bifurcation region can be better observed in case $\mu = 1.15$ (Fig. 7), because here branches which meet at an acute angle are connected. In the HNR case [1] with $\mu = 1.0$ we get obtuse-angled branches.

With GE_1 (valley floor path) and GE_2 (ridge crest path) the representation in Fig. 9 gives a pattern as in Example 2. Only the GE_3 seems very strange, showing a strong deviation from the straight path of Example 2 in the central region.

The mathematical question of how to calculate the μ value where a bifurcation exists is quite a difficult one. Eq. (7) cannot directly be used because we do not know the corresponding gradient and contour line tangent at this point. So we cannot calculate the direction derivatives in condition equation (7). For a more detailed discussion see [9].

Now, let us discuss the HNR case $\mu = 1$ in some detail. In valley v_1 with $U(x, y) < -1.5$ we find normal behaviour; namely the existence of a unique GE_1 tracing the valley floor and coming from minus infinity. GE_1 then turns up to the right near the point $(0.4, -0.05)$ toward SP_2. On contour lines lower than -1.5 in v_1, at the crossing with GE_1, we have by definition an extremal of σ, namely a minimal value. Indeed, if we leave the GE_1 crossing along a contour line to the right or to the left we get an increase of σ. At first sight, we can check this by a continuous shortening of the distance to the next contour line. But on the contour line (-1.5) walking through $(0, 0)$ the σ profile suffers a change: a shoulder emerges. We have $U_x(0, 0) = 0$, $U_y = \frac{3}{2}$, and the Hessian elements

$$U_{xx} = 1, \qquad U_{yy} = 0, \qquad U_{xy} = U_{yx} = 0. \tag{10}$$

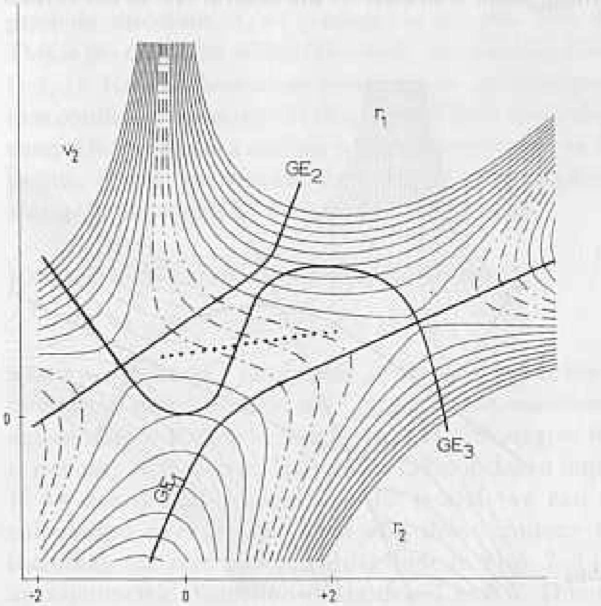

Fig. 9. Model surface $U(x, y) = \frac{1}{2}(xy(y-x) + x^2 + 2y - 3)$, cf [1], including GE's and a piece of the contact line (*dotted*)

The gradient actually pointing in the y-direction is parallel to an eigenvector (which in addition is a zero eigenvector). Hence, the point $(0, 0)$ is a GE point. It is a double point in the Basilevsky classification [8]. If we go uphill to the left from $(0, 0)$ we can now trace a new valley floor of GE_3 up to SP_1 which must be characterized by a second minimum of σ. The σ shoulder of the σ profile over $(0, 0)$ on the (-1.5)-contour line splits on higher contour lines into a minimum and a maximum. In Fig. 10 the σ profile along the contour line (-1) is given. Clearly, between the two minima there exists a maximum, but in contrast to Example 3 and to the case $\mu = 1.08$, the origin of the two new σ extremals is outside the old GE_1. The maximal σ value belongs to the central arc of GE_3 between $(0, 0)$ and $(2, 2)$. It begins at $(0, 0)$ in tracing the end cirque of v_1. Going uphill it does not find a simple CCI point but a broad intermediate region which is characterized by contact of a flank of the central ridge r_1 and of a flank of v_1 leading to SP_2. We can define a contact line of the two flanks by connecting those points on contour lines having curvature zero. If we walk along a contour line (cl) we go from convex behaviour in valley v_1 to concave behaviour on ridge r_1. Thus, the equipotential line somewhere shows a change of curvature from plus to minus: this is the point on the contact line. We again treat $U(x, y) =$ constant as an implicit definition of the contour line $y = \mathrm{cl}(x)$ and get the condition

$$\mathrm{cl}''(x) = (2U_xU_yU_{xy} - U_y^2U_{xx} - U_x^2U_{yy}) / U_y^3 = 0, \tag{11}$$

(if $U_y \neq 0$). We set $\tilde{\boldsymbol{t}} = (U_y, -U_x)$, which is a non normed tangential vector orthogonal to the gradient of U. Eq. (11) gives the condition

$$\tilde{\boldsymbol{t}} \cdot \boldsymbol{H} \cdot \tilde{\boldsymbol{t}} = 0 \tag{12}$$

for a point (x, y) to be an inflection point on a contour line where two flanks stick together. In general, the GE_3 passing the CCI belt and the contact line dividing the flanks of ridge r_1 and valley v_1 do not coincide. They do, however, cross at the point $(0.685, 0.920)$, cf [1], where the Hessian $\boldsymbol{H}$ has a zero eigenvector parallel to $\tilde{\boldsymbol{t}}$. At this point for Eq. (3) we get

$$(\boldsymbol{t} \cdot \boldsymbol{H}) \cdot \boldsymbol{g} = \boldsymbol{0} \cdot \boldsymbol{g} = 0.$$

Thus, it is on the GE_3. On the rest of the contact line we still have a zero eigenvalue of $\boldsymbol{H}$, but the corresponding eigenvector is not parallel to $\boldsymbol{t}$, (a behaviour

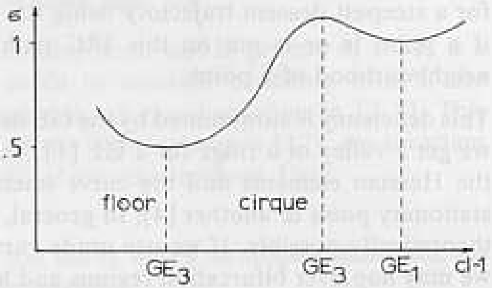

Fig. 10. σ profile over the contour line (-1) in Fig. 9