

# Abstract

With the aim of establishing a heat flow approach to the existence of Dirac-harmonic maps on spin manifolds with nonempty boundaries, we develop the existence, uniqueness and regularity for Dirac equations under a class of local elliptic boundary conditions (including chiral boundary conditions, MIT bag boundary conditions and J-boundary conditions). These results need to be sharp, since the Dirac-harmonic map problem is a borderline case in geometric analysis. Therefore, we improve, sharpen and unify the results for boundary value problems for Dirac equations known in the literature. In particular, we derive a general existence, uniqueness and regularity theorem for solutions of Dirac equations with such boundary conditions. We also show that in a particular example, the existence and regularity problem for the Dirac equation can be converted into a complex analysis problem and solved by the Cauchy integral representation formulae.

The Dirac-harmonic map problem combines a second order harmonic map type system with a first order Dirac equation. For our heat flow approach, we convert that second order system into a parabolic one and carry the solution of the first order equation along as a constraint. Since the theory of second order parabolic systems is well developed, we can focus on the first order inhomogeneous Dirac equation.

Dirac-geodesics are Dirac-harmonic maps from one dimensional domains. We introduce the heat flow of Dirac-geodesics, establish its long-time existence theorem and an asymptotic property of the global solution. We give all the possible Dirac-geodesics on the standard 2-sphere  $S^2(1)$  and the hyperbolic plane  $\mathbb{H}^2$ , and also existence results on topological spheres and hyperbolic surfaces, these solutions constitute new examples of the coupled Dirac-harmonic maps (in the sense that the map part is not the usual harmonic map).

Dirac-harmonic maps between Riemann surfaces will be considered in this thesis. First we consider Dirac-harmonic maps between closed Riemann surfaces and get some existences and criterion of uncoupled Dirac-harmonic maps. Then we consider the case of Dirac-harmonic maps from Riemann surface with nonempty boundary to Riemann surface. We derive some criterion of uncoupled Dirac-harmonics similar to the closed case and also proved the structure theorem of Dirac-harmonic map with suitable boundary condition.

Finally, we derive gradient estimates for Dirac-harmonic maps from complete spin manifolds into regular balls in Riemannian manifolds. With these estimates, we can prove Liouville theorems for Dirac-harmonic maps under curvature or energy conditions.