

# Gradient Flows for non-convex energies: Asymptotics and stability for two models from fluid mechanics

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## Abstract

This thesis mainly deals with two models from fluid mechanics which can be formulated as Wasserstein gradient flows: the Burgers' and the thin-film equation. We are proving results about the long-time behavior of solutions to those equations using the gradient flow structure. These kind of results are usually obtained as a consequence of the semi-convexity of the corresponding energies, which does not hold true in our case. By a finer analysis of the energy landscape and the trajectories, we are able to circumvent this problem.

The Burgers' equation is the simplest nonlinear scalar conservation law, and the prime example of an equation which develops singularities in finite time. The right notion of solutions is the so called entropy solution, characterized by a one-sided estimate of the distributional derivative which goes by the name of Oleinik's principle. As was already noted before, the Burgers' equation can be formally seen as a gradient flow on a modified Wasserstein space. The observation which is the basis of this part is that although the energy landscape is highly non-convex, using the bound given by Oleinik's principle, we obtain that along an entropy solution we do have an estimate on the Hessian which improves over time. This turns into an algebraic contraction rate between entropy solutions to scalar conservation laws with strictly convex fluxes.

In the following part we study the stability of a stationary solution to the thin-film equation with partial wetting boundary conditions. Even though the energy is not globally convex, we can show that the energy is nevertheless convex in a neighborhood around the stationary solution. Using this we obtain natural relaxation rates of perturbations to the stationary solution by looking at the time evolution of three crucial quantities: distance, energy and dissipation. The convexity implies certain relationships between these quantities, which in turn imply the desired rates.

A third part deals with Dirichlet minimizers of a variational problem, the Aviles-Giga limit functional. The only known method for proving minimality for a configuration  $m$  is by constructing a corresponding entropy (or calibration)  $\Phi^m$ . The main result of this part is that even in the simple case of the viscosity solution on the unit square this cannot be done via the above entropy approach, since there does not exist an entropy in this case.