Unbounded Induced ∗-Representations.

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by Yuriy Savchuk.

Induced modules are a fundamental tool in representation theory of groups and algebras. If $\mathcal{B}$ is a subring of a ring $\mathcal{A}$ and V is a left $\mathcal{B}$-module, then the left A-module $\mathcal{A}\otimes_{\mathcal{B}} V$ with action defined by $a_0(a\otimes v) := a_0 a\otimes v$ is called *induced module* of V.

Let $\mathcal{B}\subset\mathcal{A}$ be associative algebras over $\mathbb{C}$ with involution * and let V be a Hermitian $\mathcal{B}$-module. That is, we have $\langle bx, y\rangle = \langle x, b^* y\rangle$ for all $x, y\in V$ and $b\in\mathcal{B}$. In the first part of the thesis we investigate the problem of defining a corresponding induced *Hermitian* module. In order to do so, we have to introduce an appropriate inner product on the space $\mathcal{A}\otimes_{\mathcal{B}} V$ or on some quotient space.

This is done by means of conditional expectations. Let $\mathcal{A}$ be a unital ∗-algebra and let $\mathcal{B}$ be a unital ∗-subalgebra of $\mathcal{A}$. A linear mapping $p : \mathcal{A}\to\mathcal{B}$ is called a *conditional expectation from $\mathcal{A}$ onto $\mathcal{B}$* if the following conditions are satisfied:

(i) $p(a^*) = p(a)^*,\ p(b_1 a b_2) = b_1 p(a) b_2$ for all $a\in\mathcal{A},\ b_1, b_2\in\mathcal{B}$, $p(\mathbf{1}_{\mathcal{A}}) = \mathbf{1}_{\mathcal{B}}$.
(ii) $p(\sum\mathcal{A}^2)\subseteq\mathcal{B}\cap\sum\mathcal{A}^2$,

where $\sum\mathcal{A}^2$ denotes the cone of all finite sums of elements $a^*a,\ a\in\mathcal{A}$.

If there exists a conditional expectation p from a ∗-algebra $\mathcal{A}$ onto a ∗-subalgebra $\mathcal{B}$ and if a unitary space $(V,\langle\cdot,\cdot\rangle)$ is a hermitian $\mathcal{B}$-module, then there exists a sesquilinear form $\langle\cdot,\cdot\rangle_0$ on $\mathcal{A}\otimes_{\mathcal{B}} V$ defined by

$$\langle a_1\otimes v_1, a_2\otimes v_2\rangle_0 := \langle p(a_2^* a_1)v_1, v_2\rangle. \tag{1}$$

The module V is called *inducible* if the form (1) is positive semidefinite. In this case the quotient space of $\mathcal{A}\otimes_{\mathcal{B}} V$ by the null space of the form $\langle\cdot,\cdot\rangle_0$ is a Hermitian $\mathcal{A}$-module $\mathcal{D}$. It extends uniquely to a ∗-representation of $\mathcal{A}$ on the Hilbert space completion of $\mathcal{D}$.

A standard method for the construction of conditional expectations of C^*-algebra is based on groups of ∗-automorphisms. We develop an analogue of this method for general ∗-algebras. Let G be a compact group acting by ∗-automorphisms $\alpha_g,\ g\in G$, on a ∗-algebra $\mathcal{A}$ and let μ be the normalized Haar measure of G. We say that the action α_g is locally finite-dimensional if for every $a\in\mathcal{A}$ the linear hull of the set $\{\alpha_g(a),\ g\in G\}$ is finite-dimensional. Then the map $a\mapsto\int\alpha_g(a)d\mu,\ a\in\mathcal{A}$, is a well-defined conditional expectation from $\mathcal{A}$ onto the ∗-subalgebra $\mathcal{B}$ of G-stable elements.

An important class of algebras where conditional expectations arise in a natural manner are group graded ∗-algebras. Let G be a discrete group. We say that a ∗-algebra $\mathcal{A}$ is a *G-graded* if $\mathcal{A}$ is a direct sum of subspaces $\mathcal{A}_g,\ g\in G$, such that

$$\mathcal{A}_g\cdot\mathcal{A}_h\subseteq\mathcal{A}_{g\cdot h} \text{ and } (\mathcal{A}_g)^*\subseteq\mathcal{A}_{g^{-1}} \text{ for all } g, h\in G.$$

For any subgroup $H\subseteq G$ the canonical projection p_H from $\mathcal{A}$ onto $\mathcal{A}_H = \oplus_{g\in H}\mathcal{A}_g$ is a conditional expectation. In this context we develop a theory of induced representations. Among others we prove various versions of the Imprimitivity Theorem.

