Definition 3.12, we define classical Besov spaces on Γ as follows:

$$\mathbf{B}^s_{p,q}(\Gamma) := \{f \in L_p(\Gamma) : g_j \in \mathbf{B}^s_{p,q}(W_j),\ j = 1, \dots, N\},$$

equipped with the quasi-norm

$$\|f|\mathbf{B}^s_{p,q}(\Gamma)\| := \sum_{j=1}^N \|g_j|\mathbf{B}^s_{p,q}(W_j)\|.$$

By Theorem 3.10 it is clear that we have

$$\mathfrak{B}^s_{p,q}(\Gamma) = \mathbf{B}^s_{p,q}(\Gamma).$$

We complement our notation. Let $C^k(\mathbb{R}^n)$, $k \in \mathbb{N}$, stand for the collection of all complex-valued functions $h \in \mathbb{R}^n$ having bounded classical derivatives $\mathrm{D}^\alpha h(x)$ for all α with $0 \leq |\alpha| \leq k$.

Before we come to the main trace theorem, we collect a few results that will be indispensable for our proof.

Proposition 3.14 *Let $0 < p, q \leq \infty$, $s > 0$, $k \in \mathbb{N}$ with $k > s$.*

(i) (*Diffeomorphisms*)
Let ψ be a k-diffeomorphism. Then

$$f \longrightarrow f \circ \psi$$

is a linear and bounded operator from $\mathfrak{B}^s_{p,q}(\mathbb{R}^n)$ onto itself.

(ii) (*Pointwise multipliers*)
Let $h \in C^k(\mathbb{R}^n)$. Then

$$f \longrightarrow hf$$

is a linear and bounded operator from $\mathfrak{B}^s_{p,q}(\mathbb{R}^n)$ into itself.

Proof : We make use of the atomic decomposition according to Definition 1.3 with $K = k$. Concerning (i), if a is a 1_K-atom in the sense of Definition 1.2, then $a \circ \psi$ is also a 1_K-atom based on a new cube, and multiplied with a constant depending on ψ. But this is just what we need and we arrive at the desired assertion.
Similar for (ii). The atomic decomposition (1.3) multiplied with $h \in C^k$ gives an atomic decomposition of hf, which completes the proof. In particular,

$$\|hf|\mathfrak{B}^s_{p,q}(\mathbb{R}^n)\| \leq \sum_{|\alpha| \leq k} \sup_{x \in \mathbb{R}^n} |\mathrm{D}^\alpha h(x)| \cdot \|f|\mathfrak{B}^s_{p,q}(\mathbb{R}^n)\|.$$

□

We make use of an equivalent norm for $\mathfrak{B}^s_{p,q}(\Omega)$.

Proposition 3.15 *Let $0 < p, q \leq \infty$, $s > 0$, and Ω be a bounded C^k domain with $k > s$. Then*

$$\|\varphi_0 f|\mathfrak{B}^s_{p,q}(\mathbb{R}^n)\| + \sum_{j=1}^N \|(\varphi_j f)(\psi^{(j)}(\cdot))^{-1}|\mathfrak{B}^s_{p,q}(\mathbb{R}^n_+)\| \tag{3.8}$$

is an equivalent quasi-norm in $\mathfrak{B}^s_{p,q}(\Omega)$.

Proof : Let Ω_1 be a bounded domain with

$$\bar{\Omega}_1 \subset \left\{ x : x \in \mathbb{R}^n, \sum_{j=0}^N \varphi_j(x) = 1 \right\}$$

and $\bar{\Omega} \subset \Omega_1$. Let $f \in \mathfrak{B}^s_{p,q}(\Omega)$. If we restrict the infimum in Remark 3.9 to $g \in \mathfrak{B}^s_{p,q}(\mathbb{R}^n)$ with

$$g\big|_\Omega = f \qquad \text{and} \qquad \operatorname{supp} g \subset \Omega_1, \tag{3.9}$$

then we obtain a new equivalent quasi-norm in $\mathfrak{B}^s_{p,q}(\Omega)$. This follows from Proposition 3.14(ii) if one multiplies an arbitrary element $g \in \mathfrak{B}^s_{p,q}(\mathbb{R}^n)$ with a fixed infinitely differentiable function $\varkappa(x)$ with

$$\varkappa(x) = 1 \quad \text{if} \quad x \in \Omega \qquad \text{and} \qquad \operatorname{supp} \varkappa \subset \Omega_1.$$

For elements $g \in \mathfrak{B}^s_{p,q}(\mathbb{R}^n)$ with (3.9)

$$\sum_{k=0}^N \|\varphi_k g|\mathfrak{B}^s_{p,q}(\mathbb{R}^n)\|$$

is an equivalent quasi-norm. This is also a consequence of Proposition 3.14(ii). Applying part (i) of that proposition to $g(x) \to g(\psi^{(j)}(x))$, we see that

$$\|\varphi_0 g|\mathfrak{B}^s_{p,q}(\mathbb{R}^n)\| + \sum_{k=1}^{N} \|(\varphi_k g)(\psi^{(k)}(\cdot))^{-1}|\mathfrak{B}^s_{p,q}(\mathbb{R}^n)\|$$

is an equivalent quasi-norm for all $g \in \mathfrak{B}^s_{p,q}(\mathbb{R}^n)$ with (3.9). But the infimum over all admissible g with (3.9) yields (3.8). □

