Mathematics 3, WS 2014/2015

Prof. F. Brock

Summary

I. Existence and uniqueness of IVP's

Theorems of Picard–Lindelöf and of Peano for systems of first order.

Examples.

II. Integral Calculus in $\mathbb{R}^n$

1. Domain integrals: (see: A. Schüler, Chapter 9; Fischer/Kaul 1, pp. 443–469)

Volume of an interval in $\mathbb{R}^n$.

Integral of a step function.

Bounded continuous functions on intervals can be uniformly approximated by step functions.

Definition of an integral of a continuous function over an interval as the limit of a sequence of integrals of step functions.

Every open set can be represented as a union of compact intervals with mutually disjoint interior.

Definition of the integral of a continuous function over an open set, volume of an open set.

Linearity, monotonicity, additivity of the integral.

Iterated integration, Theorem of Fubini, integration over normal domains, Cavalieri's principle, examples, bodies of revolution.

Change of variables formula.

Examples, polar and spherical coordinates, problems with radial symmetry, center of mass, moments of inertia.

2. Line integrals: (see A. Schüler, Chapter 8, Fischer/Kaul 1, pp. 470–492)

Simple curves, orientation, examples.

Regular curves, tangent vectors, angle between curves.

Rectifiable curves, length of a curve, arc length differential, examples.

Definition of a line integral, properties, independence of the parametrization, dependence on the orientation, examples, work done by a force field along a curve.

Scalar line integral, mass of a curve.

Conservative vector fields, path independence of the line integral.

Every conservative vector field is a gradient field.
Potential of a gradient field, example: gravitational force.
Evaluation of the potential by using line integrals with variable endpoints, or by integration, examples.
Integrability conditions. Example of a vectorfield which is not conservative, but satisfies the integrability conditions.
Smooth vector fields that satisfy the integrability conditions are conservative in simply connected domains.

3. Surface integrals: (see A. Schüler, Chapter 10, Fischer/Kaul 1, pp. 493–503)
Regular surfaces in $\mathbb{R}^3$, tangent vectors, normal vector, explicit and implicit description of surfaces.
Area of a surface. Example: surface of revolution.
Scalar surface integrals. Examples.
Integration over (concentric) spheres.
Center of mass of a surface.
Potential of a charged surface.
Orientation of a surface, unit normal fields.
Vectorial surface integral; several writings.
Flow of a vectorfield through a surface.
Examples.
Nabla calculus : ∇. Gradient, divergence, curl. Some differentiation formulas.
Gauss' Divergence Theorem. Proof in a special case. Example.
Applications: Volume, using surface integrals.
Physical interpretation of the divergence as a source density of a vectorfield.
Green's formulas. Application to Laplace' equation.
Green's Theorem. Application: area of a plane region.
Curl- and divergence form of Green's Theorem.
Surfaces in $\mathbb{R}^3$ with boundary. Induced orientation.
Stokes' Theorem.
Circulation. Curl as an infinitesimal circulation and density of vorticity.
Application: path independence of line integrals.
Source-free vector fields and vector potentials.
The inverse problem of vector analysis.

III. Complex Analysis

1. Complex differentiation: (see A. Schüler, 363–364, 366–369; Fischer/Kaul 1, pp. 533–537)

Complex-valued functions of a complex variable.

Limits, continuity.

Complex differentiability. Examples.

Holomorphic functions. Cauchy-Riemann differential equations.

If $f = u + iv$ is holomorphic, then the vector fields $(u, -v)$ and (v, u) satisfy the integrability conditions.

2. Complex line integrals:

Arc length, independence of the parametrization, reduction to real line integrals.

2 fundamental formulas: $\int_{|z-z_0|=r} dz/(z - z_0) = 2\pi i$, $\int_{|z-z_0|=r} (z - z_0)^n\, dz = 0$ for $n \in \mathbb{Z} \setminus \{-1\}$.

Path independence of line integrals.

Antiderivatives.

2 multi-valued functions: logarithm, n-th root.

Compact convergence.

Interchange of limit and integration over lines.

Term-by-term differentiation of power series.

Analytic functions. Identity theorem for power series.

Zeros of analytic functions.

Cauchy's integral theorem. Homological pathes.

