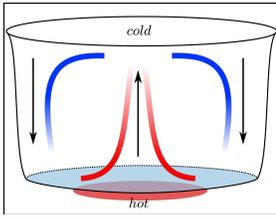# Rayleigh Bénard convection: bounds on the Nusselt number

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A model of thermal convection

Rayleigh-Bénard convection is the buoyancy-driven flow of a fluid heated from below and cooled from above. This model of thermal convection is a paradigm for pattern formation and turbulence [1] and it plays an important role in a large range of phenomena in geophysics, astrophysics, meteorology, oceanography and engineering. The problem under investigation is: given an incompressible fluid enclosed in a rectangular container heated from below and cooled from above, what are the flow dynamics? In particular, what is the heat transfer from the bottom to the top? The two main mechanisms associated to the temperature variations are conduction, which tends to remove local temperature differences, and convection, which transports fluid at macroscopic scales. In the specific, convection is generated by the interplay of the buoyancy force that arise from density variations and the limiting effect of inner friction, especially near the plates.

If the driving forces, as measures by the Rayleigh number defined below, are small the fluid is at rest in the bulk (i.e the pure conduction state is the global attractor). Above an explicitly known critical Rayleigh number the global attractor consists of stationary *convection rolls* which appear due to the density differences between layers: hot parcels move upwards and cold parcels, denser, move downwards. As the Rayleigh number increases further the convection pattern is destabilized by finger-like structures that detach from the boundary layers, called *plumes*. This ultimate stage is classified as turbulent.
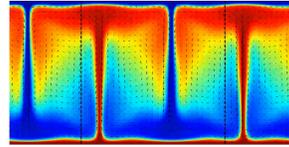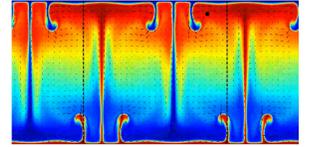

Figure 1: Convection rolls

Figure 2: Plumes

In the $d-$dimensional container we follow the evolution equation of the velocity vector field $u(x,t)$, the temperature scalar field $T(x,t)$ and the pressure scalar field $p(x,t)$ where we indicate with x the $d-$dimensional spatial variable and with t the time variable. In the Boussinesq (1903) approximation, where the density ρ is supposed to be linearly depending on T, the dimensionless equations of motion are :

$$\begin{cases} \frac{\partial}{\partial t}T + \mathbf{u}\cdot\nabla T = \Delta T & (1) \\ \frac{1}{\Pr}\left(\frac{\partial}{\partial t}\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}\right) - \Delta\mathbf{u} + \nabla p = \mathrm{Ra}T\mathbf{e_z} & (2) \\ \nabla\cdot\mathbf{u} = 0 & (3) \end{cases}$$

On the two plates at height $z = 0$ and $z = 1$, respectively, the velocity field satisfies *no-slip* boundary conditions and the temperature is 1 at $z = 0$ and 0 at $z = 1$. All the quantities $(\mathbf{u}, T, p)$ are assumed to be periodic in the the horizontal variables $x' \in [0, L]^{d-1}$. From the non-dimensionalization, only two nondi-

mensional parameters are left:

$$\mathrm{Pr} = \frac{\nu}{\chi} \quad \text{and} \quad \mathrm{Ra} \propto (T_{hot} - T_{cold})h^3 \,,$$

where ν is the fluid's kinematic viscosity, χ is the thermal diffusivity and h is the original height of the container. The two parameters are called *Prandtl number* and *Rayleigh number*, respectively.

The Nusselt number

In the applications, it is of interest to investigate the transport properties of the flow. The measure that quantifies the enhancement of vertical heat flux due to convection is the *Nusselt number*, Nu, defined as

$$\mathrm{Nu} = \limsup_{t_0 \to \infty} \int_0^{t_0} \int_0^1 \langle (\mathbf{u}T - \nabla T) \cdot \mathbf{e_z} \rangle' dz \, dt \,,$$

where $\langle \cdot \rangle'$ stands for the horizontal-space average and $\mathbf{e_z}$ is the upward unit normal. Since the convective fluid flow increases vertical heat transport beyond the purely conductive flux, our aim is to get bounds on the Nusselt number in the large Rayleigh-number regime. At *sufficiently large* Ra, when the flow is turbulent, the presumed functional relation between Nu, Pr and Ra is of the form

$$\mathrm{Nu} \approx \mathrm{Pr}^\gamma \mathrm{Ra}^\beta.$$

In 1954 Malkus [2] predicted the scaling law $\mathrm{Nu} \sim \mathrm{Ra}^{\frac{1}{3}}$ by a *marginally stable boundary layer* argument, based on the concept that the boundary layer thickness δ adjusts so as to be (as a convection layer itself) stable. The scaling $\mathrm{Nu} \sim \mathrm{Pr}^{\frac{1}{2}}\mathrm{Ra}^{\frac{1}{2}}$ has been postulated by Spiegel (1971) [3] and Kraichnan (1962) [4] as an asymptotic regime in which the heat transfer and the strength of turbulence become independent of the kinematic viscosity and the thermometric conductivity.

