Closed Geodesics and the Free Loop Space

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Table of Contents

Goals, Method and Motivation

- Goals
  - Results about the *Existence* and *Stability* of closed (periodic) geodesics on closed manifolds (compact, without boundary) carrying a Riemannian resp. Finsler metric.
  - Dependence of the results on the *Reversibility* of the metric
- Method:
  - Variational methods resp. *Morse-Theory on the Free Loop Space*
- Motivation:
  - With the help of periodic geodesics resp. periodic orbits of a mechanical system one can investigate the qualitative behaviour of dynamical systems for large times.
    These investigations start with Henri Poincaré,

Concepts from Riemannian Geometry, Part I

Let M be a differentiable (i.e. C^∞) manifold with tangent bundle $TM = \bigcup_{p \in M} T_pM$, here T_pM is the tangent space at $p \in M$.

We assume that the manifold is carrying a *Riemannian metric* $g = \langle .,. \rangle$, i.e. a family $p \in M \mapsto g_p$ of inner products

$$g_p : T_pM \times T_pM \longrightarrow \mathbb{R}\,,$$

depending smoothly on $p \in M$.

We denote by $\mathcal{V}M$ the space of vector fields X on the manifold, i.e. sections $p \in M \mapsto X(p) \in T_pM$,
The *Levi-Civita connection* ∇ is the unique metric and torsionfree connection

$$X, Y \in \mathcal{V}M \mapsto \nabla_X Y \in \mathcal{V}M$$

Concepts form Riemannian Geometry, Part II

The *Levi-Civita connection* (also called *canonical connection*) is the unique connection which is *torsionfree*, i.e.

$$\nabla_X Y - \nabla_Y X - [X, Y] = 0$$

and *metric,* i.e.

$$Z\langle X, Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle$$

The connection is uniquely determined by the *Koszul formula:*

$$\begin{aligned} 2\langle \nabla_X Y, Z\rangle &= X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle \\ &- \langle X, [Y, Z]\rangle + \langle Y, [Z, X]\rangle - \langle Z, [X, Y]\rangle \end{aligned}$$

Concepts from Riemannian Geometry, Part III

In coordinates $x = (x_1, x_2, \dots, x_n)$ with coordinates fields $\partial_j = \frac{\partial}{\partial x_j}$ the connection can be expressed using *Christoffel symbols* Γ_{ij}^k :

$$\nabla_{\partial_i} \partial_j = \Gamma_{ij}^k \partial_k$$

Here we use Einstein's sum convention.
With the *metric coefficients*

$$g_{ij}(x) = g_x \left(\partial_i(x), \partial_j(x)\right)$$

and its derivatives

$$g_{ij,l}(x) = \frac{\partial g_{ij}(x)}{\partial x_l}$$

Koszul's formula is expressed as:

$$\Gamma_{ij}^k(x) = \frac{1}{2} g^{kl}(x) \left\{ g_{jl,i}(x) + g_{li,j}(x) - g_{ij,l}(x) \right\} .$$

We introduce the *covariant derivative of a vector field* $t \mapsto V(t) \in T_{c(t)}M$ along a curve $t \mapsto c(t) \in M$: Let $\overline{V}$ be an extension of V in a neighborhood of a point $p = c(t_1) \in M$. Then

$$\frac{\nabla}{dt} V(t_1) := \nabla_{c'(t_1)} \overline{V} \,.$$

Using coordinates $x = (x_1, \ldots, x_n)$ with
$V = V^j \partial_j \,;\, c(t) = (c_1(t), \ldots, c_n(t))$:

$$\frac{\nabla V}{dt}(t) = \left\{ \frac{dV^k}{dt} + \Gamma^k_{ij}\,(c(t))\, \frac{dc_i}{dt} V^j(t) \right\} \partial_k$$

Parallel Transport

Geometrically the covariant derivative along a curve $c : [a, b] \to M$ defines *parallel transport*

$$P_{a,b}c : T_{c(a)}M \longrightarrow T_{c(b)}M$$

along c.

For a given $X_a \in T_{c(a)}M$ there is an unique *parallel vector field* X along c with $X(a) = X_a$, i.e. $\frac{\nabla X}{dt} = 0$. Then

$$P_{a,b}c\,(X_a) = X(b)\,.$$

Since the Levi Civita connection is metric parallel transport is an *isometry.*

Curvature

On a curved space parallel transport in general depends on the curve.
Flat (Euclidean) space is characterized by the property that locally parallel transport is path-independent.
Curvature can be measured using the *Riemann curvature tensor*:

$$R(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \, .$$

The *symmetries of the curvature tensor* are the following relations:

$$\begin{aligned} R(X,Y)Z + R(Y,X)Z &= 0 \\ R(X,Y)Z + R(Y,Z)X + R(Z,X)Y &= 0 \\ \langle R(X,Y)Z, W\rangle + \langle R(X,Y)W, Z\rangle &= 0 \\ \langle R(X,Y)Z, W\rangle &= \langle R(Z,W)X, Y\rangle \end{aligned}$$

Curvature, Part II

The *sectional curvature* $K(\sigma) = K(X, Y)$ of a two-dimensional plane $\sigma = \mathrm{span}\{X, Y\}$ is defined by

$$K(\sigma) = K(X, Y) = \frac{\langle R(X, Y)Y, X\rangle}{\|X\|^2\|Y\|^2 - \langle X, Y\rangle^2}$$

If the sectional curvature is *constant* $K(\sigma) = k \in \mathbb{R}$ then the curvature tensor has the following form:

$$R(X, Y)Z = k\,\{\langle Y, Z\rangle X - \langle X, Z\rangle Y\}$$

and the manifold is *locally isometric* to one of the model spaces $S^n_k; \mathbb{R}^n, H^n_k$.

Surfaces

Let

$$(x, y) \in \mathbb{R}^2 \mapsto f(x, y) \in M$$

be a *singular parametrized surface*, i.e. a smooth mapping and V a vector field along f, i.e. $V(x, y) \in T_{f(x,y)}M$. Then the covariant derivatives $\frac{\nabla V}{\partial x}$; $\frac{\nabla V}{\partial y}$ are defined as covariant derivatives along the coordinate lines $x \mapsto (x, y_0)$ for a fixed y_0 resp $y \mapsto ((x_0, y)$ for a fixed x_0.

