

Partial Differential Equations II

SS 2012

Prof. Dr. László Székelyhidi

May 25, 2018

Contents

1	Function spaces	1
1.1	Spaces of continuous functions	1
1.2	L^p spaces	2
1.3	Sobolev spaces	8
2	Elliptic Equations	12
2.1	Weak solutions	12
2.2	Regularity	14
2.3	Fredholm theory and spectrum	17

1 Function spaces

1.1 Spaces of continuous functions

Let X be a metric space and $K \subset X$ a compact subset.

$$C(K) = \{f : K \rightarrow \mathbb{R} : f \text{ continuous} \}$$

with norm

$$\|f\|_{C^0} = \sup_{x \in K} |f(x)|$$

is a Banach space. More generally,

$$BC(K) = \{f : X \rightarrow \mathbb{R} : f \text{ continuous and bounded}\}$$

is a Banach space.

Let $0 < \gamma \leq 1$ and $U \subset \mathbb{R}^n$. A function $f : U \rightarrow \mathbb{R}$ is Hölder continuous with exponent γ if there exists a constant $C > 0$ so that

$$|f(x) - f(y)| \leq C|x - y|^\gamma.$$

Definition 1.1. (i) The Hölder seminorm is defined as

$$[f]_{C^{0,\gamma}(U)} := \sup_{\substack{x,y \in U \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|^\gamma}.$$

The Hölder norm is defined as

$$\|f\|_{C^{0,\gamma}(U)} := \|f\|_{C^0(U)} + [f]_{C^{0,\gamma}(U)}.$$

(ii) The Hölder spaces are defined as

$$C^{0,\gamma}(\overline{U}) = \{f \in C(\overline{U}) : \|f\|_{C^{0,\gamma}(U)} < \infty\},$$

and, more generally

$$C^{k,\gamma}(\overline{U}) = \{f \in C(\overline{U}) : \|f\|_{C^{k,\gamma}(U)} < \infty\},$$

where

$$\|f\|_{C^{k,\gamma}(U)} := \sum_{|\alpha| \leq k} \|D^\alpha f\|_{C^0(U)} + \sum_{|\alpha|=k} [D^\alpha f]_{C^{0,\gamma}(U)}.$$

Theorem 1.2 (Arzela-Ascoli). Let X be a separable metric space and $f_k \in C(X)$ a sequence which is

(i) locally uniformly bounded, i.e.

$$\sup_k |f_k(x)| < \infty \quad \forall x \in X;$$

(ii) locally uniformly continuous, i.e. $\forall \varepsilon > 0, x \in X \exists \delta = \delta(x, \varepsilon) > 0$ such that

$$\sup_k |f_k(x) - f_k(y)| < \varepsilon \quad \forall y \in B_\delta(x).$$

Then there exists a subsequence f_{k_j} and $f \in C(X)$ such that

$$f_{k_j} \rightarrow f \quad \text{uniformly on } K$$

for all compact subsets $K \subset X$.

1.2 L^p spaces

In the following, let $U \subset \mathbb{R}^n$ be open, and $1 \leq p \leq \infty$. Let

$$\begin{aligned} \mathcal{M} &= \{f : U \rightarrow \mathbb{R} \text{ measurable} \} \\ \mathcal{N} &= \{f \in \mathcal{M} : f(x) = 0 \text{ for a.e. } x\}. \end{aligned}$$

We define the space of measurable functions as the quotient space

$$M = \mathcal{M}/\mathcal{N}.$$

In other words measurable functions that differ on a set of zero Lebesgue measure are identified.

Definition 1.3. The L^p norm is defined as

$$\|f\|_{L^p(U)} = \begin{cases} (\int_U |f(x)|^p dx)^{1/p} & p < \infty \\ \text{ess sup}_U |f| & p = \infty. \end{cases}$$

The L^p spaces are defined as

$$L^p(U) = \{f \in M : \|f\|_{L^p(U)} < \infty\}.$$

For the L^p norm we have the *Minkowski inequality*

$$\|f + g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}$$

(this is indeed why it is called a norm), and the *Hölder inequality*

$$\int |fg| dx \leq \|f\|_{L^p} \|g\|_{L^q},$$

where $\frac{1}{p} + \frac{1}{q} = 1$. The

For $1 \leq p \leq \infty$ we denote by p' the *dual exponent*, defined by

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Theorem 1.4 (Riesz-Fischer). $L^p(U)$ is complete.

Proof. Let $\{f_k\}$ be a Cauchy sequence in $L^p(U)$.

Step 1. It suffices to show that a subsequence f_{k_j} converges. Indeed let $f_{k_j} \rightarrow f$ in $L^p(U)$ for some subsequence f_{k_j} and let $\epsilon > 0$. Choose $k_0 \in \mathbb{N}$ with:

$$d(f_k, f_l) < \frac{\epsilon}{2} \quad \forall k, l \geq k_0$$

and

$$d(f_{k_j}, f) < \frac{\epsilon}{2} \quad \forall k_j \geq k_0. \quad (\text{with } d(f, g) := \|f - g\|_{L^p})$$

Choose also j_0 so that $k_j \geq k_0 \quad \forall j \geq j_0$. Now let $k \geq k_0$ and $j \geq j_0$. We get

$$d(f_k, f) \leq d(f_k, f_{k_j}) + d(f_{k_j}, f) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

and so we have $f_k \rightarrow f$ in $L^p(U)$.

Step 2. For each $i = 1, 2, 3, \dots$ there exists $k_i \in \mathbb{N}$ such that

$$\|f_{k_i} - f_n\|_{L^p} \leq 2^{-i} \quad \text{for all } n \geq k_i.$$

We claim that this sequence f_{k_j} converges. To this end define

$$F_l(x) = |f_{k_1}(x)| + \sum_{j=1}^l |f_{k_j}(x) - f_{k_{j+1}}(x)|.$$

Then $F_l(x)$ is monotone increasing and

$$\|F_l\|_{L^p} \leq \|f_{k_1}\|_{L^p} + 1.$$

Therefore $F_l(x) \rightarrow F(x)$ almost everywhere, and the monotone convergence theorem implies that $F \in L^p(U)$. In particular this means that $F(x) < \infty$ for a.e. x . Consequently

$$f_{k_l}(x) = f_{k_1}(x) + \sum_{j=1}^{l-1} (f_{k_{j+1}}(x) - f_{k_j}(x))$$

converges absolutely for a.e. x , and hence $f_{k_l}(x) \rightarrow f(x)$ a.e. Since

$$|f_{k_l}(x)| \leq F(x),$$

by dominated convergence we deduce that $f_{k_l} \rightarrow f$ in $L^p(U)$. □

Proposition 1.5. $C_c(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, for $1 \leq p < \infty$.

Proof. **Step 1:** $p = 1$.