Infinite Prandtl number limit

When the fluid is very viscous (e.g, Earth's mantel and engine oils), the inertia of the fluid can be neglected. Setting $\mathrm{Pr} = \infty$ we study

$$\begin{cases} \frac{\partial}{\partial t}T + \mathbf{u} \cdot \nabla T = \Delta T \,, \\ -\Delta \mathbf{u} + \nabla p = \mathrm{Ra}T\mathbf{e_z} \,, \\ \nabla \cdot \mathbf{u} = 0 \,, \end{cases}$$

for which the scaling $\mathrm{Nu} \sim \mathrm{Ra}^{\frac{1}{3}}$ is believed to be valid in the asymptotically high-Rayleigh number regime (Grossmann & Lohse [5]). The derivation of physically relevant upper bounds has a long history, that goes back to the sixties (Howard (1963) and Busse (1969)). Several decades after, the introduction of the *background field method* (Doering & Constantin [6]) has produced significant results. This method consists of decomposing the temperature field into a background profile and a perturbation term,

$$T(x', z, t) = \tau(z) + \theta(x', z, t)$$

where $\tau(0) = 1$, $\tau(1) = 0$ and $\theta(x', 0, t) = 0 = \theta(x', 1, t)$. The background field method reduces the problem of finding upper bounds for Nu to the problem of constructing a background profile τ that satisfies a stability constraint. Then the Dirichlet integral of a stable background temperature profile generates an upper bound for the Nusselt number. Optimizing on the class of stable profiles, the optimal upper bound is

$$\mathrm{Nu} \leq \widetilde{\mathrm{Nu}},$$

where $\widetilde{\mathrm{Nu}}$ is defined as

$$\widetilde{\mathrm{Nu}} := \inf_{\substack{\tau:(0,1)\to\mathbb{R} \\ \tau(0)=1, \tau(1)=0}} \left\{ \int_0^1 \left(\frac{d\tau}{dz} \right)^2 dz \, \big| \, \tau \text{ marginally stable} \right\}.$$

Here, the construction of stable background profiles is favored by the *instantaneous slaving* of the velocity to the temperature field. The first upper bound

that reproduce the physical scalings up to logarithmic correction, was derived by Constantin and Doering [7]. They proved $\mathrm{Nu} \lesssim (\ln \mathrm{Ra})^{\frac{2}{3}} \mathrm{Ra}^{\frac{1}{3}}$, by a (logarithmically failing) maximal regularity estimate for the Stokes equation in L^∞ together with bounds on the average Laplacian squared of the temperature. Doering, Otto and Westdickenberg [8] improved the bound in [7] obtaining $\widetilde{\mathrm{Nu}} \lesssim (\ln \mathrm{Ra})^{\frac{1}{3}} \mathrm{Ra}^{\frac{1}{3}}$, by the construction of a non-monotonic background profile with a logarithmic layer in the bulk. A refinement of the argument in [8] was used by Otto and Seis in 2011 [9] to improve the last bound obtaining $\widetilde{\mathrm{Nu}} \lesssim (\ln \mathrm{Ra})^{\frac{1}{15}} \mathrm{Ra}^{\frac{1}{3}}$. Despite the fact that the background field method has produced many significant upper bounds, it is not optimal. Indeed the author in collaboration with Felix Otto [10] showed that the Nusselt number produced by the background field method $\widetilde{\mathrm{Nu}}$, is bounded from below by $(\ln \mathrm{Ra})^{\frac{1}{15}} \mathrm{Ra}^{\frac{1}{3}}$. This implies that $\widetilde{\mathrm{Nu}} \sim (\ln \mathrm{Ra})^{\frac{1}{15}} \mathrm{Ra}^{\frac{1}{3}}$ and no better bound can be produced within this method. By the combination of the background field method with the L^∞ maximal regularity for Stokes equation, Otto and Seis [9] derived the upper bound $\mathrm{Nu} \lesssim (\ln \ln \mathrm{Ra})^{\frac{1}{3}} \mathrm{Ra}^{\frac{1}{3}}$. which is, to our knowledge, optimal. The combination of the lower bound on $\widetilde{\mathrm{Nu}}$ with the last upper bound yields

$$\mathrm{Nu} \ll \widetilde{\mathrm{Nu}},$$

implying that $\widetilde{\mathrm{Nu}}$ does not carry any physical meaning.

