THE EXISTENCE OF CALIBRATIONS FOR THE MUMFORD-SHAH FUNCTIONAL AND THE REINITIALIZATION OF THE DISTANCE FUNCTION IN THE FRAMEWORK OF THE CHAN AND VESE ALGORITHMS

MARCELLO CARIONI

The first part of the thesis is devoted to the study of one of the most important variational model for image segmentation and edge detection, called the Mumford Shah functional and introduced in the late 80's by Mumford and Shah. It is an edge detection model that can compute the best image segmentation. In particular we focus on the calibration technique for the Mumford-Shah functional, introduced by Alberti, Bouchitté and Dal Maso that was introduced to prove that specific SBV functions are local minimizers of the functional and we address the problem of the existence of such a calibration for a given minimizer. For this purpose, we extend the Mumford-Shah functional to the space of linear combination of graphs in such a way that the extension is convex. The key point is to prove that given a local minimizer for the Mumford-Shah functional, then its graph is a local minimizer for the extended functional; we accomplish that, in dimension one, by means of a generalized coarea formula for graphs. Thanks to it we are able to prove the equivalence of the minimum problems and then to answer the original question, proving the existence of calibrations in a weak sense as a consequence of the Hahn-Banach theorem, adapting Federer's approach to calibrations for minimal surfaces.

In the second part of the thesis we focus on the Chan and Vese algorithms that are designed in order to compute the approximation of a variant of the Mumford-Shah functional called *piecewise constant* Mumford-Shah functional. The drawback of the algorithms is that the gradient of the flow ϕ becomes more and more degenerate (and numerically unstable) when $|\nabla\phi|$ approaches zero. This is the reason why it is customary in the Chan and Vese algorithms to reinitialize the function ϕ to be the signed distance function from the interface $\{\phi = 0\}$ at every step of the algorithm.

The most classical way to obtain such a result is to compute asymptotically the solution an non-coercive evolutive Hamilton-Jacobi equation. This equation was introduced in the context of the level set methods for flows and it is employed to reconstruct the signed distance function from $\{\phi = 0\}$ asymptotically. In particular we prove, in the framework of viscosity solutions of Hamilton-Jacobi equation, that its solution converges uniformly to the signed distance function $\{\phi = 0\}$ as $t \to +\infty$.

In order deduce the preservation of the zero level set of the solution in the limit without relying on uniform estimate of the gradient $\nabla\phi$ (due to the lack of coercivity of the Hamiltonian) we construct barriers for the viscosity solution that close to Γ do not depend on t in such a way that even without a control on the gradient of the solution we can still infer the uniform convergence as $t \to +\infty$.