

# On a spin conformal invariant on open manifolds

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In this thesis we examine a spin conformal invariant on open manifolds. For that, let  $(M, g, \sigma)$  be an  $n$ -dimensional Riemannian spin manifold of dimension  $n \geq 2$  and let  $D_g$  be the corresponding Dirac operator. We define

$$\lambda_{min}^+(M, g, \sigma) = \inf \left\{ \frac{\|D_g \phi\|_{L^q}^2}{(D_g \phi, \phi)} \mid \phi \in C_c^\infty(M, S), (D_g \phi, \phi) > 0 \right\},$$

where  $q = \frac{2n}{n+1}$  and  $C_c^\infty(M, S)$  is the set of compactly supported smooth spinors. The  $\lambda_{min}^+$ -invariant is conformally invariant, that means that it does not depend directly on the metric but on its conformal class  $[g] = \{f^2 g \mid 0 < f \in C^\infty(M)\}$ . This invariance comes from the transformation behaviour of the Dirac operator under conformal changes. There are other operators with similar transformations. The most important is probably the conformal Laplacian  $L_g = 4\frac{n-1}{n-2}\Delta_g + s_g$ , where  $s_g$  is the scalar curvature, that gave rise to the Yamabe invariant

$$Q(M, g) = \left\{ \frac{(v, L_g v)}{\|v\|_p^2} \mid v \in C_c^\infty(M), p = \frac{2n}{n-2} \right\}.$$

Thus, a natural question is whether results for the Yamabe invariant have their counterparts for the  $\lambda_{min}^+$ -invariant. Much work is done in literature considering these and more aspects on closed manifolds. In this thesis we mainly examine the  $\lambda_{min}^+$ -invariant on open manifolds.

In the first part of the thesis we study an obstruction on spin conformal compactification:

**Theorem.** *Let  $(M, g, \sigma)$  be a Riemannian spin manifold of dimension  $n \geq 2$  with*

$$\lim_{r \rightarrow \infty} \lambda_{min}^+(M \setminus B_r(p), g, \sigma) < \lambda_{min}^+(S^n, g_{st}, \chi_{st})$$

*for a fixed  $p \in M$  and  $B_r(p)$  a ball around  $p$  of radius  $r$  with respect to the metric  $g$ . Then  $(M, g, \sigma)$  is not spin conformally compactifiable.*

Note that the  $\lambda_{min}^+$ -invariant of the standard  $n$ -sphere with its unique spin structure is the highest possible value for the  $\lambda_{min}^+$ -invariant on  $n$ -dimensional manifolds.

Furthermore, we prove the invariance of the  $\lambda_{min}^+$ -invariance when reducing a manifold by  $(n - 2)$ -dimensional open subsets. All this is done by using the Lichnerowicz formula, adequate cut-off functions and an Aubin-type inequality that we will prove on closed manifolds.

Moreover, the Bär inequality that holds for surfaces homeomorphic to  $S^2$  will be proven for open manifolds homeomorphic to  $\mathbb{R}^2$ . Additionally, we study the  $\lambda_{min}^+$ -invariant on open surfaces with cusps to obtain a class of examples of manifolds with vanishing  $\lambda_{min}^+$ -invariant. In contrast, the  $\lambda_{min}^+$ -invariant on closed manifolds is always positive.

In the second part we give some estimates for the  $\lambda_{min}^+$ -invariant depending on the underlying manifold. On closed manifolds such estimates are well-known. There is, e.g., the conformal Hijazi inequality that gives a lower bound of the  $\lambda_{min}^+$ -invariant in terms of the Yamabe invariant. We extend this result to classes of open manifolds, e.g.

**Theorem.** *Let  $(M, g, \sigma)$  be a conformally parabolic Riemannian spin manifold of dimension  $n > 2$ . If there exists a complete conformal metric  $\bar{g}$  of finite volume and 0 is not in the essential spectrum of  $D_{\bar{g}}$ , then the conformal Hijazi inequality is valid:*

$$\lambda_{min}^+(M, g, \sigma)^2 \geq \frac{n}{4(n-1)} Q(M, g).$$

Here, conformally parabolic means that in the conformal class  $[g]$  there exists a complete metric of finite volume. This Theorem is obtained by proving first the (non-conformal) Hijazi inequality on a complete Riemannian spin manifold of finite volume that compares Dirac eigenvalues with the infimum of the spectrum of the conformal Laplacian. A similar result is obtained when replacing the assumption on the essential spectrum by conditions on the scalar curvature and the dimension of the manifold.

Furthermore, we generalize another estimate, the Friedrich inequality, to arbitrary open manifolds.

For these results important tools are the Lichnerowicz formula for the Friedrich connection and the refined Kato inequality for the Dirac operator.