On Conformal Connections and Infinitesimal Conformal Transformations

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Summary

A vector field on a manifold M with a metric g of arbitrary signature is called conformal if its local flow preserves the class of all conformal metrics $e^\sigma g$ on M. The present thesis is concerned with Lie algebras of conformal vector fields that cannot be reduced by a conformal change of the metric to an algebra of homothetic vector fields. The study is based on the theory of conformal connections. In particular, it uses the correspondence between the conformal algebra of a manifold (M, g) with $\dim M = n \geq 3$ and the algebra of infinitesimal automorphisms of the normal conformal connection which is defined on the principal prolongation of the bundle of conformal frames over M.

Necessary and sufficient conditions are obtained answering when a given conformal isotropy algebra of a point m is reducible by a conformal change of the metric to an algebra of homothetic (or isometric) vector fields near m. The conditions are expressed both in terms of a family of isomorphisms from the conformal isotropy algebra into the Lie algebra of the structure group of the said prolongation and in terms of the first and second derivatives of its constituent vector fields at m. They generalise results of Beig and Capocci for the reduction of a single conformal vector field.

The given criteria enable the study of the zero set of an irreducible conformal vector field. For a conformal vector field X that is not reducible at m and a point m_1 lying in the domain of a certain normal chart at m, sufficient conditions are given for m_1 to be a zero of X. These conditions are also necessary, at least, in the case when m and m_1 are joined by a null geodesic. With respect to the normal chart, the zeros of X so described form plane sections of the 'null cone' of m, i.e. the set of all points joined to m by a null geodesic.

In the case of a positive definite metric, a new proof is proposed of a result due to Alekseevski which asserts the existence of a special neighbourhood for a fixed point of a 1-parameter group of essential conformal transformations. From this it is shown that if a conformal vector field on a Riemannian manifold is not isometric with respect to some conformal metric on a neighbourhood of a given point, then that point is an isolated zero of the vector field and some neighbourhood of it is conformally flat. By a combination of this conclusion with results of Kobayashi and Blair, the structure of the zero set of a conformal vector field on a Riemannian manifold is characterised.

To contrast this, examples are provided of Lorentzian manifolds admitting an irreducible conformal vector field that are not conformally flat. A method for the construction of such examples is given.

The thesis also contains a study of the curvature of the normal conformal connection and of its relation to the curvature of a Weyl connection, as well as a study of conformal normal charts and their relation to normal charts associated with normal Weyl connections.