Because of the definition of $L^1(\mathbb{R}^n)$ the step functions are dense in $L^1(\mathbb{R}^n)$. Therefore it suffices to show that step functions can be approximated by $C_c(\mathbb{R}^n)$, i.e. for every measurable $A \subset \mathbb{R}^n$ with $|A| < \infty$ and $\varepsilon > 0$ we find $\phi \in C_c(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} |\phi - \mathbf{1}_A| dx < \varepsilon$. Indeed, by the regularity of Lebesgue measure we find sets K compact and U open such that $K \subset A \subset U$ with $|U \setminus K| < \varepsilon$. Because $|A| < \infty$ we can assume $U \subset [-R, R]^n$ for R big enough. Moreover, $\text{dist}(K, \mathbb{R}^n \setminus U) > 0$ since $\mathbb{R}^n \setminus U$ is closed. Define now

$$\phi(x) = \max \left(0, 1 - \frac{\text{dist}(x, K)}{\text{dist}(K, \mathbb{R}^n \setminus U)} \right).$$

We have $\phi|_K = 1$, $\phi|_{\mathbb{R}^n \setminus U} = 0$ and $0 \leq \phi \leq 1$ on $U \setminus K$. Finally

$$\begin{aligned} \int_{\mathbb{R}^n} |\phi - \mathbf{1}_A| dx &= \int_K |\phi - \mathbf{1}_A| dx + \int_{U \setminus K} |\phi - \mathbf{1}_A| dx + \int_{\mathbb{R}^n \setminus U} |\phi - \mathbf{1}_A| dx \\ &= \int_{U \setminus K} |\phi - \mathbf{1}_A| dx \leq |U \setminus K| < \varepsilon. \end{aligned}$$

Since $\text{supp } \phi \subset U \subset [-R, R]^n$ and $\text{dist}(x, K)$ is continuous for compact K we have $\phi \in C_c(\mathbb{R}^n)$.

Step 2: $1 < p < \infty$.

Let $f \in L^p(\mathbb{R}^n)$ and $\varepsilon > 0$. Because $|f|, f_+, f_- \in L^p(\mathbb{R}^n)$ we may assume without loss of generality $f \geq 0$. Define

$$f_k(x) := \begin{cases} \min\{f(x), k\}, & x \in B_k(0) \\ 0, & \text{otherwise.} \end{cases}$$

Then $f_k \in L^1(\mathbb{R}^n)$ for all $k \in \mathbb{N}$ and by the dominated convergence theorem $\|f_k - f\|_{L^p} \rightarrow 0$. We find $k_0 \in \mathbb{N}$ with $\|f_{k_0} - f\|_{L^p} < \frac{\varepsilon}{2}$. By Step 1 there exists $\phi \in C_c(\mathbb{R}^n)$ such that

$$\|f_{k_0} - \phi\|_{L^1} < \frac{\varepsilon^p}{2^p k_0^{p-1}}.$$

Since $0 \leq f_{k_0} \leq k_0$, wlog $0 \leq \phi \leq k_0$ and therefore also $|f_{k_0} - \phi| \leq k_0$. Thus

$$|f_{k_0} - \phi|^p \leq k_0^{p-1} |f_{k_0} - \phi|,$$

$$\|f_{k_0} - \phi\|_{L^p}^p \leq \int_{\mathbb{R}^n} k_0^{p-1} |f_{k_0} - \phi|(x) dx = k_0^{p-1} \|f_{k_0} - \phi\|_{L^1} < k_0^{p-1} \frac{\varepsilon^p}{2^p k_0^{p-1}} = \left(\frac{\varepsilon}{2}\right)^p.$$

Finally

$$\|f - \phi\|_{L^p} \leq \|f - f_{k_0}\|_{L^p} + \|f_{k_0} - \phi\|_{L^p} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

□

Lemma 1.6. Let $f \in L^p(\mathbb{R}^n)$ with $p < \infty$. Then

$$\|f(\cdot + h) - f\|_{L^p(\mathbb{R}^n)} \rightarrow 0 \quad \text{as } |h| \rightarrow 0.$$

Proof. Let $\epsilon > 0$. By Proposition 1.5 there exists $\phi \in C_c(\mathbb{R}^n)$ with $\|f - \phi\|_{L^p} < \frac{\epsilon}{3}$. We then have, by the Minkowski inequality,

$$\begin{aligned} \|f(\cdot + h) - f\|_{L^p} &\leq \|f(\cdot + h) - \phi(\cdot + h)\|_{L^p} + \|\phi(\cdot + h) - \phi\|_{L^p} + \|\phi - f\|_{L^p} \\ &= 2\|f - \phi\|_{L^p} + \|\phi(\cdot + h) - \phi\|_{L^p} \end{aligned}$$

Since $\text{supp } \phi$ is compact, ϕ is uniformly continuous on $\text{supp } \phi$, hence there exists $\delta > 0$ such that

$$|\phi(x) - \phi(y)| < \frac{\epsilon}{3|\text{supp } \phi|^{\frac{1}{p}}} \quad \forall x, y \in \text{supp } \phi \text{ with } |x - y| < \delta.$$

Thus for all $|h| < \delta$

$$\|\phi(\cdot + h) - \phi\|_{L^p} = \left(\int_{\Omega} |\phi(x + h) - \phi(x)|^p dx \right)^{\frac{1}{p}} < \frac{\epsilon}{3}$$

and we get

$$\|f(\cdot + h) - f\|_{L^p} \leq 2\|f - \phi\|_{L^p} + \|\phi(\cdot + h) - \phi\|_{L^p} < \epsilon \quad \forall |h| < \delta$$

□

Definition 1.7. Let $f, g \in M$. The convolution $f * g \in M$ is defined as

$$f * g(x) = \int_{\mathbb{R}^n} f(x - y)g(y) dy.$$

Observe that, if $f \in L^p$ and $g \in L^{p'}$ then the Hölder inequality implies that $f * g(x)$ is defined for all $x \in \mathbb{R}^n$. Under weaker conditions we can still define it as an L^p function.

Lemma 1.8. Let $f \in L^1(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$ for some $1 \leq p \leq \infty$. Then $f * g \in L^p(\mathbb{R}^n)$ and $\|f * g\|_{L^p} \leq \|f\|_{L^1} \|g\|_{L^p}$.