Then

$$\frac{\nabla}{\partial x}\frac{\partial f}{\partial y} = \frac{\nabla}{\partial y}\frac{\partial f}{\partial x}$$

and

$$R\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) V = \frac{\nabla}{\partial x}\frac{\nabla}{\partial y} V - \frac{\nabla}{\partial y}\frac{\nabla}{\partial x} V \, .$$

Geodesics

A smooth curve $c : I \longrightarrow M$ is called *geodesic* if its velocity field c' is parallel, i.e.

$$\frac{\nabla c'}{dt} = 0 \,.$$

In local coordinates $c(t) = \left(x^1(t); \ldots, x^n(t)\right)$ the *geodesic equation* is given as:

$$(x^k)''(t) + \Gamma^k_{ij}(x(t))(x^i)'(t)(x^j)'(t) = 0 \,, k = 1, \ldots n$$

This is a system of ODEs (ordinary differential equations) of second order. It has an unique solution for the initial value problem. Hence for a given tangent vector $X \in T_pM$ there is an unique geodesic $c_X : I_X \longrightarrow M$ with $c(0) = p\,;\, c'(0) = X$ defined on the maximal interval $I_X \subset \mathbb{R}$ of definition.

Exponential mapping

Since the solutions of this system depend smoothly on the initial values the *exponential map*

$$\exp : \mathcal{U} \longrightarrow M\,;\, \exp(X) = c_X(1)$$

is well-defined and smooth in an neighborhood $\mathcal{U}$ of the zero section of the tangent bundle $\tau : TM \longrightarrow M$.

For sufficiently small $\mathcal{U}$ the mapping

$$\tau \times \exp : \mathcal{U} \longrightarrow M \times M; X \mapsto (\tau(X), \exp(X))$$

resp. the restriction

$$\exp_p : T_pM \cap \mathcal{U} \longrightarrow M$$

has maximal rank. Geodesics are parametrized proportional to arc length:

$$\frac{d}{dt}\|c'(t)\|^2 = 2\,\langle \frac{\nabla c'}{dt}, c'\rangle = 0$$

Geodesics on hypersurfaces

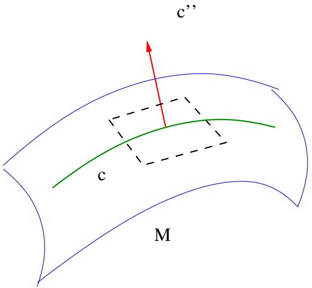
Let $M^n \subset \mathbb{R}^{n+1} =$ be a hypersurface in Euclidean space with the induced metric.

- $c(t) \in M$ curve on M (trajectory of a particle)
- $c'(t) \in T_{c(t)}M$ velocity vector field
- $c''(t) \in T_{c(t)}\mathbb{R}^{n+1}$ acceleration vector field

Then projection of $c''(t)$ onto the tangent space $T_{c(t)}M \subset T_{c(t)}\mathbb{R}^{n+1}$ gives the acceleration vector field on the hypersurface:

$$\frac{\nabla c'}{dt}(t) = \left(c''(t)\right)^{\tan}$$

- Geometric interpretation:
  A geodesic line on a (hyper)surface $M^n \subset \mathbb{R}^{n+1}$ in Euclidean space is a curve $c : I \to M$, whose acceleration c'' is orthogonal to the (hyper)surface, i.e. the acceleration on the (hyper)surface vanishes.
- Physics interpretation:
  A geodesic line describes *the trajectory of a particle* on a (hyper)surface which moves without external forces.



Geodesics on the standard sphere, Part I

Consider the *standard sphere*

$$S^n := \left\{ x \in \mathbb{R}^{n+1} ;\ \|x\|^2 = 1 \right\}$$

If $X \in T_p^1 S^n$ and $p \in S^n$, i.e. $p, X \in \mathbb{R}^{n+1}, \|X\|^2 = \|p\|^2 = 1, \langle X, p \rangle = 0$ and if $E \subset \mathbb{R}^{n+1}$ is the plane spanned by p and X then

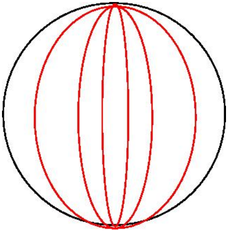
$$c_X(t) = (\sin t) \cdot X + (\cos t) \cdot p$$

is a parametrization by arc length of the great circle with $c(0) = p, c'(0) = X$ in $S^n \cap E$. Since

$$c_X''(t) = -c_X(t)$$

the great circle is a geodesic.

On the sphere S^n of dimension n with the standard metric the geodesics are great circles.



This is the example of a space, *all of whose geodesics are periodic.*

For a *generic Riemannian metric* on a compact manifold there are below a given length L only finitely many geometrically distinct closed geodesics.

This follows from the *bumpy metrics theorem* (ABRAHAM 1970, ANOSOV 1983)

First variation formula, Part I

Fix a partition $0 = t_0 < t_1 < \cdots < t_{k-1} < t_k = 1$ of the unit interval $[0,1]$ and define

$$\mathcal{C} = \{ c : [0,1] \to M \,;\, c \text{ continuous, piecewise smooth } \}$$

i.e. $c\,|[t_j, t_{j+1}]$ is smooth.