Now we can look for traces of $f \in \mathfrak{B}^s_{p,q}(\Omega)$ on Γ by the same type of reasoning as before. Since $S(\Omega)$ is dense in $\mathfrak{B}^s_{p,q}(\Omega)$ for $0 < p, q < \infty$ (both spaces can be interpreted as restrictions on their counterparts defined on $\mathbb{R}^n$), one asks first whether there is a constant $c > 0$ such that

$$\|\mathrm{Tr}_\Gamma \varphi|\mathfrak{B}^{s-\frac{1}{p}}_{pq}(\Gamma)\| \le c\|\varphi|\mathfrak{B}^s_{p,q}(\Omega)\| \quad \text{for all } \varphi \in S(\Omega).$$

If this is the case, then one defines $\mathrm{Tr}_\Gamma f \in \mathfrak{B}^{s-\frac{1}{p}}_{pq}(\Gamma)$ for $f \in \mathfrak{B}^s_{p,q}(\Omega)$ by completion and obtains

$$\|\mathrm{Tr}_\Gamma f|\mathfrak{B}^{s-\frac{1}{p}}_{pq}(\Gamma)\| \le c\|f|\mathfrak{B}^s_{p,q}(\Omega)\|, \quad f \in \mathfrak{B}^s_{p,q}(\Omega),$$

for the linear and bounded trace operator

$$\mathrm{Tr}_\Gamma : \mathfrak{B}^s_{p,q}(\Omega) \hookrightarrow \mathfrak{B}^{s-\frac{1}{p}}_{pq}(\Gamma).$$

The understanding of the trace operator when $p = \infty$ and/or $q = \infty$ is similar as for traces on hyperplanes. We refer to Remark 2.1.

Theorem 3.16 *Let $n \ge 2$, $0 < p, q \le \infty$, $s - \frac{1}{p} > 0$, and let Ω be a bounded C^k domain, $k > s$, in $\mathbb{R}^n$ with boundary $\Gamma = \partial\Omega$. Then $\mathrm{Tr}_\Gamma = \mathrm{Tr}$ is a linear and bounded operator from $\mathfrak{B}^s_{p,q}(\Omega)$ onto $\mathfrak{B}^{s-\frac{1}{p}}_{p,q}(\Gamma)$,*

$$\mathrm{Tr}\,\mathfrak{B}^s_{p,q}(\Omega) = \mathfrak{B}^{s-\frac{1}{p}}_{p,q}(\Gamma).$$

Proof : Step 1. We may restrict ourselves to smooth functions $f \in S(\Omega)$. Hence,

$$\mathrm{Tr}_\Gamma f = f\big|_\Gamma.$$

We wish to prove in this step that

$$\mathrm{Tr}_\Gamma \mathfrak{B}^s_{p,q}(\Omega) \subset \mathfrak{B}^{s-\frac{1}{p}}_{p,q}(\Gamma). \tag{3.10}$$

According to Theorem 3.5 there exists a bounded extension operator

$$\mathrm{Ex} : \mathfrak{B}^s_{p,q}(\mathbb{R}^n_+) \longrightarrow \mathfrak{B}^s_{p,q}(\mathbb{R}^n) \qquad \text{with} \qquad \|\mathrm{Ex}\, f|\mathfrak{B}^s_{p,q}(\mathbb{R}^n)\| \sim \|f|\mathfrak{B}^s_{p,q}(\mathbb{R}^n_+)\|.$$

In particular, for the trace operator $\mathrm{Tr}_{x_n=0}$ we see that

$$\mathrm{Tr}_{x_n=0}(\mathrm{Ex}\, h)(x) = \mathrm{Tr}_{x_n=0} h(x) = h(x', 0),$$

whenever h is smooth enough and the pointwise trace makes sense.
Using Theorem 2.2 we have

$$\|\mathrm{Tr}_{x_n=0} h|\mathfrak{B}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})\| = \|\mathrm{Tr}_{x_n=0}\mathrm{Ex}\, h|\mathfrak{B}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})\| \le c\|\mathrm{Ex}\, h|\mathfrak{B}^s_{p,q}(\mathbb{R}^n)\| \sim \|h|\mathfrak{B}^s_{p,q}(\mathbb{R}^n_+)\|, \tag{3.11}$$

which shows that

$$\mathrm{Tr}\,\mathfrak{B}^s_{p,q}(\mathbb{R}^n_+) \subset \mathfrak{B}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1}). \tag{3.12}$$

With this we calculate

$$\begin{aligned}
\|\mathrm{Tr}_\Gamma f|\mathfrak{B}^{s-\frac{1}{p}}_{p,q}(\Gamma)\| &= \|f\big|_\Gamma|\mathfrak{B}^{s-\frac{1}{p}}_{p,q}(\Gamma)\| \\
&= \sum_{j=1}^{N} \|\varphi_j f \circ (\psi^{(j)})^{-1}(\cdot, 0)|\mathfrak{B}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})\| \\
&\le \sum_{j=1}^{N} \|\varphi_j f \circ (\psi^{(j)})^{-1}|\mathfrak{B}^s_{p,q}(\mathbb{R}^n_+)\| \\
&\le c\|f|\mathfrak{B}^s_{p,q}(\Omega)\|, \qquad (3.13)
\end{aligned}$$

where in the third step we used (3.11) and the last step is a consequence of Proposition 3.15. In fact the calulations in (3.13) show that our problem (3.10) reduces to (3.12).