In this thesis we are dealing with unbounded ∗-representations of ∗-algebras on Hilbert space. There is a striking difference to the theory of bounded representations: the problem of classifying all irreducible unbounded ∗-representations of a general ∗-algebra is not well-posed.

The second part of the thesis deals with "well-behaved" $*$-representations. The context in which we define well-behaved representations is the following. We take a G-graded unital $*$-algebra $\mathcal{A} = \oplus_{g\in G}\mathcal{A}_g$. Further we assume that the $*$-subalgebra $\mathcal{B} := \mathcal{A}_e$ is commutative. We denote by $\widehat{\mathcal{B}}^+$ the space of all characters on $\mathcal{B}$ which are nonnegative on the cone $\sum \mathcal{A}^2 \cap \mathcal{B}$. We will also assume that all characters $\chi \in \widehat{\mathcal{B}}^+$ satisfy the following condition:

$$\chi(c^*d)\chi(d^*c) = \chi(c^*c)\chi(d^*d) \text{ for all } \chi \in \widehat{\mathcal{B}}^+,\ g \in G,\ \text{and } c, d \in \mathcal{A}_g.$$

Then we define a *partial action* of the group G on the set $\widehat{\mathcal{B}}^+$. Let $\chi \in \widehat{\mathcal{B}}^+, g \in G$. We say that χ^g is defined if there exists an element $a_g \in \mathcal{A}_g$ such that $\chi(a_g^* a_g) > 0$. In this case we put

$$\chi^g(b) := \frac{\chi(a_g^* b a_g)}{\chi(a_g^* a_g)},\ b \in \mathcal{B}.$$

Using this partial action we define a notion of well-behaved representations of $\mathcal{A}$. An essential part of the thesis is devoted to studying of fundamental properties of these well-behaved representations. Some of the properties are collected in the following

Theorem. *Let π be a $*$-representation of $\mathcal{A}$, let H be a subgroup of G and let ρ be a $*$-representation of $\mathcal{A}_H$. Then the following statements hold.*

(i) *If π is bounded, then π is well-behaved.*

(ii) *If π is well-behaved, then π is self-adjoint and any self-adjoint sub-representation $\pi_0 \subseteq \pi$ is well-behaved. In particular, every well-behaved sub-representation of π has a well-behaved complement.*

(iii) *If π is a well-behaved representation with metrizable graph topology, then π can be decomposed into a direct orthogonal sum of cyclic well-behaved representations.*

(iv) *If ρ is a well-behaved representation with metrizable graph topology, then ρ is inducible via p_H if and only ρ is $\sum \mathcal{A}^2 \cap \mathcal{A}_H$-positive.*

(v) *If ρ is a well-behaved inducible representation with metrizable graph topology, then the induced representation $\mathrm{Ind}_{\mathcal{A}_H\uparrow\mathcal{A}}(\rho)$ is well-behaved.*

In this context we develop an analogue of the Mackey normal subgroup analysis. First we associate irreducible well-behaved representations to orbits under the partial action of G on $\widehat{\mathcal{B}}^+$. A central result of our Mackey analysis is the following

Theorem. *Let $\chi \in \widehat{\mathcal{B}}^+$ and let $H \subseteq G$ be its stabilizing subgroup. Then every irreducible well-behaved representation π associated to $\mathrm{Orb}\chi$ is induced from a bounded irreducible $*$-representation ρ of the algebra $\mathcal{A}_H$ satisfying the following condition:*

(2) $\mathrm{Res}_{\mathcal{B}}\rho$ *corresponds to a multiple of the character χ.*

Moreover, the representation ρ is uniquely by π and χ up to unitary equivalence.

It is shown that large classes of important examples fit into the latter context. Among them are Weyl algebras, enveloping algebras of $su(2)$ and $su(1,1)$, quantized enveloping algebras $U_q(su(2))$ and $U_q(su(1,1))$, quotients of the enveloping algebra of the Virasoro algebra, $*$-algebras associated with dynamical systems, quantum disc algebras, Podles' quantum spheres, quantum algebras, and others.