A coupled system of the Reynolds', $k-\varepsilon$ and scalar concentration equations

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Summary

The thesis considers a system of nonlinear partial differential equations describing turbulent incompressible viscous flow in a bounded domain $[0,T)\times\Omega\subset\Re^+\times\Re^3$. The system consists of the Reynolds' equations, the $k-\varepsilon$ equations and the passive scalar concentration equations. We consider precisely the following system of partial differential equations with initial and boundary conditions:

(1) $\nabla.\overline{U}=g,\ \overline{U}=(\overline{U}_1,\overline{U}_2,\overline{U}_3)\in\ \Omega_T:=[0,T)\times\Omega.$

(2) $\partial_t\overline{U}+\overline{U}.\nabla\overline{U}-c_\mu\nabla.(\nu(\nabla\overline{U}+\nabla\overline{U}^T))+\dfrac{1}{\rho}\nabla\overline{P}=0\ in\ \Omega_T$

(3) $\nu:=\frac{k^2}{\varepsilon},\ \overline{U}(0,x)=\overline{U}_0(x)\in H^3(\Omega),\ \overline{U}(t,x)=0,\ y\in\partial\Omega,\ \int_\Omega\overline{P}dx=0$

(4) $\partial_t k+\overline{U}.\nabla k-\frac{c_\mu}{2}\nu|\nabla\overline{U}+\nabla\overline{U}^T|^2-c_\mu\nabla.(\nu\nabla k)+\varepsilon=0\ in\ \Omega_T$

(5) $k(0,x)=k_0(x)\in H^3(\Omega),\ k(t,y)=k_\Gamma,\ y\in\partial\Omega$

(6) $\partial_t\varepsilon+U.\nabla\varepsilon-\frac{c_1}{2}k|\nabla\overline{U}+\nabla\overline{U}^T|^2-c_\varepsilon\nabla.(\nu\nabla\varepsilon)+c_22\frac{\varepsilon^2}{k}=0\ in\ \Omega_T$

(7) $\varepsilon(0,x)=\varepsilon_0(x)\in H^3(\Omega),\ \varepsilon(t,y)=\varepsilon_\Gamma,\ y\in\partial\Omega$

(8) $\partial_t N_i+\overline{U}.\nabla N_i-c_N\nabla.(\nu\nabla N_i)=0\ in\ \Omega_T$

(9) $N_i(0,x)=N_i^0 0(x)\in H^3(\Omega),\ \dfrac{\partial N_i}{\partial\hat{n}}(t,y)=0,\ y\in\partial\Omega,i=1,...,m,$

where $\nu:=\frac{k^2}{\varepsilon}$, $\overline{U}$ is the average velocity field, k, the turbulent kinetic energy, ε, the turbulent energy of dissipation, and N_i $(i=1,...,m)$, the average concentrations of passive scalars.

We assume that Ω has a C^3 boundary and, for simplicity, connected. We are interested in the special case where the divergence $g=0$.

Inherent difficulties in the system. The presence of the two singular functions $\frac{k^2}{\varepsilon}$ and $\frac{\varepsilon^2}{k}$ calls to question, whether the problem is well posed or not. Further, it is imperative that the system comprising (1)-(7) yields positive values of k and nε for physical and mathematical reasons. Furthermore, the presence of the pressure gradient term, in the Reynolds equations, further complicates the system; since we do not have a separate equation nor the boundary condition for the pressure, even though it is necessary we work in higher order Banach spaces.

Method of solution.

Useful tools. As a preamble to the main work at hand, we proved the nonlinear versions of the differential and the integral forms of the Gronwall's inequality in Lemmas 2.1 and 2.2 respectively. We comment that the bound in each estimate of the lemmas blows up after some time, so that the estimates are valid only within some time intervals.

The motivations for Lemma 2.3 are the estimates of the integrals over $[0,\ t]\times\Omega$ $(t\leq T)$, of the numerous squares of nonlinear terms encountered in the thesis.

The lemma serves to introduce small values as multipliers of certain terms appearing on the right sides of some inequalities, so as be able to absorb them by the corresponding terms on the left.

Furthermore, we obtained a positivity result in Lemma 2.4 which ensures that k and ε remain strictly positive within a time interval, provided that the initial conditions are strictly positive.

A priori estimates. Estimates for the quantities of the linearized non-homogeneous system (1)-(7) are obtained as integral equations with the non-homogeneous terms assumed to be in $L^2\left[0, T; L^2(\Omega)\right]$. To obtain the estimates for the quantities of the full system (1)-(7), the non-homogeneous terms are equated to the nonlinear terms, estimated and then used in the integral equations obtained for the linearized non-homogeneous equations. The integral equations are then simplified and the integral form of the linear or nonlinear version (Lemma 2.2) of the Gronwall's inequality applied as appropriate. We remark that higher order estimates for $\overline{U}$ and $\overline{P}$; and the estimates for k and ε (Theorems 3.1 and 3.3)are obtained, mostly, by analogies with certain analogous partial differential equations arising from the proof of previous lower order estimates (Theorem 2.5). Estimates for N_i $(i = 1, ..., m)$ are obtained directly for the full problem (8)-(9).

Existence of solution. The equivalent fixed point arguments for the system (1)-(9) are obtained and an appropriate linear mapping defined. The existence of a local-in-time strong solution to the system (1)-(9), in a bounded set of a Banach space, defined in sympathy with the apriori estimates, is proved using Banach's fixed point theorem.