A *piecwise smooth variation* of $c \in \mathcal{C}$ is a continuous and piecewise smooth function

$$F : (-\epsilon, \epsilon) \times [0,1] \longrightarrow M; (s,t) \mapsto c_s(t) = F(s,t)$$

with $c_s\,|[t_j, t_{j+1}]$ is smooth and $c = c_0$. Then the *variation vector field* V is given by

$$V(t) = \left.\frac{\partial c_s(t)}{\partial s}\right|_{s=0}$$

First variation formula, Part II

Proposition (First variation formula)

Given a variation $t \in [0,1] \mapsto c_s(t) \in M$ *of a smooth curve* $c = c_0$ *we obtain for the* energy $E(c_s) = \frac{1}{2}\int_0^1 \|c_s'(t)\|^2\, dt$:

$$\left.\frac{dE(c_s)}{ds}\right|_{s=0} = \langle V(t), c'(t)\rangle\Big|_0^1 - \sum_{i=1}^{k} \langle V(t_i), c'(t_i+) - c'(t_i-)\rangle - \int_0^1 \left\langle V(t), \frac{\nabla c'}{dt}\right\rangle dt$$

First variation formula: Proof

$c_s \left|[t_i, t_{i+1}]\right.$ is smooth:

$$\frac{1}{2}\frac{\partial}{\partial s}\left\langle c_s', c_s'\right\rangle = \left\langle \frac{\nabla}{\partial s}\frac{\partial c_s}{\partial t}, \frac{\partial c_s}{\partial t}\right\rangle =$$

$$\left\langle \frac{\nabla}{\partial t}\frac{\partial c_s}{\partial s}, \frac{\partial c_s}{\partial t}\right\rangle = \frac{\partial}{\partial t}\left\langle \frac{\partial c_s}{\partial s}, \frac{\partial c_s}{\partial t}\right\rangle - \left\langle \frac{\partial c_s}{\partial s}, \frac{\nabla}{\partial t} c_s'\right\rangle$$

hence

$$\frac{1}{2}\left.\frac{d}{ds}\right|_{s=0} E(c_s) = \int_{t_i}^{t_{i+1}} \left.\frac{\partial}{\partial s}\right|_{s=0} \left\langle c_s', c_s'\right\rangle\, dt =$$

$$\left\langle V(t), c'(t)\right\rangle\Big|_{t_i}^{t_{i+1}} - \int_{t_i}^{t_{i+1}} \left\langle V, \frac{\nabla c'}{dt}\right\rangle\, dt$$

Summung up over $i = 1, \ldots, k$ yields the result

Geometric interpretation

- Geometric meaning of the second term: A variation in direction of the jump $c'(t_i+) - c'(t_i-)$ reduces the energy (and length).

- Geometric meaning of the third term: variation in direction of the acceleration field $\frac{\nabla c'}{dt}$ reduces the energy (and length).

Consequences of the first variational formula

- *Shortest curves* are geodesics resp. geodesics are *locally energy (resp. length) minimizing.*
- Let c be a piecewise smooth, closed curve. If for every variation $c_s, s \in (-\epsilon, \epsilon)$ with piecewise smooth, closed curves

$$\left.\frac{dE(c_s)}{ds}\right|_{s=0} = 0$$

holds, then $c = c_0$ is a smooth closed geodesic.

Variational setting, Part I

On the space $C^{\infty}\left(S^1, M\right)$ the energy functional

$$E : C^{\infty}\left(S^1, M\right) \longrightarrow \mathbb{R}\,;\; E(c) = \frac{1}{2}\int_0^1 \langle c', c'\rangle dt$$

is defined. We use a completion of this space:

The *free loop space* resp. *Hilbert manifold of closed curves*:

$$\Lambda M := \left\{ c : S^1 \longrightarrow M\,;\; c \text{ absolutely continuous },\ \int_0^1 \langle c', c'\rangle < \infty \right\}$$

Variational setting, Part II

A map $c : [0,1] \to M$ is called *absolutely continuous* if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$0 \leq t_0 < t_1 < \cdots < t_{2k+1} \leq 1 \; ; \; \sum_{i=0}^{k} |t_{2i+1} - t_{2i}| < \delta$$

implies

$$\sum_{i=0}^{k} d\left(c\left(t_{2i+1}\right), c\left(t_{2i}\right)\right) < \epsilon \, .$$

For a smooth closed curve $c : S^1 \to M$ and a sufficiently small tubular neighborhood U of the zero section in the tangent bundle $c^*(TM)$ over the closed curve the space $H^1(c^*(U))$ of *short* H^1-vector fields $t \in S^1 \mapsto X(t)$ with $X(t) \in U$ can be used as domain for manifold charts:

The *model space* is the separable Hilbert space $H^1(c^*(TM))$ of H^1-vector field along c. Then we can define a *chart*

$$\begin{aligned} \psi_c = \exp_c : H^1(c^*(U)) \longrightarrow U(c) = \exp_c \left(H^1(c^*(U)\right) \subset \Lambda M \\ \psi_c(\xi)(t) = \left(\exp_c \xi\right)(t) = \exp_{c(t)} \xi(t) \end{aligned}$$

The free loop space as a Riemannian manifold

The Riemannian metric g on M induces a Riemannian metric g_1 on the free loop space:

$$\langle X, Y\rangle_1 = \int_0^1 \langle X(t), Y(t)\rangle dt + \int_0^1 \left\langle \frac{\nabla}{dt}X(t), \frac{\nabla}{dt}Y(t)\right\rangle dt$$

Theorem

The free loop space $(\Lambda M, g_1)$ of a compact Riemannian manifold (M, g) is a complete separable Riemannian manifold. The energy functional

$$E : \Lambda M \longrightarrow \mathbb{R}, E(c) = \frac{1}{2}\int_0^1 \langle c', c'\rangle dt$$

is differentiable with derivative

$$dE(c).V = \int_0^1 \left\langle \frac{\nabla c'}{dt}, V\right\rangle dt$$

Closed geodesics as critical points

Hence the critical points of the energy functional are the closed geodesics and the point curves.

The *gradient vector field* $\mathrm{grad} E$ on the free loop space is defined by

$$\langle \mathrm{grad} E(c), V \rangle_1 = dE(c).V$$

for all vector fields $V \in T_c\Lambda$.

The free loop space is not locally compact but we have:

Theorem (Palais-Smale condition)

If $(c_m)_{m\geq 1} \subset \Lambda M$ is a sequence for which $E(c_m)$ is bounded and $\lim_{m\to\infty} \|\mathrm{grad} E(c_m)\|_1 = 0$ then (c_m) has a subsequence converging to a critical point of the energy functional E.