Proof. **Step 1.** $p = \infty$

$$|(f * g)(x)| \leq \int_{\mathbb{R}^n} |f(x - y)| |g(y)| dy \leq \|g\|_{L^\infty} \int_{\mathbb{R}^n} |f(x - y)| dy = \|g\|_{L^\infty} \|f\|_{L^1} \quad \forall x \in \mathbb{R}^n$$

Step 2. $p = 1$. With Fubini follows

$$\|(f * g)(x)\|_{L^1} \leq \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(x - y)| |g(y)| dy \right) dx = \int_{\mathbb{R}^n} |g(y)| \left(\int_{\mathbb{R}^n} |f(x - y)| dx \right) dy = \|g\|_{L^1} \|f\|_{L^1}$$

Step 3. $1 < p < \infty$ Let $1 = \frac{1}{p} + \frac{1}{q}$. Fubini and the Hölder inequality implies

$$\begin{aligned} \|(f * g)(x)\|_{L^p}^p &= \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} f(x - y)g(y) dy \right|^p dx \\ &\leq \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |g(y)| |f(x - y)|^{\frac{1}{p}} |f(x - y)|^{\frac{1}{q}} dy \right)^p dx \\ &\leq \int_{\mathbb{R}^n} \left(\left(\int_{\mathbb{R}^n} |f(x - y)| dy \right)^{\frac{p}{q}} \int_{\mathbb{R}^n} |f(x - y)| |g(y)|^p dy \right) dx \\ &= \|f\|_{L^1}^{\frac{p}{q}} \int_{\mathbb{R}^n} |g(y)|^p \left(\int_{\mathbb{R}^n} |f(x - y)| dx \right) dy \\ &= \|g\|_{L^p}^p \|f\|_{L^1}^{1 + \frac{p}{q}} = \|g\|_{L^p}^p \|f\|_{L^1}^p. \end{aligned}$$

□

Proposition 1.9 (Dirac sequence). Let $\phi \in L^1(\mathbb{R}^n)$ with $\phi \geq 0$ and $\int_{\mathbb{R}^n} \phi = 1$, and define for $\epsilon > 0$

$$\phi_\epsilon(x) = \frac{1}{\epsilon^n} \phi\left(\frac{x}{\epsilon}\right).$$

Then for every $f \in L^p(\mathbb{R}^n)$ with $p < \infty$

$$\phi_\epsilon * f \rightarrow f \quad \text{in } L^p(\mathbb{R}^n).$$

Proof. We have

$$(\phi_\epsilon * f)(x) - f(x) = \int_{\mathbb{R}^n} \phi_\epsilon(x-y)(f(y) - f(x))dy = (\phi_\epsilon * (f - f(x)))(x)$$

Divide $\phi_\epsilon(x) = \phi_\epsilon(x)\chi_{B(0,\delta)}(x) + \phi_\epsilon(x)\chi_{\mathbb{R}^n \setminus B(0,\delta)}(x) =: \phi_\epsilon^1 + \phi_\epsilon^2$.

Let $\eta > 0$. Lemma 8 says

$$\begin{aligned} \|\phi_\epsilon * f - f\|_{L^p(\mathbb{R}^n)} &\leq \|\phi_\epsilon^1\|_{L^1(\mathbb{R}^n)} \cdot \sup_{|h| \leq \delta} \|f(\cdot + h) - f\|_{L^p(\mathbb{R}^n)} \\ &\quad + \|\phi_\epsilon^2\|_{L^1(\mathbb{R}^n)} \cdot \sup_{h \in \mathbb{R}^n} \|f(\cdot + h) - f\|_{L^p(\mathbb{R}^n)} \\ &\leq 1 \cdot \sup_{|h| \leq \delta} \|f(\cdot + h) - f\|_{L^p(\mathbb{R}^n)} + \int_{\mathbb{R}^n \setminus B(0,\delta)} \phi_\epsilon dx \cdot 2\|f\|_{L^p(\mathbb{R}^n)} \\ &\leq \sup_{|h| \leq \delta} (\|f(\cdot + h) - g(\cdot + h)\|_{L^p} + \|g(\cdot + h) - g\|_{L^p} + \|g - f\|_{L^p}) \\ &\quad + 2\|f\|_{L^p} \int_{\mathbb{R}^n \setminus B(0,\frac{\delta}{\epsilon})} \phi dx < \eta, \end{aligned}$$

if we choose:

- (i) $g \in C_c(\mathbb{R}^n)$ such that $\|g - f\|_{L^p} < \frac{\eta}{4}$ (Proposition 5)
- (ii) $\delta > 0$ such that $\|g(\cdot + \delta) - g\|_{L^p} < \frac{\eta}{4}$ (Lemma 6)
- (iii) $\epsilon > 0$ such that $\int_{\mathbb{R}^n \setminus B(0,\frac{\delta}{\epsilon})} \phi dx < \frac{\eta}{8\|f\|_{L^p}}$.

□

Proposition 1.10. Let $\phi \in C_c^\infty(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \phi(x)dx = 1$ and $\phi_\epsilon(x) = \frac{1}{\epsilon^n} \phi(\frac{x}{\epsilon})$. If $f \in L_{loc}^1(\mathbb{R}^n)$, then $\phi_\epsilon * f \in C^\infty(\mathbb{R}^n)$ and $D^\alpha(\phi_\epsilon * f) = (D^\alpha \phi_\epsilon) * f$ for all multiindices α .

Proof. Show: $\phi_\epsilon * f \in C^1(\mathbb{R}^n)$. (rest can be shown similarly by induction)

Let $f_\epsilon := \phi_\epsilon * f$.

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f_\epsilon(x + he_i) - f_\epsilon(x)}{h} &= \lim_{h \rightarrow 0} \frac{1}{\epsilon^n} \int_{\mathbb{R}^n} \underbrace{\frac{1}{h} \left(\phi\left(\frac{x + he_i - y}{\epsilon}\right) - \phi\left(\frac{x - y}{\epsilon}\right) \right)}_{\text{compact supp and } \Rightarrow \partial_i \phi(\frac{x-y}{\epsilon})} f(y) dy \\ &= \int_{\mathbb{R}^n} \partial_i \phi_\epsilon(x - y) f(y) dy = (\partial_i \phi_\epsilon) * f \end{aligned}$$

by dominated convergence theorem. □

Corollary 1.11. $C^\infty(\overline{U})$ is dense in $L^p(U)$ for all $U \subset \mathbb{R}^n$ open and all $p < \infty$.

Proof. Let $f \in L^p(U)$. Continue f by 0 on $\mathbb{R}^n \setminus U$. $\Rightarrow f \in L^1_{loc}(\mathbb{R}^n)$.
Proposition 9 and 10 $\Rightarrow \|\phi_\epsilon * f - f\|_{L^p(\mathbb{R}^n)} \rightarrow 0$ and $\phi_\epsilon * f \in C^\infty(\mathbb{R}^n)$.
Define $g_\epsilon := (\phi_\epsilon * f)|_{\overline{U}} \in C^\infty(\overline{U})$.
 $\Rightarrow \|f - g_\epsilon\|_{L^p(U)} = \|f - \phi_\epsilon * f\|_{L^p(U)} \leq \|f - \phi_\epsilon * f\|_{L^p(\mathbb{R}^n)} \rightarrow 0$. $\square$

Theorem 1.12 (Riesz-Kolmogorov). Let $1 \leq p < \infty$ and $\mathcal{F} \subset L^p(\mathbb{R}^n)$. Then $\mathcal{F}$ is totally bounded if and only if the following three conditions hold:

- (i) $\mathcal{F}$ is bounded, i.e. $\sup_{f \in \mathcal{F}} \|f\|_{L^p} < \infty$;
- (ii) $\mathcal{F}$ is uniformly L^p -continuous, i.e. $\sup_{f \in \mathcal{F}} \|f(\cdot + h) - f\|_{L^p} \rightarrow 0$ as $|h| \rightarrow 0$;
- (iii) $\mathcal{F}$ is tight, i.e. $\sup_{f \in \mathcal{F}} \|f\|_{L^p(\mathbb{R}^n \setminus B_R(0))} \rightarrow 0$ as $R \rightarrow \infty$.