Step 2. In order to see that the trace operator Tr_Γ is onto $\mathfrak{B}^{s-\frac{1}{p}}_{p,q}(\Gamma)$, we establish the existence of a bounded extension operator

$$\widetilde{\mathrm{Ex}}\, : \mathfrak{B}^{s-\frac{1}{p}}_{p,q}(\Gamma) \longrightarrow \mathfrak{B}^{s}_{p,q}(\Omega), \qquad \widetilde{\mathrm{Ex}}\, g\big|_\Gamma = g,$$

such that for $g \in \mathfrak{B}^{s-\frac{1}{p}}_{p,q}(\Gamma)$ we have

$$\|\widetilde{\mathrm{Ex}}\, g|\mathfrak{B}^{s}_{p,q}(\Omega)\| \leq c\|g|\mathfrak{B}^{s-\frac{1}{p}}_{p,q}(\Gamma)\|.$$

We choose functions $\eta_j \in D(\mathbb{R}^n)$, $j = 1, \ldots, N$ with

$$\mathrm{supp}\,\eta_j \subset K_j, \qquad \eta_j = 1, \quad \text{if} \quad x \in \mathrm{supp}\,\varphi_j.$$

Put

$$\widetilde{\mathrm{Ex}}\, g(x) = \sum_{j=1}^{N} \eta_j(x) \cdot \mathrm{Ex}\, \Big((\varphi_j g)(\psi^{(j)})^{-1}(\cdot, 0)\Big) \Big(\psi^{(j)}(x)\Big), \qquad x \in \Omega,$$

where

$$\mathrm{Ex}\, : \mathfrak{B}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1}) \longrightarrow \mathfrak{B}^{s}_{p,q}(\mathbb{R}^n)$$

stands for the extension operator constructed in Theorem 2.2, Step 3. In particular, our construction can be extended from Ω to $\mathbb{R}^n$ by putting $\eta_j(x) \cdot \mathrm{Ex}\,(\ldots)\big(\psi^{(j)}(x)\big) = 0$ outside K_j. This yields

$$\begin{aligned}
\|\widetilde{\mathrm{Ex}}\, g|\mathfrak{B}^{s}_{p,q}(\Omega)\| &= \inf \|h|\mathfrak{B}^{s}_{p,q}(\mathbb{R}^n)\|, \qquad h\big|_\Omega = \widetilde{\mathrm{Ex}}\, g \\
&\leq \left\| \sum_{j=1}^{N} \eta_j(\cdot)\mathrm{Ex}\, \Big((\varphi_j g)(\psi^{(j)})^{-1}(\cdot, 0)\Big)\, (\psi^{(j)}(\cdot))|\mathfrak{B}^{s}_{p,q}(\mathbb{R}^n) \right\| \\
&\sim \sum_{j=1}^{N} \|\eta_j(\cdot)\mathrm{Ex}\, \Big((\varphi_j g)(\psi^{(j)})^{-1}(\cdot, 0)\Big)\, (\psi^{(j)}(\cdot))|\mathfrak{B}^{s}_{p,q}(\mathbb{R}^n)\| \\
&\sim \sum_{j=1}^{N} \|\mathrm{Ex}\, \Big((\varphi_j g)(\psi^{(j)})^{-1}(\cdot, 0)\Big)\, |\mathfrak{B}^{s}_{p,q}(\mathbb{R}^n)\| \\
&\leq c \sum_{j=1}^{N} \|(\varphi_j g)(\psi^{(j)})^{-1}(\cdot, 0)|\mathfrak{B}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})\| \\
&= c\|g|\mathfrak{B}^{s-\frac{1}{p}}_{p,q}(\Gamma)\|,
\end{aligned}$$

where in the 4th step we used Proposition 3.14(i), (ii), since $\psi^{(j)}$ is a k-diffeomorphism from $\mathbb{R}^n$ onto itself $(k > s)$, and $\eta_j \in D(\mathbb{R}^n)$ if we put $\eta_j(x) = 0$ if $x \in \mathbb{R}^n \setminus K_j$. This completes the proof. □

We obtain similar results for the limiting case.

Corollary 3.17 *Let* $0 < p < \infty$ *and* $0 < q \leq \infty$. *Then*

$$\mathrm{Tr}\,\mathfrak{B}^{1/p}_{p,q}(\Omega) = L_p(\Gamma), \qquad 0 < q \leq \min(1, p).$$

Proof : The proof is the same as the proof of Theorem 3.16 – using Corollary 2.5 and Remark 2.6 instead of Theorem 2.2 this time. □

Remark 3.18 By Theorem 3.10 and Remark 3.13 for bounded C^k domains Ω with boundary $\Gamma = \partial\Omega$ we have

$$\mathbf{B}^{s}_{p,q}(\Omega) = \mathfrak{B}^{s}_{p,q}(\Omega) \qquad \text{and} \qquad \mathbf{B}^{s}_{p,q}(\Gamma) = \mathfrak{B}^{s}_{p,q}(\Gamma).$$

Therefore our trace results in Theorem 3.16 and Corollary 3.17 hold for the classical spaces $\mathbf{B}^{s}_{p,q}$ as well.

We are now able to consider traces on the spaces $\mathfrak{F}^s_{p,q}(\Omega)$.