The *flow*

$$\Phi_s : \Lambda M \to \Lambda M \,;\; \left.\frac{d\Phi_s(c)}{ds}\right|_{s=t} = -\mathrm{grad} E\left(\Phi_t(c)\right)$$

of the *negative gradient field* $-\mathrm{grad} E$ is defined for all $s \geq 0$. For any $c \in \Lambda M$ the limit

$$\lim_{s\to\infty} \Phi_s(c) \in \mathrm{Cr}$$

exists and lies in the *critical set* $\mathrm{Cr} := \{c \in \Lambda M \,;\, dE(c) = 0\}$ consisting of point curves and closed geodesics.

Finsler metric, Part I

A Finsler metric $F : T_pM \to \mathbb{R}$ defines a *norm* in any tangent space.

Definition

A *Finsler metric* on a differentiable manifold M with tangent bundle TM is a continuous map $F : TM \to \mathbb{R}$ which is smooth outside the zero section and satisfies the following:

(a) $F(y) > 0$ for all $y \neq 0$.

(b) $F(ay) = aF(y)\,, a > 0$.

(c) *Legendre condition:* For all $V \neq 0$ the bilinear symmetric form

$$g^V(X, Y) := \left.\frac{\partial^2}{\partial s \partial t}\right|_{s=t=0} F^2(V + sX + tY)$$

is *positive definite.*

Finsler metrics, Part II

A Finsler metric can be characterized in any tangent space by its unit vectors, which form a *strictly convex hypersurface* in the tangent space. For a Riemannian metric this hypersurface is an *ellipsoid.*
Then the *length* of curve $c : [0,1] \to \mathbb{R}$ is defined:

$$L(c) = \int_0^1 F\left(c'(t)\right)\, dt\,.$$

We call a Finsler metric *reversible* if $F(-y) = F(y)$ for all tangent vectors y. In general we call the number

$$\lambda := \max\left\{\frac{F(-y)}{F(y)}\,;\, y \neq 0\right\} \geq 1$$

the *reversibility* of the Finsler metric.

Finsler metrics, Part III

In *physics terminology:* The Riemannian metric describes a kinetic energy on a manifold, which is quadratic in velocity.

The orbits of the corresponding Lagrangian system are the geodesics.

A Finsler metric $F = F(x, \dot{x}))$ describes a more general class of kinetic energy with Lagrangian $L(x, \dot{x}) = F^2(x, \dot{x})/2$.
In contrast to the quadratic case (Riemannian metric) the coefficients

$$g_{ij} = g_{ij}(x, \dot{x}) = \frac{\partial^2}{\partial \dot{x}_i \partial \dot{x}_j} L^2(x, \dot{x})$$

do not depend only on x but also the velocity $\dot{x}$.

Finsler metrics, Part IV

For a fixed geodesic $c:(0,1)\to M$ with velocity vector field $c'(t)$ one can choose an extension V without zeros of the velocity vector field in a tubular neighborhood $U\subset M$ of c. Then

$$g^V(X,Y) := \left.\frac{\partial^2}{\partial s\partial t}\right|_{s=t=0} F^2(V+sX+tY)$$

defines a *Riemannian metric,* i.e.

$$g^V_{ij}(x_1,\dots,x_n) = g_{ij}(x_1,\dots,x_n,V_1(x),\dots,V_n(x)) \,.$$

Then c is also a geodesic of the Riemannian manifold $\left(U,g^V\right)$, sometimes called the *osculating Riemannian metric.*

A *flag* (V,σ) consists of a non-zero vector V and a two-dimensional plane σ containing V. Then the *flag curvature* $K(V;\sigma)$ of the flag $(V;\sigma)$ equals the sectional curvature $K(\sigma)$ of the plane σ with respect to the Riemannian metric g^V.

A modification of a Riemannian metric g leads to a particular Finsler metric, it is called *Randers metric*: For a vector field V with $g(V,V)<1$ the Randers metric is defined by

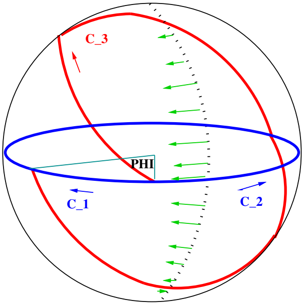
$$F(X)=\sqrt{g\left(X,X\right)}+g\left(X,V\right).$$

Then the Randers metric measures the *travelling time* on the Riemannian manifold (M,g) under the influence of a weak wind.
Then

$$\lambda:=\frac{1+\max\|V\|}{1-\max\|V\|}\in(1,\infty)\,.$$

Katok metrics

The *Katok metric* is a Randers metric measuring the time, which a particle on a sphere needs under the additional influence of *wind*. For a generic choice the angle ϕ is an irrational multiple of π then the *geodesic* c_3 does not close. Hence there are exactly *two closed geodesics* c_1, c_2 which differ only by orientation and length.



Variational setting for Finsler metrics, Part I

The *energy* $E(c)$ of a curve $c : [0,1] \to M$ on a manifold equipped with a Finsler metric F is given by

$$E(c) = \frac{1}{2} \int_0^1 F^2 \left(c'(t)\right) \, dt$$

and geodesics are locally length-minimizing and energy-minimzing.

The *induced distance*

$$d : M \times M \longrightarrow \mathbb{R} \, ; \; d(p,q) = \inf \left\{ L(c) \, ; \; c(0) = p, c(1) = q \right\}$$

is in general not symmetric if the metric is non-reversible.

Variational setting for Finsler metrics, Part II

The critical points of the energy functional

$$E : \Lambda M \longrightarrow \mathbb{R}\,;\; E(c) = \frac{1}{2}\int_0^1 F^2\left(c'(t)\right)\,dt$$

are the closed geodesics and the point curves.

Critical value of a homology class, Part I

We denote by

$$\Lambda^a := \{\sigma \in \Lambda M, E(\sigma) \leq a\}$$

the sublevel set. For a *non-trivial homology class*

$$0 \neq h \in H_k\left(\Lambda M, \Lambda^0 M\right)$$

the *critical value* is defined

$$\mathrm{cr}(h) := \inf\left\{a > 0\,;\, h \in Im\left(H_k\left(\Lambda^{\leq a} M, \Lambda^0 M\right) \to H_k\left(\Lambda M, \Lambda^0 M\right)\right)\right\} > 0$$

Given a singular chain $u \in C_k\left(\Lambda M, \Lambda^0 M\right)$ representing h and $\epsilon > 0$ for sufficiently large $s > 0$ we obtain $E\left(\Phi_s(u)\right) < a + \epsilon$.

