Concerning Triangulations of Products of Simplices

Abstract

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In this thesis, we undertake a combinatorial study of certain aspects of triangulations of cartesian products of simplices, particularly in relation to their relevance in toric algebra and to their underlying product structure.

The first chapter reports joint work with Samu Potka. The object of study is a class of homogeneous toric ideals called cut ideals of graphs, that were introduced by Sturmfels and Sullivant 2006. Apart from their inherent appeal to combinatorial commutative algebra, these ideals also generalize graph statistical models for binary data and are related to some statistical models for phylogenetic trees.

Specifically, we consider minimal free resolutions for the cut ideals of trees. We propose a method to combinatorially estimate the Betti numbers of the ideals in this class. Using this method, we derive upper bounds for some of the Betti numbers, given by formulas exponential in the number of vertices of the tree.

Our method is based on a common technique in commutative algebra whereby arbitrary homogeneous ideals are deformed to initial monomial ideals, which are easier to analyze while conserving some of the information of the original ideals. The cut ideal of a tree on n vertices turns out to be isomorphic to the Segre product of the cut ideals of its n-1 edges (in particular, its algebraic properties do not depend on its shape). We exploit this product structure to deform the cut ideal of a tree to an initial monomial ideal with a simple combinatorial description: it coincides with the edge ideal of the incomparability graph of the power set of the edges of the tree. The vertices of the incomparability graph are subsets of the edges of the tree, and two subsets form an edge whenever they are incomparable.

In order to obtain algebraic information about these edge ideals, we apply an idea introduced by Dochtermann and Engström in 2009 that consists in regarding the edge ideal of a graph as the (monomial) Stanley-Reisner ideal of the independence complex of the graph. Using Hochster's formula for computting Betti numbers of Stanley-Reisner ideals by means of simplicial homology, the computation of the Betti numbers of these monomial ideals is turned to the enumeration of induced subgraphs of the incomparability graph. That the resulting values give upper bounds for the Betti numbers of the cut ideals of trees is an important well-known result in commutative algebra.

In the second chapter, we focus on some combinatorial features of triangulations of the point configuration obtained as the cartesian product of two standard simplices. These were explored in collaboration with César Ceballos and Arnau Padrol, and had a two-fold motivation. On the one hand, we intended to understand the influence of the product structure on the set of triangulations of the cartesian product of two point configurations; on the other hand, the set of all triangulations of the product of two simplices is an intricate and interesting object that has attracted attention both in discrete geometry and in other fields of mathematics such as commutative algebra, algebraic geometry, enumerative geometry

or tropical geometry.

Our approach to both objectives is to examine the circumstances under which a triangulation of the polyhedral complex given by the the product of an (n-1)-simplex times the (k-1)-skeleton of a (d-1)-simplex extends to a triangulation of an (n-1)-simplex times a (d-1)-simplex. We refer to the former as a partial triangulation of the product of two simplices.

Our main result says that if d >= k > n, a partial triangulation always extends to a uniquely determined triangulation of the product of two simplices. A somewhat unexpected interpretation of this result is as a finiteness statement: it asserts that if d is sufficiently larger than n, then all partial triangulations are uniquely determined by the (compatible) triangulations of its faces of the form ?(n-1)-simplex times n-simplex?. Consequently, one can say that in this situation ?'triangulations of an (n-1)-simplex times a (d-1)-simplex are not much more complicated than triangulations of an (n-1)-simplex times an n-simplex".

The uniqueness assertion of our main result holds already when d>=k>=n. However, the same is not true for the existence assertion; namely, there are non extendable triangulations of an (n-1)-simplex times the boundary of an n-simplex that we explicitly construct.

A key ingredient towards this construction is a triangulation of the product of two (n-1)-simplices that can be seen as its "second simplest triangulation" (the simplest being its staircase triangulation). It seems to be knew, and we call it the Dyck path triangulation. This triangulation displays symmetry under the cyclic group of order n that acts by simultaneously cycling the indices of the points in both factors of the product.

Next, we exhibit a natural extension of the Dyck path triangulation to a triangulation of an (n-1)-simplex times an n-simplex that, in a sense, enjoys some sort of ?'rigidity" (it also seems new). Performing a ?'local modification" on the restriction of this extended triangulation to the polyhedral complex given by (n-1)-simplex times the boundary of an n-simplex yields the non-extendable partial triangulation.

The thesis includes two appendices on basic commutative algebra and triangulations of point configuration, included to make it slightly self-contained.