Proof. Step 1: Total boundedness implies (i),(ii),(iii)

Let $\varepsilon > 0$ and

$$\mathcal{F} \subset \bigcup_{i=1}^{N_\varepsilon} B_\varepsilon(f_i^\varepsilon)$$

Then, for all $f \in \mathcal{F}$ there exists $f_{i_0}^\varepsilon$ such that $f \in B_\varepsilon(f_{i_0}^\varepsilon)$. Consequently

- $\|f\|_{L^p} \leq \|f_{i_0}^\varepsilon\| + \varepsilon \leq \max_{i=1, \dots, N_\varepsilon} \|f_i^\varepsilon\|_{L^p} + \varepsilon < \infty$,
- $\|f(\cdot + h) - f\|_{L^p} \leq \max_i \|f_i^\varepsilon(\cdot + h) - f_i^\varepsilon\|_{L^p} + 2\varepsilon$,
- $\|f\|_{L^p(\mathbb{R}^n \setminus B_R(0))} \leq \max_i \|f_i^\varepsilon\|_{L^p(\mathbb{R}^n \setminus B_R(0))} + \varepsilon$.

Therefore (i) and (iii) follows immediately, whereas (ii) follows from Lemma 1.6.

Step 2: (i),(ii),(iii) implies total boundedness

Let ϕ_δ be a standard regularizing kernel, and for any $f \in \mathcal{F}$ set

$$f_\delta = \phi_\delta * f.$$

Then

$$\begin{aligned} \|f_\delta\|_{C^0(\mathbb{R}^n)} &\leq \|\phi_\delta\|_{L^{p'}} \|f\|_{L^p} \leq C\delta^{-\frac{n}{p'}} \\ \|Df_\delta\|_{C^0(\mathbb{R}^n)} &\leq \|D\phi_\delta\|_{L^{p'}} \|f\|_{L^p} \leq C\delta^{-\frac{n}{p'}-1}. \end{aligned}$$

In particular the family $\{f_\delta : f \in \mathcal{F}\}$ is uniformly bounded and uniformly continuous, hence by Theorem 1.2 (Arzela-Ascoli) it is totally bounded in $C(\overline{B_R(0)})$ for any $R > 0$. Thus, for any $\varepsilon > 0$ there exists $N = N(\varepsilon, \delta, R)$ and functions $g_1, \dots, g_N \in C(\overline{B_R(0)})$ such that

$$\min_i \|f_\delta - g_i\|_{C(\overline{B_R(0)})} < \frac{\varepsilon}{3R^{n/p}} \quad \text{for all } f \in \mathcal{F}.$$

Hence also

$$\min_i \|f_\delta - g_i\|_{L^p(B_R(0))} < \frac{\varepsilon}{3} \quad \text{for all } f \in \mathcal{F}.$$

Set

$$h_i = \begin{cases} g_i & \text{in } B_R(0) \\ 0 & \text{in } \mathbb{R}^n \setminus B_R(0) \end{cases}$$

Then

$$\begin{aligned}
\|f - h_i\|_{L^p(\mathbb{R}^n)} &\leq \|f - h_i\|_{L^p(\mathbb{R}^n \setminus B_R(0))} + \|f - h_i\|_{L^p(B_R(0))} \\
&\leq \|f\|_{L^p(\mathbb{R}^n \setminus B_R(0))} + \|f_\delta - g_i\|_{L^p(B_R(0))} + \|f_\delta - f\|_{L^p(B_R(0))} \\
&= I + II + III.
\end{aligned}$$

The term I is small uniformly over $\mathcal{F}$ for large R by assumption (iii), whereas the term II is estimated by $\varepsilon/3$ above. Concerning the term III , using Hölder we have

$$|f_\delta(x) - f(x)|^p \leq \int_{\mathbb{R}^n} |f(x-y) - f(x)|^p \phi_\delta(y) dy$$

so that

$$\begin{aligned}
\|f_\delta - f\|_{L^p}^p &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x-y) - f(x)|^p dx \phi_\delta(y) dy \\
&= \int_{B_\delta(0)} \|f(\cdot - y) - f\|_{L^p(\mathbb{R}^n)}^p \phi_\delta(y) dy \\
&\leq \sup_{|y| \leq \delta} \|f(\cdot - y) - f\|_{L^p(\mathbb{R}^n)}^p
\end{aligned}$$

Using (i) we can choose $\delta > 0$ so that $\sup_{f \in \mathcal{F}} \|f_\delta - f\|_{L^p} < \varepsilon/3$, and, using (iii) choose $R > 0$ so that $\sup_{f \in \mathcal{F}} \|f\|_{L^p(\mathbb{R}^n \setminus B_R(0))} < \varepsilon/3$. This choice of R, δ determines $N = N(\varepsilon, \delta, R)$ and the functions h_i . We deduce that

$$\mathcal{F} \subset \bigcup_{i=1}^N B_\varepsilon(h_i).$$

□

1.3 Sobolev spaces

Definition 1.13 (Weak derivatives). Let $u, v \in L^1_{loc}(U)$ and α be a multiindex. v is called the α^{th} - weak partial derivative of u , $D^\alpha u = v$, if

$$\int_U u D^\alpha \phi dx = (-1)^{|\alpha|} \int_U v \phi dx$$

for all $\phi \in C_c^\infty(U)$.

Lemma 1.14 (Uniqueness of weak derivatives). A weak α^{th} - partial derivative of u , if it exists, is uniquely defined up to a set of measure zero.