Critical value of a homology class, Part II

Using again the gradient flow $\Phi_s, s \geq 0$ one can show:

Proposition

Let $h \in H_k\left(\Lambda M, \Lambda^0 M\right)$ be a non-trivial homology class. Then there exists a non-trivial closed geodesic c with $E(c) = \mathrm{cr}(h)$.

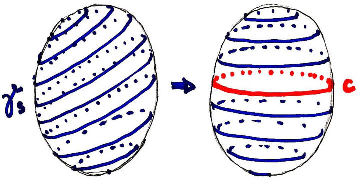
Intuitively:
The homology class h *remains hanging* at the closed geodesic c.

Existence of one closed geodesic

Theorem (Birkhoff 1927; Lusternik-Fet 1951)

On a simply-connected and compact manifold M with a Riemannian or Finsler metric there exists a non-trivial closed geodesic.

RUBBER BAND PROOF: A family of rubber bands γ_s covering a sphere of dimension n tightens until it remains hanging at a closed geodesic c.



Proof

There is a smallest number $k \in \{2, 3, \ldots, n\}$ such that there exists a homotopically nontrivial map

$$\left(S^{k+1}, p_0\right) \to (M, p) \ .$$

The manifold is called *k-connected*. Fibring the sphere S^{k+1} by circles we obtain a homotopically non-trivial map

$$\left(D^k, S^{k-1}\right) \longrightarrow (\Lambda M, \Lambda^0 M)$$

defining a non-trivial homology class $0 \neq h \in H_k\left(\Lambda M, \Lambda^0 M\right)$.

Then there exists a closed geodesic c with $E(c) = \mathrm{cr}(h) > 0$. $\square$

S^1-action on the free loop space, Part I

The group

$$S^1 = \mathbb{R}/\mathbb{Z} = \{\exp(2\pi it\,;\; t \in [0,1]\}$$

can be identified with the *special orthogonal group* $S\mathbb{O}(2)$ acting on S^1 resp. $\mathbb{R}^2$ by orientation preserving rotations. Then the *orthogonal group* $\mathbb{O}(2)$ is generated by $S^1 = S\mathbb{O}(2)$ and the element

$$\theta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The element θ acts orientation-reversing and generates a $\mathbb{Z}_2$-action on S^1 and we have the following relations for $z \in S^1 \subset \mathbb{C}$ and θ :

$$\theta(z) = \overline{z} \;\; ; \;\; z \cdot \theta = \theta \cdot z^{-1}$$

Hence we have an induced (canonical) $\mathbb{O}(2)$-action on the free loop space:

$$\mathbb{O}(2) \times \Lambda M \longrightarrow \Lambda M, (z,c) \mapsto z.c$$

with $z.c(t) = c(z.t)$

S^1-action on the free loop space, Part II

In particular

$$\theta : \Lambda M \longrightarrow \Lambda M\,;\ \theta(c)(t) = c(1-t)\,.$$

For an element $w \in \mathbb{O}(2)$ the mapping

$$w : \Lambda M \longrightarrow \Lambda M\,;\ c \mapsto w.c$$

is differentiable and an isometry of $(\Lambda M, \langle .,. \rangle_1)$.

If $ev : \Lambda M \longrightarrow M\,;\ ev(c) = c(0)$ is the differentiable evaluation map and $c \in \Lambda M$ a non-differentiable curve, then the composition

$$z \in S^1 \mapsto e(z.c) = z.c(0) = c(z)$$

is non-differentiable. This shows that the continuous S^1-action on the free loop space is non-differentiable.

S^1-action on the free loop space, Part III

We collect the facts from the last slides in the following

Theorem

The free loop space $(\Lambda M, g_1)$ of a compact Riemannian manifold (M, g) carries the structure of a complete $\mathbb{O}(2)$-Riemannian manifold (also called Hilbert manifold).
The energy functional $E : \Lambda M \longrightarrow \mathbb{R}$ is differentiable and $\mathbb{O}(2)$-invariant with derivative

$$dE(c).V = \int_0^1 \left\langle \frac{\nabla}{dt} c'(t), V(t) \right\rangle dt$$

Multiplicity of closed geodesics

For a closed curve $c \in \Lambda M$ the *isotropy subgroup*

$$I(c) = \{z \in \mathbb{O}(2)\,;\, z.c = c\}$$

is a closed subgroup of $\mathbb{O}(2)$. The fixed point set of the $\mathbb{O}(2)$-action is the set $\Lambda^0 M = \{c \in \Lambda M\,;\, E(c) = 0\}$ of point curves which we can identify with the manifold M.

For a closed geodesic c (which is not a point curve) the isotropy subgroup is a closed subgroup of S^1, if

$$I(c) = \mathbb{Z}_m = \mathbb{Z}/\,(m\mathbb{Z})$$

then $m = \mathrm{mul}(c)$ is called the *multiplicity* of the closed geodesic c. The closed geodesic is called *prime* if $\mathrm{mul}(c) = 1$.

Geometrically distinct closed geodesics

Definition

- Two closed geodesics $c_1, c_2 : S^1 \longrightarrow M$ of a Riemannian manifold resp. areversible Finsler metric are called *geometrically equivalent* if $c_1\left(S^1\right) = c_2\left(S^1\right)$.
- Two closed geodesics $c_1, c_2 : S^1 \longrightarrow M$ of a *non-reversible* Finsler metric are called *geometrically equivalent* if $c_1\left(S^1\right) = c_2\left(S^1\right)$ and if their orientation coincides.

Otherwise they are called *geometrically distinct.*

For a prime closed geodesic c and a Riemannian resp. reversible Finsler metric the set of geometrically equivalent closed geodesics is given as:

$$\bigcup_{m \geq 1} \mathbb{O}(2).c^m = \{z.c^m \,;\, z \in \mathbb{O}(2), m \geq 1\}$$

also called *tower of geometrically equivalent closed geodesics.*

Second variation formula

For a geodesic $c : [0,1] \to M$ and two vector fields X, Y along c we call

$$I_c(X,Y) = \int_0^1 \left\{ \left\langle \frac{\nabla X}{dt}, \frac{\nabla Y}{dt} \right\rangle dt - \langle R(X,Y)Y, X \rangle \right\} dt$$

the *index form* of the closed geodesic.

