On Functoriality of Homological Mirror Symmetry of Elliptic Curves

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Abstract

In this work the author uses the wellknown mirror functor for elliptic curves (Kontsevich [1], Polishchuk and Zaslow [3], Kreussler [2]) to give surprising answers to several natural questions und construct a lot of new, previously unknown functors between Fukaya categories of complexified tori.

Problem 1: Functoriality with respect to morphisms of the underlying manifolds

A given morphism in the category of complex tori or in the category of complexified symplectic tori induces a functor between the derived categories and the Fukaya categories, respectively. We ask whether a functor induced by a morphism corresponds to a functor that is itself induced by a morphism or, more generally, by a global geometric construction. The answer is unfortunately negative.

Theorem A. *Functors between derived categories of elliptic curves induced by morphisms between the underlying complex tori do not in general correspond to functors between Fukaya categories with a global geometric interpretation.*

Problem 2: Describe the impact of choices on the algebraic side involved in constructing the mirror functor

The mirror functor as constructed by Polishchuk and Zaslow depends on various choices. This calls for a description of how the constructed functor depends on these choices. For a "good" mirror functor Φ_τ we would expect that we get a symplectic counterpart, that has a geometric explanation in terms of complexified symplectic tori. But unfortunately, the mirror functor does not have this expected property.

Theorem B. *The construction of the derived category of coherent sheaves and therefore of the mirror functor depends on non canonical choices. The functors for two such choices differ by equivalences on both the algebraic and the symplectic side. In general, on the symplectic side there is no geometric interpretation for these induced equivalences.*

Problem 3: Describe the impact of choices on the symplectic side involved in constructing the mirror functor

There are similar choices involved in the construction of the Fukaya category. For a proper understanding of the functor $\mathcal{FK}(E^\tau) \longrightarrow \mathcal{FK}(E^{\tau+1})$ describing the difference of two such

choices, we need it to be induced by an isomorphism of the tori or a comparable geometric construction, which is however not the case.

Theorem C. *The construction of the Fukaya category depends on noncanonical choices. The categories for two such choices differ by an equivalence. There is no geometric interpretation for these equivalences.*

New functors between Fukaya categories

Finally, this method allows us to construct many new functors between Fukaya categories. We can write down explicitly and explain by geometrical constructions many more algebraic functors than symplectic functors. With the help of the mirror symmetry functor and the resulting new functors on the symplectic side, we can shed more light onto the still quite unknown Fukaya category and its internal symmetries.

References

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[2] Bernd Kreußler. Homological mirror symmetry in dimension one. In *Advances in algebraic geometry motivated by physics (Lowell, MA, 2000)*, volume 276 of *Contemp. Math.*, pages 179–198. Amer. Math. Soc., Providence, RI, 2001.

[3] Alexander Polishchuk and Eric Zaslow. Categorical mirror symmetry: the elliptic curve. *Adv. Theor. Math. Phys.*, 2(2):443–470, 1998.