Theorem 3.19 *Let* $0 < p < \infty$, $0 < q \leq \infty$, $s > \frac{1}{p}$, *and* $\Omega \subset \mathbb{R}^n$ *a* C^k *domain with* $k > s$. *Then*

$$\operatorname{Tr} \mathfrak{F}^s_{p,q}(\Omega) = \mathfrak{B}^{s-\frac{1}{p}}_{p,p}(\Gamma). \tag{3.14}$$

Furthermore, in the limiting case $s = \frac{1}{p}$ *we have*

$$\operatorname{Tr} \mathfrak{F}^{1/p}_{p,q}(\Omega) = L_p(\Gamma), \qquad 0 < p \leq 1. \tag{3.15}$$

Proof : The first assertion (3.14) is proven similar to the corresponding assertion on $\mathbb{R}^n$, cf. Theorem 2.10. We only need to show that the trace is independent of q. Since

$$\|f|\mathfrak{F}^s_{p,q}(\Omega)\| = \inf \|g|\mathfrak{F}^s_{p,q}(\mathbb{R}^n)\|, \qquad g\big|_\Omega = f,$$

for $0 < q \leq r \leq \infty$ we have

$$\mathfrak{F}^s_{p,q}(\Omega) \hookrightarrow \mathfrak{F}^s_{p,r}(\Omega),$$

leading to

$$\operatorname{Tr} \mathfrak{F}^s_{p,q}(\Omega) \subset \operatorname{Tr} \mathfrak{F}^s_{p,r}(\Omega).$$

For the other direction let $g \in \mathfrak{F}^s_{p,r}(\Omega)$ with optimal atomic decomposition

$$g = \sum_j \sum_m{}^{\Omega,j} \lambda_{j,m} a_{j,m}, \qquad \text{i.e,} \qquad \|f|\mathfrak{F}^s_{p,r}(\Omega)\| \sim \|\lambda|f^s_{p,r}(\Omega)\|.$$

Setting

$$f(x) = \sum_{j,m} \Lambda_{j,m} a_{j,m}(x), \qquad x \in \mathbb{R}^n,$$

with

$$\Lambda_{j,m} = \begin{cases} \lambda_{j,m}, & dQ_{j,m} \cap \Omega \neq \emptyset \\ 0, & \text{otherwise}, \end{cases}$$

we see that $f\big|_\Omega = g$ and

$$\|g|\mathfrak{F}^s_{p,r}(\Omega)\| \sim \|\Lambda|f^s_{p,r}\| \sim \|f|\mathfrak{F}^s_{p,r}(\mathbb{R}^n)\|.$$

We need to show the existence of $\tilde{g} \in \mathfrak{F}^s_{p,q}(\Omega)$ such that

$$\operatorname{Tr} g = \operatorname{Tr} \tilde{g}.$$

We put

$$\tilde{g} = \sum_{j,m}{}^{\Omega,j} \tilde{\lambda}_{j,m} a_{j,m},$$

where

$$\tilde{\lambda}_{j,m} = \begin{cases} \lambda_{j,m}, & dQ_{j,m} \cap \Omega \cap \Gamma \neq \emptyset \\ 0, & dQ_{j,m} \cap \Omega \neq \emptyset. \end{cases}$$

Again we extend $\tilde{g}$ to $\mathbb{R}^n$ by considering

$$f = \sum_{j,m} \tilde{\Lambda}_{j,m} a_{j,m},$$

with

$$\tilde{\Lambda}_{j,m} = \begin{cases} \lambda_{j,m}, & dQ_{j,m} \cap \Omega \cap \Gamma \neq \emptyset \\ 0, & \text{otherwise}. \end{cases}$$

In particular, $\tilde{f}\big|_\Omega = \tilde{g}$. Note that since $\Gamma = \partial\Omega$ is compact we only sum over finitely many $m \in \mathbb{Z}^n$. By the same observations as in Theorem 2.10 we obtain

$$\|\tilde{g}|\mathfrak{F}^s_{p,q}(\Omega)\| \leq \|\tilde{f}|\mathfrak{F}^s_{p,q}(\mathbb{R}^n)\| \leq \|f|\mathfrak{F}^s_{p,r}(\mathbb{R}^n)\| \sim \|g|\mathfrak{F}^s_{p,r}(\Omega)\|.$$

This proves (3.14).
For the second assertion we use the above observations and apply Corollary 3.17. This yields

$$\operatorname{Tr} \mathfrak{F}^{1/p}_{p,q}(\Omega) = \operatorname{Tr} \mathfrak{F}^{1/p}_{p,p}(\Omega) = \operatorname{Tr} \mathfrak{B}^{1/p}_{p,p}(\Omega) = L_p(\Gamma), \qquad 0 < p \leq 1.$$

□

4 Dichotomy: traces versus density

4.1 Preliminaries

So far we were concerned with exact traces of spaces $\mathfrak{A}^s_{p,q}$, where $\mathfrak{A} \in \{\mathfrak{B}, \mathfrak{F}\}$, with $n \geq 2$, $s > 0$, and $0 < p, q < \infty$ either on hyper-planes $\mathbb{R}^m$, $n > m \in \mathbb{N}$ or on boundaries $\Gamma = \partial\Omega$ of bounded C^k domains Ω. In the sequel let Ω and Γ denote either $\mathbb{R}^n$ and $\mathbb{R}^m$ or a bounded C^k domain Ω and its boundary $\partial\Omega$, respectively, i.e.,

$$\begin{cases} \Omega = \mathbb{R}^n & \text{and} \quad \Gamma = \mathbb{R}^m \quad \text{or} \\ \Omega \quad C^k\text{-domain} & \text{and} \quad \Gamma = \partial\Omega. \end{cases}$$