Finite Prandtl numbers

When Pr is finite, the equation for the temperature is coupled with the full Navier-Stokes equation for the velocity field, i.e

$$\frac{1}{\mathrm{Pr}} \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) - \Delta \mathbf{u} + \nabla p = \mathrm{Ra} T \mathbf{e_z},$$

$$\nabla \cdot u = 0.$$

As opposed to the case $\mathrm{Pr} = \infty$, here a major difficulty comes from the fact that the velocity and the temperature field are not instantaneously slaved to each other. We are interested in deriving rigorous upper bounds for Nu that reproduce both physical scalings $\mathrm{Nu} \sim \mathrm{Ra}^{\frac{1}{3}}$ and $\mathrm{Nu} \sim \mathrm{Ra}^{\frac{1}{2}}$ in some parameter regimes, up to logarithms. In 1996 Doering and Constantin [6] applied the background field method to the problem, finding $\mathrm{Nu} \lesssim \mathrm{Ra}^{\frac{1}{2}}$ for no-slip (or stress-free) boundary condition. In 2006 Wang [11] proved the upper bound $\mathrm{Nu} \lesssim (\ln \mathrm{Ra})^{\frac{2}{3}} \mathrm{Ra}^{\frac{1}{3}}$ for $\mathrm{Pr} \gtrsim \mathrm{Ra}$ with a perturbation argument on the Stokes equation, assuming that Pr is very large. Combining (logarithmically failing) maximal regularity estimates in L^∞ and weighted$-L^1$ for the nonstationary Stokes equation with forcing terms given by the buoyancy term and the nonlinear term respectively, the author in collaboration with Antoine Choffrut and Felix Otto proved [12]

$$\mathrm{Nu} \lesssim \begin{cases} (\ln \mathrm{Ra})^{\frac{1}{3}} \mathrm{Ra}^{\frac{1}{3}} & \text{for } \mathrm{Pr} \gtrsim (\ln \mathrm{Ra})^{\frac{1}{3}} \mathrm{Ra}^{\frac{1}{3}}, \\ (\ln \mathrm{Ra})^{\frac{1}{2}} \mathrm{Pr}^{-\frac{1}{2}} \mathrm{Ra}^{\frac{1}{2}} & \text{for } \mathrm{Pr} \lesssim (\ln \mathrm{Ra})^{\frac{1}{3}} \mathrm{Ra}^{\frac{1}{3}}. \end{cases}$$

This result improves the upper bound in [11] and captures the turbulence coming from the acceleration of the fluid in the bulk.

References

[1] David J Tritton. *Physical fluid dynamics*, volume 1. 1988.

[2] Willem VR Malkus. The heat transport and spectrum of thermal turbulence. *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences*, 225(1161):196–212, 1954.

[3] Edward A Spiegel. Convection in stars: I. Basic Boussinesq convection. *Annual review of astronomy and astrophysics*, 9:323, 1971.

[4] Robert H Kraichnan. Turbulent thermal convection at arbitrary Prandtl number. *Physics of Fluids (1958-1988)*, 5(11):1374–1389, 1962.

[5] Siegfried Grossman and Detlef Lohse. Scaling in thermal convection: a unifying theory. *Journal of Fluid Mechanics*, 407:27–56, 3 2000.

[6] Charles R Doering and Peter Constantin. Variational bounds on energy dissipation in incompressible flows. III. Convection. *Physical Review E*, 53(6):5957, 1996.

[7] Peter Constantin and Charles R Doering. Infinite Prandtl number convection. *Journal of Statistical Physics*, 94(1-2):159–172, 1999.

[8] Charles R Doering, Felix Otto, and Maria G Reznikoff. Bounds on vertical heat transport for infinite-Prandtl-number Rayleigh-Bénard convection. *Journal of Fluid Mechanics*, 560:229–242, 2006.

[9] Felix Otto and Christian Seis. Rayleigh–Bénard convection: improved bounds on the Nusselt number. *Journal of Mathematical Physics*, 52(8):083702, 2011.

[10] Camilla Nobili and Felix Otto. Lower bound on the temperature background field method for Rayleigh-Bénard convection. *to be submitted.*

[11] Xiaoming Wang. Asymptotic behavior of the global attractors to the Boussinesq system for Rayleigh-Bénard convection at large Prandtl number. *Communications on Pure and Applied Mathematics*, 60(9):1293–1318, 2007.

[12] Antoine Choffrut, Camilla Nobili, and Felix Otto. Upper bounds on Nusselt number at finite Prandtl number. *Submitted to the Journal of differential equations.*