Then one can show that for a closed geodesic the *Hessian* $d^2E(c)$ of the energy functional $E : \Lambda M \to \mathbb{R}$ equals the index form, i.e. for all $X, Y \in T_c\Lambda M :$

$$d^2E(c)(X,Y) = I_c(X,Y)$$

Index and Nullity of a closed geodesic

- The *index* of a closed geodesic is the maximal dimension of a subspace of $T_c\Lambda M$ on which the Hessian $d^2E(c) = I_c$ is negative definite,

$$\mathrm{ind}(c) = \mathrm{ind}\left(d^2E(c)\right)$$

- The *nullity* of a closed geodesic is the dimension of the kernel of the index form minus 1.

$$\mathrm{null}(c) = \dim\ker d^2E(c) - 1 \geq 0$$

The nullity is the dimension of the space of *periodic Jacobi fields.*

The metric is called *bumpy* if all closed geodesics are non-degenerate (i.e. $\mathrm{null}(c) = 0$.)

The index form

with the inner product

$$\langle X, Y \rangle_0 = \int_0^1 \langle X(t), Y(t) \rangle \; dt$$

on ΛM we can write the index form and the induced self-adjoint operator $A_c : T_c\Lambda \to T_c\Lambda :$

$$\langle A_c X, Y \rangle_1 = d^2 E(c)\,(X, Y) = \langle X, Y \rangle_1 - \langle (Id + R_c)\, X, Y \rangle_0 \; .$$

Here $R_c(X) = R\,(X, c')\, c'$ is the *curvature (Jacobi) operator:* Therefore the eigenvectors of A_c to the eigenvalue λ are the periodic solutions of the differential equation

$$(\lambda - 1)\left(\nabla^2 - 1\right) X - (R_c + 1)\, X = 0$$

Indices of great circles on the standard sphere

Then the *index* $\mathrm{ind}(c)$ of c equals the sum of the dimensions of the eigenspaces of the self-adjoint operator A_c with negative eigenvalues.
A prime closed geodesic on the standard sphere $S^n = \left\{x \in \mathbb{R}^n; \|x\|^2 = 1\right\}$ is a *great circle,* for example
$c(t) = (\cos(2\pi t), \sin(2\pi t), 0, \ldots, 0)\,, t \in [0, 2\pi]$.

Then one can show with the above characterization and the fact that $R_c(X) = X$:

$$\mathrm{ind}\,(c^m) = (2m-1)(n-1)\,;\,\mathrm{null}(c^m) = 2n-2.$$

In this case the index $\mathrm{ind}\,(c^m)$ coincides with the index $\mathrm{ind}_\Omega\,(c^m)$ which equals the *Morse index theorem* with the *number of conjugate points* to $c(0)$ along $c\,|[0, m)$.

Morse-Bott function, Part I

The energy functional E is a *Morse-Bott function* if the set of closed geodesics $\mathrm{Cr} \subset \Lambda = \bigcup_i B_i$ decomposes into a disjoint union of *non-degenerate submanifolds* B_i.
A manifold B without boundary of critical points is called a *non-degenerate submanifold* if the following properties are satisfied:

- The index $\mathrm{ind}(c), c \in B_i$ is constant.
- The nullity $\mathrm{null}(c), c \in B_i$ is constant and $\mathrm{null}(B) = \dim B - 1$.

The -1 occurs since with a closed geodesic c the S^1-orbit $S^1.c$ also belongs to the critical submanifold.

Morse Theory of the Energy functional

Using a *Morse-Lemma* for the energy functional we obtain the following local result:

Proposition

If the set of closed geodesics of energy a *forms a non-degenerate critical submanifold* B *with* $\dim B = \mathrm{null}(B) + 1 \geq 1$ *and if* $k = \mathrm{ind}(B)$ *then for sufficiently small* $\epsilon > 0$*:*

$$H_{r+k}\left(\Lambda^{a+\epsilon}, \Lambda^{a-\epsilon}; R\right) \cong H_r\left(B; R\right)$$

with $R = \mathbb{Z}_2 = \mathbb{Z}/(2\mathbb{Z})$ resp. $R = \mathbb{Q}$ or $R = \mathbb{Z}$ if the *negative normal bundle* of the critical submanifold is orientable.

Morse-Bott functions, Examples

- *Bumpy metric:* All closed geodesics are non-degenerate, i.e. $\mathrm{null}(c) = 0$. Then the set of closed geodesics is a union of one-dimensional non-degenerate submanifolds $B_i^m = S^1.c_i^m$.
- For the *standard metric* on S^n the set of closed geodesics equals

$$\mathrm{Cr} = \bigcup_{m \geq 1} B^m$$

Here $B = T^1 S^n = V(2, n-1)$ is the set of *great circles* and $c^m(t) = c(tm)$ is the m-th cover of c.

Bumpy metric, local homology

Let c be a *prime closed geodesic* of a bumpy metric with $E(c) = a$ and $m \geq 1$: If there is no further closed geodesic of length a then

$$H_r\left(\Lambda^{m^2a^2/2+\epsilon}, \Lambda^{m^2a^2/2-\epsilon}, \mathbb{Z}_2\right) \cong \begin{cases} \mathbb{Z}_2 & ; \quad r = k, k+1 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

resp.

$$H_r\left(\Lambda^{m^2a^2/2+\epsilon}, \Lambda^{m^2a^2/2-\epsilon}, \mathbb{Q}\right) \cong \begin{cases} \mathbb{Q} & ; \quad r = k, k+1 \text{ and } m \text{ odd or} \\ & \quad \mathrm{ind}(c^2) - \mathrm{ind}(c) \text{ even} \\ 0 & ; \quad \text{otherwise} \end{cases}$$

standard metric on the sphere, local homology

Let c be a *prime great circle* on the standard sphere. Then $E(c^m) = 2m^2\pi^2$, and

$$\mathrm{ind}(c) = n-1, \mathrm{ind}(c^2) = 3(n-1).$$

Hence the negative normal bundle of the critical submanifold $B^m, m \geq 1$ is *orientable* and for sufficiently small ϵ :

$$H_r\left(\Lambda^{2m^2\pi^2+\epsilon}S^n, \Lambda^{2m^2\pi^2/2-\epsilon}S^n; \mathbb{Z}\right) \cong H_{r-(2m-1)(n-1)}\left(T^1S^n; \mathbb{Z}\right)$$