Furthermore μ stands for either the m-dimensional Lebesgue measure l_m or a surface measure $\mathrm{d}\sigma$.
We now adopt a slightly more general point of view. Again we understand traces as limits of pointwise traces of smooth functions (recall that $D(\Omega)$ and $S(\Omega)$ are dense in all the spaces $\mathfrak{A}^s_{p,q}(\Omega)$, excluding $p = \infty$ and/or $q = \infty$). Therefore, if for some $c > 0$ we have

$$\|\varphi | L_r(\Gamma)\| \leq c \|\varphi | \mathfrak{A}^s_{p,q}(\Omega)\| \qquad \text{for all } \varphi \in S(\Omega), \tag{4.1}$$

the trace operator Tr_Γ,

$$\mathrm{Tr}_\Gamma : \mathfrak{A}^s_{p,q}(\Omega) \hookrightarrow L_r(\Gamma)$$

is the completion of the pointwise trace $(\mathrm{Tr}_\Gamma \varphi)(\gamma) = \varphi(\gamma)$ with $\varphi \in S(\Omega)$ and $\gamma \in \Gamma$.

Remark 4.1 In particular, it can be shown that for individual elements f the traces are independent of the source spaces and of the target spaces as long as one has (4.1) and whenever comparison makes sense, cf. [Tri08a, Rem. 13].

Let $D_\Gamma = D(\Omega \setminus \Gamma)$ be as usual the collection of all (complex-valued) C^∞ functions in Ω with compact support in $\Omega \setminus \Gamma$.

One may ask the two mutually exclusive questions (see also Proposition 4.4 below):

(i) In which of the above spaces $\mathfrak{A}^s_{p,q}(\Omega)$ is D_Γ dense?

(ii) For which of the above spaces $\mathfrak{A}^s_{p,q}(\Omega)$ does there exist a linear and bounded trace operator

$$\mathrm{Tr}_\Gamma : \mathfrak{A}^s_{p,q}(\Omega) \hookrightarrow L_p(\Gamma) \qquad ?$$

It comes out that the above spaces divide sharply in these two contrasting classes (dichotomy).

The well-known inclusion properties of the spaces $\mathfrak{A}^s_{p,q}$ under consideration suggest the following formulation.

Definition 4.2 *Let* $0 < p < \infty$ *and let*

$$\mathfrak{A}_p(\Omega) = \{\mathfrak{A}^s_{p,q}(\Omega) : 0 < q < \infty,\ s > 0\}.$$

Let $\sigma > 0$. *Then*

$$\mathbb{D}(\mathfrak{A}_p(\Omega), L_p(\Gamma)) = (\sigma, u) \qquad \text{with} \qquad 0 < u < \infty$$

is called dichotomy of $\{\mathfrak{A}_p(\Omega), L_p(\Gamma)\}$ *if*

$$\mathrm{Tr}_\Gamma \text{ exists for } \begin{cases} s > \sigma, & 0 < q < \infty, \\ s = \sigma, & 0 < q \leq u, \end{cases}$$

and

$$D_\Gamma \text{ is dense in } \mathfrak{A}^s_{p,q}(\Omega) \text{ for } \begin{cases} s = \sigma, & u < q < \infty, \\ s < \sigma, & 0 < q < \infty. \end{cases}$$

Furthermore,

$$\mathbb{D}(\mathfrak{A}_p(\Omega), L_p(\Gamma)) = (\sigma, 0)$$

means that

$$\begin{cases} \mathrm{Tr}_\Gamma \text{ exists for } s > \sigma,\ 0 < q < \infty, \\ D_\Gamma \text{ is dense in } \mathfrak{A}^s_{p,q}(\Omega) \text{ for } s \le \sigma,\ 0 < q < \infty; \end{cases}$$

and

$$\mathbb{D}(\mathfrak{A}_p(\Omega), L_p(\Gamma)) = (\sigma, \infty)$$

means that

$$\begin{cases} \mathrm{Tr}_\Gamma \text{ exists for } s \ge \sigma,\ 0 < q < \infty, \\ D_\Gamma \text{ is dense in } \mathfrak{A}^s_{p,q}(\Omega) \text{ for } s < \sigma,\ 0 < q < \infty. \end{cases}$$

Remark 4.3 The above definition makes sense. Let $s \ge \sigma > 0$, $0 < p < \infty$, and $0 < q_1, q_2 < \infty$. We have the continuous embedding

$$\mathfrak{A}^s_{p,q_1}(\Omega) \hookrightarrow \mathfrak{A}^\sigma_{p,q_2}(\Omega), \tag{4.2}$$

whenever

$$s \ge \sigma \qquad \text{and} \qquad q_1 \le q_2 \quad \text{if} \quad s = \sigma. \tag{4.3}$$

