Algebraic and Topological Properties of Unitary Groups of a II_1-Factors

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We are concerned with several group theoretical questions in the context of unitary groups of functional analytic type. Our main focus lies on unitary groups of II_1 factors. II_1 factors are special von Neumann algebras, which are by reduction theory of von Neumann algebras often and by many concerns the most interesting case of those mathematical objects.

The thesis is structured as follows. After an introduction in Chapter 1 and clarifying the preliminaries in Chapter 2, we present an alternative and elementary proof of the extreme amenability of the unitary group the hyperfinite II_1 factor, endowed with the strong operator topology. This was first proven by Giordano and Pestov, building on results from Gromov and Milman.

In Chapter 4, we are concerned with the bounded normal generation property. For any (noncommutative) group G one can ask under which condition an element $g \in G$ is a product of conjugates of another element $h \in G$. We are able to provide a necessary and sufficient criterion for

(i) the projective unitary group $\mathrm{PU}(n)$ of the $n \times n$ matrix algebra over $\mathbb{C}$;
(ii) the connected component of the identity of the projective unitary group of the Calkin algebra;
(iii) the projective unitary group $\mathrm{PU}(\mathcal{M})$ of a (separable) II_1 factor $\mathcal{M}$.

Our criteria are formulated in terms of so called projective generalized s-numbers that can be defined in any semi-finite von Neumann algebra $\mathcal{M}$.
We further show that each of these groups is generated in finitely many steps from the conjugacy class of any nontrivial element and of its inverse. We call this property the bounded normal generation property.

In Chapter 5 we apply results obtained in Chapter 4. A group property, which recently has drawn a lot of attention of several experts in descriptive set theory, is the so called automatic continuity property. Automatic continuity comes out of a question of Cauchy, asking whether every endomorphism of the additive group of the reals is continuous. We are able to show that any homomorphism from the groups

- $\mathrm{PU}(n)$, $n \in \mathbb{N}$, endowed with the uniform topology,

- $\mathrm{PU}(\mathcal{M})$, $\mathcal{M}$ a separable II_1 factor, endowed with the strong operator topology,

into any separable SIN group is continuous. Our proof uses our results on bounded normal generation. Our techniques allow us to further show that $\mathrm{PU}(n)$ and $\mathrm{PU}(\mathcal{M})$ have a unique Polish group topology - this has previously been unknown even in the hyperfinite case. It is worthwile mentioning that $\mathrm{PU}(n)$, $n \in \mathbb{N}$, embeds discontinuously into S_∞.