Cauchy's integral formula.

Power series for holomorphic functions, Cauchy-formulas for the coefficients.

Identity theorem for holomorphic functions.

Examples of power series: e^z, $\sin z$, $\cos z$, $\log(1 + z)$.

Estimate for the coefficients in a power series.

Entire functions.

Liouville's Theorem: Every bounded entire function is constant.

Proof of the Fundamental Theorem of Algebra.

Theorem of Morera.

Compact convergence for sequences of holomorphic functions.

Isolated singularities: Removable singularities, poles, essential singularities.

Examples.
Definition of Laurent series; singular and regular part.
Convergence of Laurent series in an annulus. Cauchy-formulas for the coefficients.
Characterization of isolated singularities by means of Laurent series.
Theorem of Casorati-Weierstrass.
Entire-transcendental functions.
Residue Theorem.
Evaluation of residues. Examples.
Real integrals:
Rational functions of Sine and Cosine. Improper integrals $\int_{-\infty}^{+\infty} f(x)\,dx$.
Integrands of the type $g(z)e^{i\alpha z}$, $(\alpha > 0)$.
A basic integral for the Fourier transform.

IV. Some partial differential equations

(see Fischer, Kaul, Volume 2, Chapter III and IV)
1. Introduction: Laplace equation, heat equation, wave equation.
2. Solutions via Fourier series:
Motivation: Initial-boundary value problem for the clamped vibrating string; Ansatz with separation of variables.
Fourier series for functions on $[-\pi, \pi]$, Euler-Fourier formulas for the coefficients.
Piecewise continuous and piecewise smooth functions.
2π-periodic extension, periodic standard extension.
Theorem of Dirichlet about the pointwise convergence of Fourier series.
Special case: If u is piecewise smooth, then the Fourier series of u converges uniformly to u.
Gibbs' phenomenon.
Decay of the Fourier coefficients for piecewise continuous and for piecewise smooth functions.
Expansion of functions on $[0, L]$ into Fourier Sine and into Fourier Cosine series.
Weierstrass' Theorem about approximation of continuous functions by polynomials.
Solution of initial-boundary value problems for the vibrating string. d'Alembert's

solution formula. Conservation of energy. Uniqueness of the solution.
Vibrating string under exterior force.
Heat conduction in a wire, initial-boundary value problem for the heat equation in one space dimension. Solution via Fourier series.
Maximum principle and uniqueness for the heat equation.
Laplace equation, harmonic functions. Relation to complex analysis.
Dirichlet problem for the Laplace equation in a rectangle.
Dirichlet problem for the Laplace equation in a disc, transformation to polar coordinates, solution via separation of variables and Fourier series, Poisson's solution formula.
Maximum principle and uniqueness for the Laplace equation.
Remarks to the Lebesgue integral, the spaces $L^p(a,b)$, $(p \geq 1)$. orthonormal systems (ONS), Bessel's inequality.
Complete orthonormal bases (ONB), Parseval's equality.
Examples: trigonometric polynomial, Legendre polynomials.
3. Solutions via Fourier transform:
Definition of the Fourier transform for functions in $L^1(\mathbb{R})$.
The sets $\mathcal{D}$ and $\mathcal{D}^*$. Fourier Inversion Theorem.
Examples: Fourier transforms of $e^{-x^2/2}$, $\chi_{[-1,1]}$, $e^{-|x|}$, Dirac's Delta distribution.
Schwartz space $\mathcal{S}$.
Properties of the Fourier transform: Shift formulas, Fourier transform of a derivative.
Convolution. Properties.
Plancherel formulas.
A list of Fourier transforms.
Applications: Non-homogeneous Cauchy-problem for the heat equation in one space dimension. Heat kernel. Regularity of the solution in the homogeneous case. Decay of the energy.
Dirichlet problem for the Laplace equation in a halfplane. Poisson kernel. Maximum principle.
Cauchy problem for the wave equation in n space dimensions:
$n = 1$: D'Alembert's formula.
$n = 3$: Poisson's solution. Domain of dependence. Huygens' principle.

$n = 2$: Hadamard's method of descent. Solution formula. Domain of dependence.
Nonhomogeneous wave equation: Duhamel's principle.

–