Proof. Let $v, \tilde{v} \in L^1_{loc}(U)$ satisfy

$$\int_U u D^\alpha \phi dx = (-1)^{|\alpha|} \int_U v \phi dx = (-1)^{|\alpha|} \int_U \tilde{v} \phi dx$$

for all $\phi \in C_c^\infty(U)$. Then

$$\int_U (v - \tilde{v}) \phi dx = 0$$

for all $\phi \in C_c^\infty(U)$ and the fundamental lemma of the calculus of variations tells us $v - \tilde{v} = 0$ a.e. in U . □

Definition 1.15 (Sobolev spaces). For $1 \leq p \leq \infty$ and $k = 1, 2, \dots$ the Sobolev space of order k and integrability p is defined as

$$W^{k,p}(U) := \{f \in L^1_{loc}(U) : D^\alpha f \in L^p(U) \quad \text{for all } |\alpha| \leq k\}$$

with norm

$$\|f\|_{W^{k,p}(U)} = \sum_{|\alpha| \leq k} \|D^\alpha f\|_{L^p(U)}.$$

The subspace $W_0^{k,p}$ is defined to be the closure of $C_c^\infty(U)$ in $W^{k,p}(U)$.

Proposition 1.16 (Elementary properties). Let $u, v \in W^{k,p}(U)$, $|\alpha| \leq k$. Then

- (i) $D^\alpha u \in W^{k-|\alpha|,p}(U)$ and $D^\beta(D^\alpha u) = D^\alpha(D^\beta u) = D^{\alpha+\beta}u$ for all multi-indices α, β with $|\alpha| + |\beta| \leq k$.
- (ii) For each $\lambda, \mu \in \mathbb{R}$, $\lambda u + \mu v \in W^{k,p}(U)$ and $D^\alpha(\lambda u + \mu v) = \lambda D^\alpha u + \mu D^\alpha v$, $|\alpha| \leq k$.
- (iii) If V is an open subset of U , then $u \in W^{k,p}(V)$.
- (iv) If $\zeta \in C_c^\infty(U)$, then $\zeta u \in W^{k,p}(U)$ and

$$D^\alpha(\zeta u) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta \zeta D^{\alpha-\beta} u \quad (\text{Leibniz' formula}),$$

$$\text{where } \binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!}.$$

Proof. see [Evans, p. 247-248] □

Theorem 1.17. For any $1 \leq p \leq \infty$, $k \in \mathbb{N}$ and $U \subset \mathbb{R}^n$ open $W^{k,p}(U)$ is a Banach space.

Proof. see [Evans, p. 249] □

In the sequel $\eta_\varepsilon \in C_c^\infty(\mathbb{R}^n)$ will always denote a smooth regularizing kernel, i.e. such that

$$\eta_\varepsilon(x) = \frac{1}{\varepsilon^n} \eta\left(\frac{x}{\varepsilon}\right), \quad \text{where } \eta \in C_c^\infty(B_1(0)), \eta \geq 0, \int_{\mathbb{R}^n} \eta dx = 1.$$

Theorem 1.18 (Local approximation by smooth functions). Let $f \in W^{k,p}(U)$, $1 \leq p < \infty$ and set

$$f^\varepsilon = \eta_\varepsilon * f \quad \text{in } U_\varepsilon := \{x \in U : \text{dist}(x, \partial U) > \varepsilon\}$$

Then $f^\varepsilon \in C^\infty(U_\varepsilon)$ for all $\varepsilon > 0$ and

$$f^\varepsilon \rightarrow f \quad \text{in } W_{loc}^{k,p}(U)$$

as $\varepsilon \rightarrow 0$.

Proof. see [Evans, p. 250-251] □

Definition 1.19 (Locally finite covering). Let $S \subset \mathbb{R}^n$ be a set. We call $\{U_i\}_{i \in \mathbb{N}}$ an open, locally finite covering of S , if

- (i) $U_i \subset \mathbb{R}^n$ are open subsets
- (ii) $S \subseteq \bigcup_{i \in \mathbb{N}} U_i$
- (iii) $\forall x \in S \quad \exists \delta > 0, N \in \mathbb{N} : U_i \cap \overline{B(x, \delta)} = \emptyset \quad \forall i \geq N.$

Proposition 1.20 (Partition of unity). Let $U \subset \mathbb{R}^n$ be open, $\{U_i\}_{i \in \mathbb{N}}$ be an open, bounded, local finite covering of $U = \bigcup_{i \in \mathbb{N}} U_i$. Then $\exists \phi_i \in C_c^\infty(U_i)$ with $\phi_i \geq 0$ and $\sum_{i=1}^\infty \phi_i(x) \equiv 1 \quad \forall x \in U$.

Remark: $\{U_i\}$ local finite $\Rightarrow$ sum is finite

Theorem 1.21 (Global approximation by smooth functions). Let $U \subset \mathbb{R}^n$ open and bounded, $1 \leq p < \infty$. Then $C^\infty(U) \cap W^{k,p}(U)$ is dense in $W^{k,p}(U)$.

Proof. see [Evans, p. 251-252] □

Theorem 1.22 (Product rule). Let $U \subset \mathbb{R}^n$ open, $1 \leq p \leq \infty$, $f \in W^{1,p}(U)$, $g \in W^{1,p'}(U)$. Then $fg \in W^{1,1}(U)$ with

$$\partial_i(fg) = \partial_i f \cdot g + \partial_i g \cdot f.$$

Theorem 1.23 (Chain rule I). Let $\phi \in C^1(\mathbb{R})$ with $\phi' \in L^\infty(\mathbb{R})$, $f \in W^{1,p}(U)$ where $U \subset \mathbb{R}^n$ is open and bounded, and let $1 \leq p \leq \infty$. Then $\phi \circ f \in W^{1,p}(U)$ with

$$\partial_i(\phi \circ f) = (\phi' \circ f) \partial_i f.$$

Proposition 1.24 (Chain rule II). Let $U, \tilde{U} \subset \mathbb{R}^n$ open, $\Phi : \tilde{U} \rightarrow U$ be a C^1 -diffeomorphism with $D\Phi, D(\Phi^{-1})$ bounded. If $f \in W^{1,p}(U)$, $1 \leq p \leq \infty$ then $f \circ \Phi \in W^{1,p}(\tilde{U})$ with

$$\partial_i(f \circ \Phi) = \sum_{j=1}^n (\partial_j f \circ \Phi) \partial_i \Phi_j.$$

Proof. see [Alt, p. 124] □

Observe that $f \in L_{loc}^1(U)$ is only defined for almost every $x \in U$. The function $f \circ \Phi \in L_{loc}^1(U)$ is then also defined for almost every $x \in \tilde{U}$ because of the following lemma:

Lemma 1.25. Let $U, \tilde{U}$ and Φ be as in the previous proposition. If $N \subset \tilde{U}$ with $|N| = 0$ (Lebesgue measure zero), then $|\Phi(N)| = 0$.

Proof. see [Alt, p. 124-125] □

Theorem 1.26 (Global approximation up to the boundary). Let $U \subset \mathbb{R}^n$ be open, bounded with C^1 boundary. Then $C^\infty(\overline{U})$ is dense in $W^{k,p}(U)$

Proof. see [Evans, p. 252-254] □

In the proof we use only the *segment property*: for all $x_o \in \partial U$ there exists $r > 0, e \in \mathbb{R}^n \setminus \{0\}$ such that: whenever $x \in U \cap B(x_o, r)$ then $x + te \in U \forall 0 < t < 1$.