For $\Omega = \mathbb{R}^n$ this follows from Proposition 1.5. But the embedding results stated there can be generalized to C^k domains Ω, since the spaces $\mathfrak{A}^s_{p,q}(\Omega)$ are defined by restriction of the corresponding spaces on $\mathbb{R}^n$, cf. Remark 3.9.
If the traces exist in $\mathfrak{A}^\sigma_{p,q_2}(\Omega)$, then automatically all spaces on the left-hand side in (4.2) have traces as well.
Furthermore, if D_Γ is dense in $\mathfrak{A}^s_{p,q_1}(\Omega)$, the embedding (4.2) together with the density of $D(\Omega)$ in all spaces in (4.2) imply the density of D_Γ in $\mathfrak{A}^\sigma_{p,q_2}(\Omega)$. This can easily be seen. If $\varphi \in D(\Omega)$ and $\psi_j \in D_\Gamma$ is an approximating sequence in $\mathfrak{A}^s_{p,q_1}(\Omega)$, we have

$$\|\varphi - \psi_j | \mathfrak{A}^\sigma_{p,q_2}(\Omega)\| \le c \|\varphi - \psi_j | \mathfrak{A}^s_{p,q_1}(\Omega)\| \longrightarrow 0.$$

Additionally, one has the following almost obvious observation.

Proposition 4.4 *Let $s > 0$, $0 < p, q < \infty$, $0 < r < \infty$ and let D_Γ be dense in $\mathfrak{A}^s_{p,q}(\Omega)$. Then there is no $c > 0$ with*

$$\|\varphi | L_r(\Gamma)\| \le c \|\varphi | \mathfrak{A}^s_{p,q}(\Omega)\|, \qquad \varphi \in S(\Omega). \tag{4.4}$$

Proof: We assume that there is a constant $c > 0$ with (4.4). First let Ω be a bounded C^k domain. We approximate a function φ which is identically 1 near Γ by D_Γ-functions ψ_j, $j \in \mathbb{N}$. Then one has that

$$\mathrm{Tr}_\Gamma \varphi = \lim_{j \to \infty} \mathrm{Tr}_\Gamma \psi_j = 0 \quad \mu - a.e.$$

But this contradicts $\mu(\Gamma) > 0$. The proof is similar when $\Omega = \mathbb{R}^n$. In this case we have

$$\mathbb{R}^n \subset \bigcup_{l=0}^{\infty} K_l,$$

where K_l are appropriate compact sets. Put

$$\Gamma_l := \mathbb{R}^m \cap K_l,$$

where Γ_l may be interpreted as a subset of $\mathbb{R}^m$ (considered as a space itself and not just a hyperplane of $\mathbb{R}^n$). We approximate a function φ_l which is identically 1 near Γ_l and has support in a neighbourhood of K_l by D_Γ-functions ψ_j, $j \in \mathbb{N}$. Since

$$\mu(\Gamma) \le \sum_{l=0}^{\infty} \mu(\Gamma_l),$$

similar as above we derive a contradiction. □

4.2 Dichotomy

Our main result for $\Omega = \mathbb{R}^n$ and hyperplanes $\Gamma = \mathbb{R}^m$ is stated in the theorem below.

Theorem 4.5 *Let $n, m \in \mathbb{N}$, $n > m$, and $0 < p < \infty$. Then*

$$\mathbb{D}\left(\mathfrak{B}_p(\mathbb{R}^n), L_p(\mathbb{R}^m)\right) = \begin{cases} \left(\frac{n-m}{p}, 1\right) & \text{if } p > 1, \\ \left(\frac{n-m}{p}, p\right) & \text{if } p \le 1, \end{cases} \tag{4.5}$$

and

$$\mathbb{D}\left(\mathfrak{F}_p(\mathbb{R}^n), L_p(\mathbb{R}^m)\right) = \begin{cases} \left(\frac{n-m}{p}, 0\right) & \text{if } p > 1, \\ \left(\frac{n-m}{p}, \infty\right) & \text{if } p \le 1. \end{cases} \tag{4.6}$$

Proof : The proof is based on ideas from a similar proof in [Tri08a]. Note that D_Γ here stands for the set of all C^∞ functions defined on $\mathbb{R}^n$ with compact support in $\mathbb{R}^n \setminus \mathbb{R}^n$.
Step 1. We have to show that the breaking points (σ, u) exist and that they coincide with the right-hand sides of (4.5) and (4.6). By Corollary 2.5, Remark 2.7, and our discussions in Remark 4.3 in case of the B-spaces it remains to prove that

$$D_\Gamma \text{ is dense in } \mathfrak{B}^{\frac{n-m}{p}}_{p,q}(\mathbb{R}^n) \quad \text{if} \quad 0 < p < \infty,\ q > \min(p, 1), \tag{4.7}$$

which will be done in Steps 3 and 5 below. Concerning the F-spaces if $p \le 1$ we have

$$\mathfrak{B}^{\frac{n-m}{p}}_{p,\tilde{q}}(\mathbb{R}^n) \hookrightarrow \mathfrak{F}^{\frac{n-m}{p}-\varepsilon}_{p,q}(\mathbb{R}^n), \qquad 0 < q < \infty,\ \tilde{q} > p,\ \varepsilon > 0. \tag{4.8}$$

$D(\mathbb{R}^n)$ is dense in both spaces. Using (4.7) (with $\tilde{q}$) now yields that D_Γ is dense in all spaces on the right-hand side of (4.8). This together with Corollary 2.11 and Remark 2.12 already gives the bottom line in (4.6). As for the case $1 < p < \infty$ we have

$$\mathfrak{F}^{\frac{n-m}{p}+\varepsilon}_{p,q}(\mathbb{R}^n) \hookrightarrow \mathfrak{B}^{\frac{n-m}{p}}_{p,1}(\mathbb{R}^n), \qquad 0 < q < \infty,\ \varepsilon > 0. \tag{4.9}$$

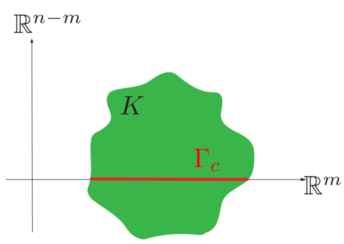
By Corollary 2.5 all spaces on the right-hand side of (4.9) have traces. It therefore remains to prove in the case of F-spaces that

$$D_\Gamma \text{ is dense in } \mathfrak{F}^{\frac{n-m}{p}}_{p,q}(\mathbb{R}^n) \quad \text{if} \quad 1 < p < \infty,\ 0 < q < \infty, \tag{4.10}$$

which we do in Step 4.

