Thesis Summary

The derivation of asymptotic theories which serve as simplified models for lower dimensional objects in three dimensional elasticity has a long history, but only recently rigorous derivations have been obtained. Their mathematical framework is that of Gamma-convergence, a concept developed by DeGiorgi and his school in the 1970s. In 2001 Friesecke, James and Müller proved that for a certain scaling of the energy with respect to the plate thickness, Kirchhoff's plate theory arises as a Γ-limit of three dimensional nonlinear elasticity.

In the first two chapters of my thesis I study two questions which are motivated by their result. One important feature of Kirchhoff's plate theory is that it only allows for limiting deformations which are locally isometric to the flat reference configuration. More precisely, the set of admissible limiting deformations is given by

$$\mathcal{A} = \{u \in W^{2,2}(S;\mathbb{R}^3) : |u_{x_1}| = |u_{x_2}| = 1 \text{ and } u_{x_1} \cdot u_{x_2} = 0\},$$

where $S \subset \mathbb{R}^2$ is a bounded Lipschitz domain. Such deformations are locally bilipschitz, and the question arises whether this property is also enjoyed by the genuinely three dimensional thin-film deformations (recovery sequences in the terminology of Γ-convergence). In the first chapter I prove an upper bound on the maximal distance between two preimages of the same point in the deformed configuration: Let $h > 0$ and let $u^{(h)} : S \times (-\frac{h}{2}, \frac{h}{2})$ be a thin-film deformation with almost optimal bending energy (a recovery sequence). Assume that that the points $z_h, z_h' \in S \times (-\frac{h}{2}, \frac{h}{2})$ satisfy

$$u^{(h)}(z_h) = u^{(h)}(z_h')$$

for a sequence $h \to 0$. Then their mutual distance satisfies $|z_h - z_h'| \ll h$ as h converges to zero.
Another central result of the first chapter is an example showing that this positive result is optimal in the sense that there exist three dimensional thin-film deformations $u^{(h)}$ with almost optimal energy (recovery sequences) which are non-injective on a set of positive volume. This example applies to arbitrary scalings of the energy with respect to the plate thickness, and it gives a negative answer to the question of local invertibility in thin elastic films.

The second chapter is devoted to the study of existence and regularity of deformations minimizing Kirchhoff's energy functional. For a certain class of materials it agrees - on the set of surfaces Σ which are locally isometric to the plane - with the Willmore functional from differential geometry. The Willmore functional is given by

$$\mathcal{E}(\Sigma) = \int_{\Sigma} |H|^2 \, d\mathcal{H}^2,$$

where H denotes the mean curvature vector of the surface Σ. For a convex smooth domain $S \subset \mathbb{R}^2$ I consider admissible parametrizations in the set $\mathcal{A}$ introduced above.
After proving existence of minimizers $u \in \mathcal{A}$ of the functional $\mathcal{E}$ under general boundary conditions, I study their regularity properties, focusing on a subclass of $\mathcal{A}$ defined by imposing certain boundary conditions. The main difficulty in the derivation of an Euler-Lagrange equation for a minimizer u in this subclass is to find proper variations. These must not only respect the boundary conditions but also the isometry constraint, which is a nonconvex constraint on the derivatives of u. Such a nonholonomic problem cannot be treated by general Lagrange Multiplier theorems. However, it can be circumvented by a proper choice of coordinates which exploits the developability of mappings in the class $\mathcal{A}$. By this choice the problem is reduced to a one-dimensional problem on lines of curvature, where the boundary conditions on the original minimizer u show up as isoperimetric constraints on the normal and geodesic curvatures of its curvature lines. Such constraints can be incorporated into the Euler-Lagrange equations by means of Lagrange Multipliers.
Then I introduce two kinds of variations and derive two sets of Euler-Lagrange equations. While for surfaces one generally expects a partial differential Euler-Lagrange equation, due to the choice of coordinates just described I obtain a pair of ordinary differential equations. These can be used to prove a partial regularity result for local minimizers u of the Willmore functional within the subclass of $\mathcal{A}$ defined by the boundary conditions. It states that the surface $u(\Omega)$ is smooth away from planar points. By a subtle analysis of the α-limit set of the flow induced by the Euler-Lagrange system near planar points, the exceptional set is further reduced to a certain kind of planar points at which, in addition, the surface approaches the shape of a cone with vertex on the boundary of the domain.

In the third chapter I focus on another instance of thin elastic films: I rigorously derive, in terms of Gamma-convergence, the energy functional that governs the behaviour of thin martensitic films within the framework of linearized elasticity. Due to their shape-memory properties, martensitic materials have recieved considerable interest over the past years. From a mathematical viewpoint, their main feature is the presence of multiple energy wells in their stored energy function, each well corresponding to a different solid phase. In 1999 Bhattacharya and James observed that for many martensitic materials the structural variety of thin films is much richer than that of bulk samples made of the same material. The reason is that two phases may be able to form an interface in the plane, while being incompatible in bulk.
This situation is modelled by a nonnegative continuous stored energy function

$$W : \mathbb{R}^{3\times 3} \to \mathbb{R}$$

that satisfies growth and coercivity conditions and whose zero set $K_1 \cup K_2$ is given by two incompatible linearized energy wells like for example

$$K_1 = \Big\{ M \in \mathbb{R}^{3\times 3} : \frac{M^T + M}{2} = 0 \Big\}$$
$$K_2 = \Big\{ M \in \mathbb{R}^{3\times 3} : \frac{M^T + M}{2} = e_2 \otimes e_1 + e_1 \otimes e_2 + e_3 \otimes e_3 \Big\}.$$

For a strictly star-shaped Lipschitz domain $S \subset \mathbb{R}^2$ and a thin film $S \times (-\frac{h}{2}, \frac{h}{2})$ of thickness h, I show that the functionals

$$I^h(u) = \begin{cases} \frac{1}{h^2} \int_{S\times(-\frac{h}{2},\frac{h}{2})} W(\nabla u) & \text{if } u \in W^{1,2}(S \times (-\frac{h}{2}, \frac{h}{2}); \mathbb{R}^3) \\ +\infty & \text{otherwise} \end{cases}$$

Gamma-converge to the functional

$$I^0(w) = \begin{cases} \int_J k(\nu(x)) d\mathcal{H}^1(x) & \text{if } w \text{ is admissible} \\ \infty & \text{otherwise.} \end{cases}$$

The admissible limiting displacements are given by the set of two dimensional displacements w whose symmetrized gradient has bounded variation and takes at most two different values, which are related to the energy wells K_1 and K_2 of the three dimensional energy density W. The symbol J denotes the jump set of the symmetrized gradient $(\nabla w)^T + (\nabla w)$, the vector $\nu \in S^1$ denotes the normal to it and k is a real-valued function on S^1. The normal can only assume two values ν_1 and ν_2.

The derivation is based upon a rigidity theorem for two incompatible linear wells which leads to a compactness theorem for sequences with finite limiting energy. For the proof of the upper bound a gluing argument for recovery sequences is applied which involves a three-step interpolation.