Theorem 1.27 (Trace theorem). Let $U \subset \mathbb{R}^n$ be open, bounded with C^1 boundary and $p < \infty$. There exists a continuous linear operator

$$T : W^{1,p}(U) \rightarrow L^p(\partial U)$$

such that $Tf = f|_{\partial U}$ for $f \in W^{1,p}(U) \cap C(\overline{U})$ and

$$\|Tf\|_{L^p(\partial U)} \leq C\|f\|_{W^{1,p}(U)} \quad \forall f \in W^{1,p}(U),$$

where $C = C(p, U)$.

Proof. see [Evans, p. 258-259] □

We call Tf the "weak" boundary-values of f or trace of f on ∂U .

Theorem 1.28 (Trace zero functions in $W^{1,p}$). Let $U \subset \mathbb{R}^n$ be bounded with C^1 boundary and $u \in W^{1,p}(U)$. Then

$$u \in W_0^{1,p}(U) \Leftrightarrow Tu = 0 \quad \text{on} \quad \partial U$$

Proof. see [Evans, p. 259-261] □

Theorem 1.29 (Extension theorem). Let $U \subset \mathbb{R}^n$ bounded with C^1 boundary. Choose V open and bounded with $U \subset \subset V$. Then there exists a bounded linear operator

$$E : W^{1,p}(U) \rightarrow W^{1,p}(\mathbb{R}^n)$$

such that for all $u \in W^{1,p}(U)$:

- (i) $Eu = u$ a.e. in U
- (ii) Eu has support within V
- (iii) $\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(U)}$ where $C = C(p, U, V)$.

We call Eu an extension of u to $\mathbb{R}^n$.

2 Elliptic Equations

2.1 Weak solutions

Let $U \subset \mathbb{R}^n$ be open and bounded. We consider the Dirichlet problem

$$\begin{aligned} Lu &= f & \text{in } U, \\ u &= 0 & \text{on } \partial U, \end{aligned} \tag{1}$$

where

$$Lu = - \sum_{i,j=1}^n \partial_j (a^{ij} \partial_i u) + \sum_{i=1}^n b^i \partial_i u + cu.$$

We assume, unless otherwise stated, that

$$a^{ij}, b^i, c \in L^\infty(U), \quad a^{ij} = a^{ji}, \quad a^{ij} \text{ uniformly elliptic.}$$

The latter condition means that there exists $\theta > 0$ so that

$$\sum_{i,j=1}^n a^{ij} \xi_i \xi_j \geq \theta |\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n.$$

The bilinear form associated to L is defined as

$$B[u, v] = \int_U \sum_{i,j} a^{ij} \partial_i u \partial_j v + \sum_i b^i \partial_i uv + cuv \, dx.$$

A weak solution of the Dirichlet problem with $f \in L^2(U)$ is defined to be

$$u \in H_0^1(U) : \quad B[u, v] = (f, v)_{L^2} \text{ for all } v \in H_0^1(U)$$

The tool for existence will be the following generalization of the Riesz Representation Theorem:

Theorem 2.1 (Lax-Milgram). Let

$$B : H \times H \rightarrow \mathbb{R}$$

be a bilinear mapping such that

$$|B[u, v]| \leq \alpha \|u\| \|v\| \quad \forall u, v \in H$$

and

$$\beta \|u\|^2 \leq B[u, u] \quad \forall u \in H,$$

where $\alpha, \beta > 0$ are constants. Furthermore $f : H \rightarrow \mathbb{R}$ is a bounded linear functional on H . Then there exists a unique $u \in H$ such that

$$B[u, v] = f(v) \quad \forall v \in H.$$

Proof. see [Evans, p. 298-299]

□

Theorem 2.2 (Poincaré inequality I). Let $U \subset \mathbb{R}^n$ be open and bounded, $1 \leq p < n$ and $u \in W_0^{1,p}(U)$. Then for all $1 \leq q \leq p^*$ we have

$$\|u\|_{L^q(U)} \leq C \|Du\|_{L^p(U)}$$

where $C = C(n, p, q, U)$. In particular

$$\|u\|_{L^p(U)} \leq C \|Du\|_{L^p(U)} \quad \forall 1 \leq p \leq \infty.$$

Proof. see [Evans, p. 265-266] □

Theorem 2.3 (Energy estimates). Let

$$B[u, v] = \int_U \sum_{i,j=1}^n a^{ij} u_{x_i} v_{x_j} + \sum_{i=1}^n b^i u_{x_i} v + cuv \quad dx \quad \forall u, v \in H_0^1(U)$$

be the specific bilinear form associated with the elliptic operator L . Then we find constants $\alpha, \beta > 0$ and $\gamma \geq 0$ such that

$$|B[u, v]| \leq \alpha \|u\|_{H_0^1(U)} \|v\|_{H_0^1(U)}$$

and

$$\beta \|u\|_{H_0^1(U)}^2 \leq B[u, u] + \gamma \|u\|_{L^2(U)}^2$$

for all $u, v \in H_0^1(U)$.

Proof. see [Evans, p. 300-301] □

This leads to

Theorem 2.4 (First existence theorem). There exists $\gamma \geq 0$ such that for each

$$\mu \geq \gamma$$

and each function

$$f \in L^2(U),$$

the boundary-value problem

$$\begin{cases} Lu + \mu u = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$

has a unique weak solution $u \in H_0^1(U)$.

Proof. see [Evans, p. 301] □

Definition 2.5 (H^{-1}). We denote by $H^{-1}(U)$ the dual space to $H_0^1(U)$, that is the space of bounded linear functionals f on $H_0^1(U)$.

We will write $\langle \cdot, \cdot \rangle$ to denote the pairing between $H^{-1}(U)$ and $H_0^1(U)$.

Let $f \in H^{-1}(U)$, then we define the norm

$$\|f\|_{H^{-1}(U)} := \sup\{\langle f, u \rangle \mid u \in H_0^1(U), \|u\|_{H_0^1(U)} \leq 1\}.$$

Theorem 2.6 (Characterization of H^{-1}). (i) If $f \in H^{-1}(U)$, then there exist functions $f^0, f^1, \dots, f^n \in L^2(U)$ such that

$$(*) \quad \langle f, v \rangle = \int_U f^0 v + \sum_{i=1}^n f^i v_{x_i} dx \quad (v \in H_0^1(U)).$$

(ii) Furthermore,

$$\|f\|_{H^{-1}(U)} = \inf \left\{ \left(\int_U \sum_{i=0}^n |f^i|^2 dx \right)^{\frac{1}{2}} \mid f \text{ satisfies } (*) \text{ for } f^0, \dots, f^n \in L^2(U) \right\}.$$

Proof. see [Evans, p. 284] □

2.2 Regularity

An important tool to prove regularity is the following characterization of Sobolev functions

Proposition 2.7 (Difference quotients). Let $u : U \rightarrow \mathbb{R}$ locally summable and $V \subset\subset U$. We define the i -th difference quotient of size h by

$$D_i^h u(x) = \frac{u(x + h e_i) - u(x)}{h}, i \in \{1, \dots, n\}$$

for $x \in V, h \in \mathbb{R}, 0 < |h| < \text{dist}(V, \partial U)$ and set $D^h u := (D_1^h u, \dots, D_n^h u)$.