Step 2. We begin with a preparation. Let $K \subset \mathbb{R}^n$ be a compact set (which in Step 3 will be chosen to be the support of $f \in D(\mathbb{R}^n)$, the function we wish to approximate) and consider

$$\Gamma_C = \mathbb{R}^m \cap K.$$



The aim is to construct a sequence $\{\varphi^J\}_{J=1}^{\infty} \in D(\mathbb{R}^n)$ with

$$\varphi^J(x) = 1 \text{ in an open neighbourhood of } \Gamma_C \tag{4.11}$$

(depending on J) and

$$\varphi^J \longrightarrow 0 \quad \text{in } \mathfrak{B}^{\frac{n-m}{p}}_{p,q}(\mathbb{R}^n) \qquad \text{if} \qquad 0 < p < \infty, \quad q > 1. \tag{4.12}$$

For given $j \in \mathbb{N}$ we cover a neighbourhood of Γ_C with balls $B_{j,k}$ in $\mathbb{R}^n$ centred at Γ_C and of radius 2^{-j}, where $k = 1, \ldots, M_j$ and $M_j \sim 2^{jm}$ (which is possible since Γ_C is compact) such that there is a resolution of unity,

$$\sum_{k=1}^{M_j} \varphi_{j,k}(x) = 1 \text{ near } \Gamma_C, \quad 0 \le \varphi_{j,k} \in D(B_{j,k}), \tag{4.13}$$

with the usual properties,

$$|\mathrm{D}^\gamma \varphi_{j,k}(x)| \le c_\gamma 2^{j|\gamma|}, \qquad \gamma \in \mathbb{N}_0^n. \tag{4.14}$$

For $2 \le J \in \mathbb{N}$, let $J' \in \mathbb{N}$ be such that

$$\sum_{j=J}^{J'+1} r_j = 1 \quad \text{with } r_j = j^{-1} \text{ if } J \le j \le J' \text{ and } 0 < r_{J'+1} \le (J'+1)^{-1}.$$

Then

$$\varphi^J(x) = \sum_{j=J}^{J'+1} r_j \sum_{k=1}^{M_j} \varphi_{j,k}(x), \qquad x \in \mathbb{R}^n, \tag{4.15}$$

is an atomic decomposition in $\mathfrak{B}^s_{p,q}(\mathbb{R}^n)$ according to Definition 1.3 for any $s > 0$, $0 < p < \infty$. Setting $s = \frac{n-m}{p}$ such that $s - \frac{n}{p} = -\frac{m}{p}$ one gets for $q > 1$

$$\|\varphi^J|\mathfrak{B}^{\frac{n-m}{p}}_{p,q}(\mathbb{R}^n)\|^q \le c \sum_{j=J}^{J'+1} r_j^q 2^{-j\frac{mq}{p}} \left(\sum_{k=1}^{M_j} 1\right)^{q/p} \le c' \sum_{j=J}^{\infty} j^{-q} \sim J^{1-q} \longrightarrow 0 \qquad \text{as} \quad J \to \infty. \tag{4.16}$$

This proves (4.12).

Step 3. We prove (4.7) for $p > 1$, $q > 1$. It is sufficient to approximate $f \in D(\mathbb{R}^n)$ in $\mathfrak{B}^s_{p,q}(\mathbb{R}^n)$, $s = \frac{n-m}{p}$, by functions $f^J \in D_\Gamma$. Put

$$K := \operatorname{supp} f \qquad \text{and} \qquad \Gamma_C := K \cap \mathbb{R}^m.$$

Let φ^J be the functions constructed in Step 2 and

$$f = f_J + f^J \quad \text{with} \quad f_J = \varphi^J f \quad \text{and} \quad f^J = (1-\varphi^J) f \in D_\Gamma.$$

(We choose a different resolution of unity φ_J for every f.)
By Proposition 3.14(ii) one has for some $c > 0$, $f \in D(\mathbb{R}^n)$, and φ^J that

$$\|f_J|\mathfrak{B}^s_{p,q}(\mathbb{R}^n)\| \le \|f|C^\infty(\mathbb{R}^n)\| \cdot \|\varphi^J|\mathfrak{B}^s_{p,q}(\mathbb{R}^n)\| \le c\|\varphi^J|\mathfrak{B}^s_{p,q}(\mathbb{R}^n)\| \longrightarrow 0 \qquad \text{as } J \to \infty, \tag{4.17}$$

using (4.12).

