The Truncated Moment Problem

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– Summary –

The *(generalized) moment problem* asks: *Given a (real) sequence $s = (s_a)_{a\in\mathsf{A}}$. Does there exists a (positive) Borel measure μ on $\mathcal{X}$ such that*

$$s_a = \int_{\mathcal{X}} a(x)\ \mathrm{d}\mu(x) \quad \textit{for all} \quad a \in \mathsf{A},\ \mathsf{A} \textit{ a set of functions.} \tag{1}$$

We concentrate our effort to two less studied aspects and fundamental questions therein: Carathéodory numbers and set of atoms. Even for the most important cases where $\mathcal{A} = \operatorname{span}\mathsf{A}$ is the set of polynomials in n variables of degree at most d little is known there.

The starting point of our investigations is a fundamental result by Richter [Ric57, Satz 4]: *Every truncated moment sequence $s = (s_i)_{i=1}^m$ has a k-atomic representing measure with $k \leq m$.* Fundamental questions arise:

Chapter 3: *How many atoms are required to represent a fixed/all truncated moment sequence(s)?*

Chapter 4: *What are the possible atom positions appearing in an atomic representing measure?*

The first fundamental result in this work: A general lower bound on the Carathéodory number $\mathcal{C}_\mathsf{A}$. Under the assumption of sufficient differentiablity with n being the number of variables we find

Theorem 3.38. $\left\lceil \frac{m}{n+1} \right\rceil \leq \mathcal{C}_\mathsf{A}$.

In Theorem 3.68 we gave a criterion when $\mathsf{A} = \{1, x^{d_2}, \dots, x^{d_m}\}$ on $\mathbb{R}$ also has Carathéodory number $\mathcal{C}_\mathsf{A} = \left\lceil \frac{m}{2} \right\rceil$ at our lower bound. In Example 3.69 with $\mathsf{A} = \{1, x^2, x^3, x^5, x^6\}$ this condition is fulfilled, i.e., the Carathéodory number is 3. A nice little byproduct is

Corollary 3.70. $p(x) = a + bx^2 + cx^3 + dx^5 + ex^6 \geq 0,\ p \neq 0 \quad \Rightarrow \quad |\mathcal{Z}(p)| \leq 2.$

Investigating the homogeneous polynomials in n homogeneous variables of degree $2d$ we improve the upper bounds further (see Theorems 3.81, 3.86, and 3.91). So for homogenuous polynomials in 3 variables of degree $2d$ we get the bounds $\frac{3}{2}d(d+1)+1$ in Theorem 3.81 and $\frac{3}{2}d(d-1)+2$ in Theorem 3.86. Both are significantly better than $m = \binom{n+2d-1}{n-1} - 1$.

In Chapter 4 we deal with the set of atoms $\mathcal{W}(s)$. Previously the set $\mathcal{V}(s)$ was introduced:

$$\mathcal{V}(s) := \{x \in \mathcal{X} \,|\, p(x) = 0\ \forall p \geq 0 : L_s(p) = 0\} = \bigcap_{p\geq 0 \wedge L_s(p)=0} \mathcal{Z}(p).$$

Our main result give descriptions of $\mathcal{V}(s)$ and $\mathcal{W}(s)$.

Theorem 4.4. $\exists p \in \mathcal{A},\ p \geq 0 : \mathcal{V}(s) = \mathcal{Z}(p)$.

Theorem 4.34. $\mathcal{W}(s) = \mathcal{Z}(p) \cap \mathcal{Z}(p_1) \cap \dots \cap \mathcal{Z}(p_k)$ *for a* $p \geq 0$ *and* p_i *indefinite.*

Theorem 4.7. $\mathcal{W}(s) = \mathcal{V}_+(s) \quad \Leftrightarrow \quad$ *s is in the relative interior of an exposed face.*

References

[Ric57] H. Richter, *Parameterfreie Abschätzung und Realisierung von Erwartungswerten*, Bl. Dtsch. Ges. Versmath. **3** (1957), 147–161.