The Leray-Serre spectral sequence in Morse homology on Hilbert manifolds and in Floer homology on cotangent bundles

Dissertation

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The Leray-Serre spectral sequence is a fundamental tool for studying singular homology of a fibration E->B with typical fiber F. It expresses H (E) in terms of H (B) and H (F). One of the classic examples of a fibration is given by the free loop space fibration, where the typical fiber is given by the based loop space
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The first part of this thesis constructs the Leray-Serre spectral sequence in Morse homology on Hilbert manifolds under certain natural conditions, valid for instance for the free loop space fibration if the base is a closed manifold. We extend the approach of Hutchings which is restricted to closed manifolds. The spectral sequence might provide answers to questions involving closed geodesics, in particular to spectral invariants for the geodesic energy functional. Furthermore we discuss another example, the free loop space of a compact G-principal bundle, where G is a connected compact Lie group. Here we encounter an additional difficulty, namely the base manifold of the fiber bundle is infinite-dimensional. Furthermore, as H (P) = HF (T P) and H (Q) =HF (T Q), where HF denotes Floer homology for periodic orbits, the spectral sequence for P -> Q might provide a stepping stone towards a similar spectral sequence defined in purely Floer-theoretic terms, possibly even for more general symplectic quotients.

Hutchings' approach to the Leray-Serre spectral sequence in Morse homology couples a fiberwise negative gradient flow with a lifted negative gradient flow on the base. We study the Morse homology of a vector field that is not of gradient type. The central issue in the Hilbert manifold setting to be resolved is compactness of the involved moduli spaces. We overcome this difficulty by utilizing the special structure of the vector field. Compactness up to breaking of the corresponding moduli spaces is proved with the help of Gronwall-type estimates. Furthermore we point out and close gaps in the standard literature, see Section 1.4 for an overview.

In the second part of this thesis we introduce a Lagrangian Floer homology on cotangent bundles with varying Lagrangian boundary condition. The corresponding complex allows us to obtain the Leray-Serre spectral sequence in Floer homology on the cotangent bundle of a closed manifold Q for Hamiltonians quadratic in the fiber directions. This corresponds to the free loop space fibration of a closed manifold of the first part. We expect applications to spectral invariants for the Hamiltonian action functional.
The main idea is to study pairs of Morse trajectories on Q and Floer strips on T Q which are non-trivially coupled by moving Lagrangian boundary conditions. Again, compactness of the moduli spaces involved forms the central issue. A modification of the compactness proof of Abbondandolo-Schwarz along the lines of the Morse theory argument from the first part of the thesis can be utilized.