Step 4. We prove (4.10). By Theorem 2.10 and Corollary 2.11 we see that the trace of f in $\mathfrak{F}^s_{p,q}$ is independent of q. Therefore,

$$\|\varphi^J|\mathfrak{F}^{\frac{n-m}{p}}_{p,q}(\mathbb{R}^n)\| \sim \|\varphi^J|\mathfrak{B}^{\frac{n-m}{p}}_{p,p}(\mathbb{R}^n)\| \longrightarrow 0 \quad \text{if} \quad J \to \infty,$$

cf. the constructions in Theorem 2.10. Then one gets (4.10) by the same arguments as in Step 3 for all $1 < p < \infty$ and $0 < q < \infty$.

Step 5. We prove (4.7) for the remaining case when $p < q$ (in particular $p \le 1$), constructing now a more refined resolution of unity as in Step 2. We cover the compact set Γ_C, say with $\mu(\Gamma_C) = 1$, for given $L \in \mathbb{N}$ by sets Γ_l such that

$$\Gamma_C = \bigcup_{l=L}^{L'} \Gamma_l, \qquad \mu(\Gamma_l) \sim l^{-1}, \qquad \sum_{l=L}^{L'} \mu(\Gamma_l) \sim \mu(\Gamma_C) = 1,$$

where $L \in \mathbb{N}$ with $L' > L$ is appropriately chosen. For the details we refer to [Tri08b, Th. 6.68]. In particular, this can be done in such a way that there are functions $\psi_l \in D(\mathbb{R}^n)$, $\psi_l \ge 0$,

$$\sum_{l=L}^{L'} \psi_l(\gamma) = 1 \text{ if } \gamma \in \Gamma_C, \quad \Gamma_l \subset \operatorname{supp} \psi_l \subset \{y \in \mathbb{R}^n : \operatorname{dist}(y, \Gamma_l) < \varepsilon_l\}$$

for some $\varepsilon_l > 0$. Let for given $l \in \mathbb{N}$ (between L and L') and appropriately chosen $j(l) \in \mathbb{N}$,

$$\sum_{k=1}^{M_{j(l)}} \varphi_{j(l),m}(x) = 1 \text{ near } \Gamma_C, \qquad 0 \le \varphi_{j(l),k} \in D(B_{j(l),k})$$

as in (4.13) with the counterpart of (4.14) and $M_{j(l)} \sim 2^{j(l)m}$. With

$$j(L) < \cdots < j(l) < j(l+1) < \cdots < j(L'),$$

we put in analogy to (4.15)

$$\varphi^L(x) = \sum_{l=L}^{L'} \psi_l(x) 2^{-\frac{j(l)m}{p}} \sum_{k=1}^{M_{j(l)}} 2^{\frac{j(l)m}{p}} \varphi_{j(l),k}(x), \qquad x \in \mathbb{R}^n.$$

With $j(l)$ chosen large enough this is an atomic decomposition which can be written as

$$\varphi^L(x) = \sum_{l=L}^{L'} 2^{-\frac{j(l)m}{p}} \sum_{k=1}^{M'_{j(l)}} 2^{\frac{j(l)m}{p}} \tilde{\varphi}_{j(l),k}(x), \qquad x \in \mathbb{R}^n,$$

with

$$M'_{j(l)} \sim \mu(\Gamma_l) 2^{j(l)m} \sim l^{-1} 2^{j(l)m}$$

counting only non-vanishing terms, where the equivalence constants are independent of l. We have $\varphi^L(x) = 1$ near Γ_C. Then one gets by Definition 1.3 for $q > p$,

$$\|\varphi^L | \mathfrak{B}^{\frac{n-m}{p}}_{p,q}(\mathbb{R}^n)\|^q \le c \sum_{l=L}^{L'} 2^{-j(l)\frac{mq}{p}} \left(\sum_{k=1}^{M'_{j(l)}} 1 \right)^{q/p} \le c' \sum_{l=L}^{L'} l^{-q/p} \sim L^{1-\frac{q}{p}}.$$

This is the counterpart of (4.16). Proceeding as in Step 3 finally proves the Theorem. □

Theorem 4.5 now can be gereralized to smooth domains with boundary $\Gamma = \partial\Omega$.

Corollary 4.6 *Let $\Omega \subset \mathbb{R}^n$ be a C^∞ domain with boundary $\Gamma = \partial\Omega$, $n \in \mathbb{N}$, and $0 < p < \infty$. Then*

$$\mathbb{D}\left(\mathfrak{B}_p(\Omega), L_p(\Gamma)\right) = \begin{cases} \left(\frac{1}{p}, 1\right) & \text{if } p > 1, \\ \left(\frac{1}{p}, p\right) & \text{if } p \le 1 \end{cases} \tag{4.18}$$

and

$$\mathbb{D}\left(\mathfrak{F}_p(\Omega), L_p(\Gamma)\right) = \begin{cases} \left(\frac{1}{p}, 0\right) & \text{if } p > 1, \\ \left(\frac{1}{p}, \infty\right) & \text{if } p \le 1. \end{cases} \tag{4.19}$$

Proof : Recalling Remark 4.3 the proof basically is the same as the proof of Theorem 4.5 now using Corollary 3.17 instead of Corollary 2.5. The proof of Steps 2,3 and 5 is even simpler since by definition the boundary Γ of Ω is compact so we can take the same sequence φ^J for all functions $f \in \mathfrak{B}^s_{p,q}(\Omega)$.
Concerning the results for the F-spaces, the proof is similar to Step 4 in Theorem 4.5, where we now use the results from Theorem 3.19. □

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