Using the homology of the manifold $B \cong T^1S^n$ one can compute the homology of the free loop space of a sphere:

One obtains for the Betti numbers for odd n

$$b_r\left(\Lambda S^n, \Lambda^0 S^n; \mathbb{Q}\right) = \begin{cases} 1 & ; \quad r = (2m-1)(n-1), m \geq 1 \\ 1 & ; \quad r = (2m+1)(n-1)+1, m \geq 1 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

For $n \geq 4$ even:

$$b_r\left(\Lambda S^n, \Lambda^0 S^n; \mathbb{Q}\right) = \begin{cases} 1 & ; \quad r = m(n-1), m \geq 1 \\ 1 & ; \quad r = m(n-1)+1, m \geq 1 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Linearized Poincaré mapping

Let c be a closed geodesic and

$$T^{\perp}_{c(0)}M = \left\{ v \in T_{c(0)}M ; \langle v, c'(0) \rangle = 0 \right\} .$$

For a closed geodesic c we define the *linearized Poincaré mapping*

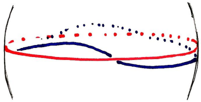
$$P_c : T^{\perp}_{c(0)}M \oplus T^{\perp}_{c(0)}M \longrightarrow T^{\perp}_{c(0)}M \oplus T^{\perp}_{c(0)}M ;$$
$$P_c \left(Y(0), \frac{\nabla}{dt} Y(0) \right) = \left(Y(1), \frac{\nabla}{dt} Y(1) \right)$$

for a *Jacobi field* $Y(t)$ along the geodesic. Jacobi fields are the variation vector fields of *geodesic variations.* Hence P_c describes the behaviour of nearby geodesics.

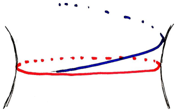
Stability of closed geodesics

The linearized Poincare mapping is a *symplectic linear mapping* $P_c \in \mathrm{Sp}(n-1)$.

- Stable closed geodesics: *Nearby geodesics* stay in a neighborhood of the *closed geodesic c*: Then any eigenvalue of P_c satisfies $|\lambda| = 1$. (the closed geodesic is called *elliptic*)



- *Unstable* closed geodesics: *Nearby geodesics* have the tendency to diverge. If any eigenvalue of P_c satisfies $|\lambda| \neq 1$ then the *closed geodesic c* is called *hyperbolic.*



The sequence of indices of iterates

It is possible to describe the sequence $\mathrm{ind}(c^m)$ of iterates using the linearized Poincaré mapping P_c: (Hedlund 1939, Bott 1956, Klingenberg 1973, Ballmann-Thorbergsson-Ziller 1982, Long 2002,...)

- *average (mean) index* $\alpha_c = \lim_{m\to\infty} \frac{\mathrm{ind}(c^m)}{m}$ exists.
- If c is *hyperbolic* then $\mathrm{ind}(c^m) = m \cdot \mathrm{ind}(c) = m\alpha_c$.
- If $n = 2$ and if the metric is *bumpy*:

$$\alpha_c \in \begin{cases} \mathbb{Z}^{\geq 0} & ; \quad c \text{ hyperbolic} \\ \mathbb{R}^+ - \mathbb{Q} & ; \quad c \text{ elliptic} \end{cases}$$

Gromoll-Meyer Theorem

Using the estimate for $\mathrm{ind}(c^m)$ and a Morse theory for *isolated degenerate critical points* one obtains:

Theorem (Gromoll-Meyer 1969)

Let the sequence $\left(b_k\left(\Lambda M\right)\right)_{k\geq 1}$ of Betti numbers of the free loop space on a compact manifold M be unbounded.
Then for any Riemannian or Finsler metric there are infinitely many closed geodesics.

Rational homotopy theory shows that the assumption of the Theorem is satisfied, if the rational cohomology ring $H^*\left(M;\mathbb{Q}\right)$ has at least two generators (Sullivan, Vigue-Poirrier 1976).

The assumption is not satisfied for spheres and complex resp. quaternionic projective spaces. For example for all $n \geq 2$:

$$b_k\left(\Lambda S^n\right) \leq 2,$$

Eigen frequency of a closed geodesic

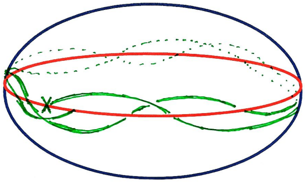
The *eigen frequency* of a closed geodesic $c : \mathbb{R} \longrightarrow M$ with $c(t+1) = c(t)$ and length $L(c)$ is given by

$$\overline{\alpha}_c = \lim_{m\to\infty} \frac{\#\{\text{ conjugate points } c(t)\,;\, t \le m\}}{mL(c)}$$

Then

$$\overline{\alpha}_c = \alpha_c / L(c)$$

i.e. the average index is the *mean value of conjugate points per period.*



Metrics with only finitely many closed geodesics, I

Theorem (Hingston 1984)

Let M be a simply-connected and compact manifold carrying a Riemannian or Finsler metric all of whose closed geodesics are hyperbolic. Then there are infinitely many closed geodesics.