Then the following holds:

(i) Suppose $1 \leq p < \infty$ and $u \in W^{1,p}(U)$. For each $V \subset\subset U$ and $0 < |h| < \frac{1}{2} \text{dist}(V, \partial U)$ there exists a constant C such that

$$\|D^h u\|_{L^p(V)} \leq C \|Du\|_{L^p(U)}.$$

(ii) Now assume $1 < p < \infty, u \in L^p(V)$ and

$$\|D^h u\|_{L^p(V)} \leq C$$

for a constant C and all $0 < |h| < \frac{1}{2} \text{dist}(V, \partial U)$. Then

$$u \in W^{1,p}(V) \text{ and } \|Du\|_{L^p(V)} \leq C.$$

Proof. see [Evans, p.277-278] □

Theorem 2.8 (Interior H^2 -regularity). Suppose

$$a^{ij} \in C^1(U), b^i, c \in L^\infty(U) \quad \forall i, j = 1, \dots, n \quad \text{and} \quad f \in L^2(U)$$

and let $u \in H^1(U)$ be a weak solution of the elliptic PDE

$$Lu = f \quad \text{in } U.$$

Then

$$u \in H_{loc}^2(U)$$

and we have the estimate

$$\|u\|_{H^2(V)} \leq C(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}),$$

for each open subset $V \subset\subset U$, where C depends only on V , U and the coefficients of L .

Proof. see [Evans, p. 310-314] □

Theorem 2.9 (Boundary H^2 -regularity). Assume

$$a^{ij} \in C^1(\overline{U}), b^i, c \in L^\infty(U) \quad \forall i, j = 1, \dots, n \quad \text{and} \quad f \in L^2(U)$$

and let $u \in H_0^1(U)$ be a weak solution of the elliptic boundary-value problem

$$\begin{cases} Lu = f & \text{in } U \\ u = 0 & \text{on } \partial U. \end{cases}$$

Suppose furthermore ∂U is C^2 . Then

$$u \in H^2(U)$$

and we have the estimate

$$\|u\|_{H^2(U)} \leq C(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}),$$

where C depends only on U and the coefficients of L .

Proof. see [Evans, p. 317-322] □

Theorem 2.10 (Higher interior regularity). Suppose

$$a^{ij}, b^i, c \in C^{m+1}(U) \quad \forall i, j = 1, \dots, n \quad \text{and} \quad f \in H^m(U)$$

for a nonnegative integer m . Let $u \in H^1(U)$ be a weak solution of the elliptic PDE

$$Lu = f \quad \text{in } U.$$

Then

$$u \in H_{loc}^{m+2}(U)$$

and we have the estimate

$$\|u\|_{H^{m+2}(V)} \leq C(\|f\|_{H^m(U)} + \|u\|_{L^2(U)}),$$

for each $V \subset\subset U$, where C depends only on m , V , U and the coefficients of L .

Proof. see [Evans, p. 314-316] □

Corollary 2.11 (Infinite differentiability in the interior). Assume now

$$a^{ij}, b^i, c \in C^\infty(U) \quad \forall i, j = 1, \dots, n \quad \text{and} \quad f \in C^\infty(U)$$

and let $u \in H^1(U)$ be a weak solution of the elliptic PDE

$$Lu = f \quad \text{in } U.$$

Then

$$u \in C^\infty(U).$$

Theorem 2.12 (Higher boundary regularity). Suppose

$$a^{ij}, b^i, c \in C^{m+1}(\bar{U}) \quad \forall i, j = 1, \dots, n \quad \text{and} \quad f \in H^m(U)$$

for a nonnegative integer m . Let $u \in H_0^1(U)$ be a weak solution of the boundary-value problem

$$\begin{cases} Lu = f & \text{in } U \\ u = 0 & \text{on } \partial U. \end{cases}$$

Suppose furthermore ∂U is C^{m+2} . Then

$$u \in H^{m+2}(U)$$

and we have the estimate

$$\|u\|_{H^{m+2}(U)} \leq C(\|f\|_{H^m(U)} + \|u\|_{L^2(U)}),$$

where C depends only on m , U and the coefficients of L .

Proof. see [Evans, p. 323-325] □

Corollary 2.13 (Infinite differentiability up to the boundary). Assume now

$$a^{ij}, b^i, c \in C^\infty(\bar{U}) \quad \forall i, j = 1, \dots, n \quad \text{and} \quad f \in C^\infty(\bar{U})$$

and let $u \in H_0^1(U)$ be a weak solution of the boundary-value problem

$$\begin{cases} Lu = f & \text{in } U \\ u = 0 & \text{on } \partial U. \end{cases}$$

Suppose furthermore ∂U is C^∞ . Then

$$u \in C^\infty(\bar{U}).$$

2.3 Fredholm theory and spectrum

The embedding $X \hookrightarrow Y$ is compact, denoted by

$$X \xrightarrow{\text{cpt}} Y$$

if the identity map $id : X \rightarrow Y$ is compact.

Theorem 2.14 (Rellich-Kondrachov). Let $U \subset \mathbb{R}^n$ be open, bounded, with C^1 boundary. Then for all $1 \leq p \leq \infty$

$$W^{1,p}(U) \xrightarrow{\text{cpt}} L^p(U).$$

Proof. Let $\mathcal{F} \subset W^{1,p}(U)$ be a bounded set. It suffices to prove that $\mathcal{F}$ is totally bounded in $L^p(U)$. First of all observe, that using the extension theorem (Theorem 1.29) we may assume that $\mathcal{F} \subset W^{1,p}(\mathbb{R}^n)$ and $\text{supp } u \subset B_R(0)$ for some $R > 0$ and all $u \in \mathcal{F}$.

