PERIODIC AND HOMOCLINIC MOTIONS IN INFINITE LATTICES

(THESIS SUMMARY)

PERCY D. MAKITA

This thesis is concerned with the study of periodic and homoclinic motions in one-dimensional infinite lattices with nearest-neighbour interaction(s) and on-site potential(s). Physically, this can be interpreted as a chain of an infinite number of particles subjected to some potential(s) and such that each particle interacts with its nearest neighbours. If the particle at the j-th site is subjected to a potential f_j and the interaction between the j-th and $(j+1)$-th particles is governed by V_j, the displacement q_j of the j-th particle satisfies the second order ODE

$$\ddot{q}_j + f_j'(q_j) = V_j'(q_{j+1} - q_j) - V_{j-1}'(q_j - q_{j-1}),\ j \in \mathbb{Z}. \tag{1}$$

This is naturally an inifnite-dimensional Hamiltonian system with 'formal' Hamiltonian

$$H = \sum_{j\in\mathbb{Z}} \left[\frac{1}{2}p_j^2 + f_j(q_j) + V_j(q_{j+1} - q_j) \right].$$

In the particular case when $V_j = V$ and $f_j = f$ for all j, we are mainly interested in the so-called travelling wave solutions, i.e. solutions of the type $q_j(t) = u(j - ct)$, $j \in \mathbb{Z}$. The wave profile u satisfies the second order forward-backward ODE:

$$c^2\ddot{u}(z) + f'(u(z)) = V'(u(z+1) - u(z)) - V'(u(z) - u(z-1)), \quad z \in \mathbb{R}. \tag{2}$$

This thesis addresses and answers questions on the existence of periodic and homoclinic solutions for the equations (1), (2) and a non-autonomous generalization of (2), namely

$$\ddot{u}(t) + f_0(t, u(t)) = \sum_{i=1}^{N} \left[f_i(t, u(t+\tau_i) - u(t)) - f_i(t - \tau_i, u(t) - u(t - \tau_i)) \right] \tag{3}$$

where the τ_i are positive constants.

Due to their variational structure, periodic and homoclinic solutions of the above equations can be found as critical points of certain functionals. Denote by Φ_T (resp. Φ_∞) the usual functional whose critical points correspond to the T-periodic (resp. homoclinic) solutions of (1) (resp. (2), (3)).

The critical points of Φ_T can be found using standard methods. Under some appropriate conditions on the potentials, Φ_T satisfies all the conditions of the mountain pass theorem of Ambrosetti and Rabinowitz, except the Palais-Smale compactness condition in the case of (1). We use a discrete version of P.-L. Lions' concentration compactness lemma to detect the critical points of Φ_T which are non-trivial T-periodic solutions of (1). These periodic solutions are non-constants provided the

period be relatively high. In the case of (2) (resp. (3)), under some appropriate conditions on the "potentials", a linking theorem of Rabinowitz or the standard mountain pass theorem allows us to detect periodic solutions for (2) (resp. (3)). Those given by the linking theorem are non-constant, while those given by the mountain pass theorem are non-trivial and become non-constant when the period is "large enough".

When dealing with homoclinic solutions, the situation is completely different. The functional Φ_∞ for either (1), (2) or (3) possesses the mountain pass geometry but do not satisfy the Palais-Smale condition. However we give a constructive method for constructing non-trivial critical points of Φ_∞. Namely, we construct a family $\mathcal{Q}^{[0]}$ of periodic solutions of (1) (resp. (2), (3)). The family $\mathcal{Q}^{[0]}$ has the properties of being uniformly bounded from above as well as from below in the space C_b of bounded continuous functions equipped with the sup-norm, and the sets $\mathcal{Q}^{[1]}$, and $\mathcal{Q}^{[2]}$ containing the first and second derivatives of the members of $\mathcal{Q}^{[0]}$ are uniformly bounded from above in C_b. We can then extract, thanks to Arzelà-Ascoli's theorem, a sequence of members of $\mathcal{Q}^{[0]}$ whose limit is shown to be a non-trivial critical point of Φ_∞.