Next, note that for $u \in C_c^1(\mathbb{R}^n)$

$$\begin{aligned} \int_{\mathbb{R}^n} |u(x+y) - u(x)|^p dx &= \int_{\mathbb{R}^n} \left| \int_0^1 \frac{d}{dt} u(x+ty) dt \right|^p dx \\ &= \int_{\mathbb{R}^n} \left| \int_0^1 Du(x+ty) \cdot y dt \right|^p dx \\ &\leq \int_{\mathbb{R}^n} \int_0^1 |Du(x+ty)|^p dt dx |y|^p \\ &= \int_{\mathbb{R}^n} |Du|^p dx |y|^p. \end{aligned}$$

By approximation it follows for all $u \in W^{1,p}(\mathbb{R}^n)$ that

$$\|u(\cdot + y) - u\|_{L^p} \leq \|Du\|_{L^p} |y|.$$

Consequently, $\mathcal{F}$ satisfies condition (ii) of the Riesz-Kolmogorov compactness criterion (Theorem 1.12). By the assumption on the supports condition (iii) is also satisfied, and condition (i) is trivially true. Therefore, by Theorem 1.12 $\mathcal{F}$ is totally bounded in $L^p(\mathbb{R}^n)$. □

The assumption on the boundary regularity may be dropped for the compact embedding

$$W_0^{1,p}(U) \xrightarrow{\text{cpt}} L^p(U).$$

Let $\mu > 0$ be as in (2.4). Then $L_\mu = L + \mu I$ is invertible, with continuous inverse

$$L_\mu^{-1} : L^2(U) \rightarrow H_0^1(U)$$

Then we can reformulate (1) as

$$u - Ku = g,$$

where $K = \mu L_\mu^{-1}$ and $g = L_\mu^{-1} f$. By Rellich-Kondrachov $K : L^2(U) \rightarrow L^2(U)$ is compact, so we can apply the theory of Fredholm operators. In general we have

Theorem 2.15 (Fredholm Alternative). Let $K : H \rightarrow H$ be a compact linear operator. Then

- (i) $N(I - K)$ is finite dimensional
- (ii) $R(I - K)$ is closed
- (iii) $R(I - K) = N(I - K^*)^\perp$
- (iv) $N(I - K) = \{0\}$ if and only if $R(I - K) = H$
- (v) $\dim N(I - K) = \dim N(I - K^*)$.

Proof. see [Evans, p. 641-643] □

Remark: In particular we have either

$$(\alpha) \begin{cases} \text{for each } f \in H, \text{ the equation } u - Ku = f \\ \text{has a unique solution} \end{cases}$$

or else

$$(\beta) \begin{cases} \text{the homogeneous equation } u - Ku = 0 \\ \text{has solutions } u \neq 0. \end{cases}$$

This dichotomy is the *Fredholm alternative*.

In our situation the adjoint $I - K^*$ is related to the formal adjoint of L :

$$L^*v = - \sum_{i,j=1}^n \partial_j(a^{ij}\partial_i u) - \sum_{i=1}^n b^i \partial_i v + \left(c - \sum_{i=1}^n \partial_i b^i \right) v$$

with associated bilinear form

$$B^*[u, v] = B[v, u].$$

Let

$$N = \{u \in H_0^1(U) : Lu = 0\}, \quad N^* = \{v \in H_0^1(U) : L^*v = 0\}.$$

Lemma 2.16 (Characterizing N^*).

$$N^* = \{v \in L^2(U) : v = K^*v\}.$$

Proof. First of all observe that K extends to a continuous linear operator $H^{-1}(U) \rightarrow L^2(U)$. Then, for any $u, v \in L^2(U)$ we have

$$(u, K^*v)_{L^2} = (Ku, v)_{L^2} \leq \|v\|_{L^2} \|Ku\|_{L^2} \leq C \|v\|_{L^2} \|u\|_{H^{-1}}.$$

This shows that $K^*v \in H_0^1(U)$ for any $v \in L^2(U)$.

Next, let $v \in L^2(U)$ with $v = K^*v$. Then $v \in H_0^1(U)$. Recalling the definition of K , we see that (formally) $u - Ku = L(L_\mu^{-1}u)$ and hence

$$B[L_\mu^{-1}u, v] = (u - Ku, v) \quad \text{for all } u \in L^2(U). \quad (2)$$

Since K extends to a continuous operator $H^{-1}(U) \rightarrow L^2(U)$, also $I - K$ extends to a continuous operator $H^{-1} \rightarrow H^{-1}$. Moreover, L_μ^{-1} is an isomorphism $H^{-1}(U) \rightarrow H^1(U)$. Therefore (2) holds for all $u \in H^{-1}(U)$. It follows that $B[u, v] = 0$ for all $u \in H_0^1(U)$, hence that $v \in N^*$.

Conversely, if $v \in N^*$ then it follows from (2) that $K^*v = v$. □

Theorem 2.17 (Second existence theorem). (i) Precisely one of the following statements holds:
either

$$(\alpha) \quad \begin{cases} \text{for each } f \in L^2(U) \text{ there exists a unique} \\ \text{weak solution } u \text{ of the boundary-value problem} \\ (1) \quad \begin{cases} Lu = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases} \end{cases}$$

or else

$$(\beta) \quad \begin{cases} \text{there exists a weak solution } u \neq 0 \text{ of} \\ \text{the homogeneous problem} \\ (2) \quad \begin{cases} Lu = 0 & \text{in } U \\ u = 0 & \text{on } \partial U. \end{cases} \end{cases}$$

(ii) Furthermore, should assertion (β) hold, the dimension of the subspace $N \subset H_0^1(U)$ of weak solutions of (2) is finite and equals the dimension of the subspace $N^* \subset H_0^1(U)$ of weak solutions of

$$(3) \quad \begin{cases} L^*v = 0 & \text{in } U \\ v = 0 & \text{on } \partial U. \end{cases}$$

(iii) Finally, the boundary-value problem (1) has a weak solution if and only if

$$(f, v) = 0 \quad \forall v \in N^*.$$

Proof. see [Evans, p. 303-305] □

Consider now an elliptic operator having the form

$$Lu = - \sum_{i,j=1}^n (a^{ij}(x)u_{x_i})_{x_j},$$

where $a^{ij} \in L^\infty(U) \quad \forall \quad i, j = 1, \dots, n$.

As usual we suppose the uniform ellipticity condition to hold and

$$a^{ji} = a^{ij} \quad \forall \quad i, j = 1, \dots, n.$$

Theorem 2.18 (Eigenvalues of symmetric elliptic operators). (i) Each eigenvalue of L is real.

(ii) If we repeat each eigenvalue according to its finite multiplicity, we have

$$\Sigma = \{\lambda_k\}_{k=1}^\infty$$

where

$$0 < \lambda_1 \leq \lambda_2 \leq \dots, \quad \text{and} \quad \lambda_k \rightarrow \infty \quad \text{as} \quad k \rightarrow \infty.$$

(iii) Let $w_k \in H_0^1(U)$ be an eigenfunction corresponding to λ_k ,

$$\begin{cases} Lw_k = \lambda_k w_k & \text{in } U \\ w_k = 0 & \text{on } \partial U \end{cases}, \quad \text{for } k = 1, 2, \dots$$

normalized to $\|w_k\|_{L^2} = 1$. Then $\{w_k\}_{k=1}^\infty$ is an orthonormal basis of $L^2(U)$.