Hence a Finsler metric with only finitely many closed geodesics carries a non-hyperbolic closed geodesic.
Using a generalization of the Euler characteristic

$$B(M) := \lim_{m\to\infty} \left\{ \frac{1}{m} \sum_{j=0}^{m} (-1)^j b_j \left(\Lambda M/S^1, \Lambda^0 M; \mathbb{Q} \right) \right\}$$

for the quotient $\Lambda M/S^1$ of the free loop space of a manifold for which the Betti numbers of $\Lambda M/S^1$ are bounded (in particular spheres) we obtain:

Metrics with only finitely many closed geodesics, II

Theorem (R. 1989)

For a bumpy Finsler metric on a compact and simply-connected manifold with only finitely many closed geodesics $c_1, c_2, \ldots, c_r$, with average indices $\alpha_1, \alpha_2 \ldots, \alpha_r$ and invariants $\gamma_1, \gamma_2, \ldots, \gamma_r \in \{\pm 1, \pm 1/2\}$: we obtain

$$\frac{\gamma_1}{\alpha_1} + \cdots + \frac{\gamma_r}{\alpha_r} = B(M) \neq 0$$

(e.g. $B(S^n) = 1/2 + 1/(2n-2)$ for even dimension n)

Here $\gamma_1 = \pm 1$, iff $\mathrm{ind}\left(c_1^2\right) - \mathrm{ind}\left(c_1\right)$ is even, and $\gamma_1 > 0$ iff $\mathrm{ind}\left(c_1\right)$ is even. There is also a generalization to the case of isolated degenerate closed geodesics.

Xiao-Long 2014: Computation of this invariant for the component of non-contractible closed curves on $\mathbb{R}P^{2m+1}$.

Infinitely many closed geodesics

Consequence

If there are only finitely many closed geodesics on a compact and simply-connected manifold then their average indices are *algebraically dependent.*

Perturbing the metric one can destroy the algebraic dependence, i.e. one obtains:

Theorem (R.1989/92)

For a C^2-generic Riemannian metric on a compact and simply-connected manifold there are infinitely many closed geodesics.

Two closed geodesics on a bumpy 2–sphere

As a consequence of the formula for the average indices we obtain and Hingston's result we obtain directly:

Corollary (R. 1989)

A bumpy Finsler metric on S^2 has at least two closed geodesics. If $N < \infty$ then there are two elliptic closed geodesics.

For surfaces with a *Riemannian metric* there is an even stronger result, which combines methods from dynamical systems and Morse theory resp. variational methods:

Theorem (Birkhoff 1925, Franks 1982/Hingston 1983, Bangert 1983)

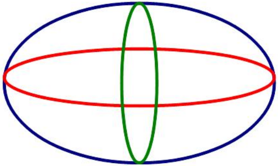
For any Riemannian metric on the sphere of dimension 2 there are infinitely many closed geodesics.

Either the geodesic flow can be described by an area-preserving annulus map (which is the case for a convex metric) or there exists a closed geodesic which is a local minimum for the length.

Ellipsoid

The *Ellipsoid* is defined by the Equation:

$$\frac{x_1^2}{a_1^2}+\frac{x_2^2}{a_2^2}+\frac{x_3^2}{a_3^2}=1 \quad ; \quad 1 \le a_1 < a_2 < a_3$$



There are exactly *three simple* closed geodesics c_1, c_2, c_3, the intersections with the coordinate planes.

The geodesic flow is *integrable.* There are *infinitely* many closed geodesics.

The length L_4 of the *fourth* closed geodesic c_4 (i.e. c_4 is a shortest closed geodesic which is longer than c_3.) goes to infinity, when the ellipsoid gets *rounder,* i.e.:

$$\lim_{a_3 \to 1} L_4 = \infty.$$

Results for the n-dimensional sphere, Part I

N = Number of geometrically distinct closed geodesics

n-dimensional Ellipsoid:

$\frac{n(n+1)}{2}$ *simple, closed geodesics*

$N = \infty$

n-dimensional Katok metrics:

n resp. $n+1$ *simple, closed geodesics*

$$N = \begin{cases} n & ; \quad n \text{ even} \\ n+1 & ; \quad n \text{ odd} \end{cases}$$

Results for 2-dimensional spheres

Riemannian metrics, $n = 2$

- Lusternik-Schnirelmann 1929, Ballmann 1978, Jost 1989, Grayson 1989: There are *three simple* closed geodesics.
- Grjuntal 1979: There is a convex metric all of whose *simple* closed geodesics are *hyperbolic.*
- R. 1992: There are Riemannian metrics all of whose *homologically visible* closed geodesics are *hyperbolic.*
- Birkhoff 1920, Bangert 1993, Franks 1992/Hingston 1993 : For any Riemannian metric $N = \infty$
- Contreras & Oliveira 2010: There is an open and dense set of Riemannian metrics carrying an *elliptic* closed geodesic.

Existence of closed geodesics on n-dimensional spheres, $n \geq 2$.

Riemannian metrics

- BALLMANN-THORBERGSSON-ZILLER 1982:
  If the sectional curvature K satisfies $1/4 < K \leq 1$ then there exist $g(n) \in [3n/2, 2n]$ *short* closed geodesics and there exists one *short non-hyperbolic* closed geodesic (*short:* $L(c) \leq 4\pi$.) If all closed geodesics of length $< 4\pi$ are *non-degenerate* then there are $n(n+1)/2$ short closed geodesics.
  (similar results by ALBER, ANOSOV, HINGSTON, KLINGENBERG,...)
- HINGSTON 1983:
  Let $N(l)$ be the number of closed geodesics with length $< l$. Then for a C^4-generic metric: $\liminf_{l\to\infty} N(l)\frac{\log(l)}{l} > 0$.

Finsler metrics

- BANGERT-LONG 2005:
  For *any* Finsler metric there are *two* closed geodesics, i.e. $N \geq 2$..

- LONG–WANG 2008:
  If $N < \infty$ then there are *two irrationally elliptic* closed geodesics.

- HARRIS & PATERNAIN 2008, HOFER-WYSOCKI-ZEHNDER 2003:
  If $\lambda^2/(\lambda+1)^2 < K \leq 1$ then $N \in \{2, \infty\}$.

Finsler metrics

- DUAN-LONG 2007, R. 2010:
  For a bumpy metric: $N \geq 2$.

- If the flag curvature satisfies: $\lambda^2/(\lambda+1)^2 < K \leq 1$ we have:

  R. 2007: There are $n/2 - 1$ closed geodesics with length $< 2n\pi$.

  WANG 2012: For any *bumpy* metric there are n (resp. $(n+1)$) closed geodesics for even n (resp. odd n.)

The n-dimensional sphere: Open problems

Riemannian metrics

$n > 2$:

Is there a metric with $N < \infty$?

Finsler metrics

- $n = 2$:

  Does for any metric $N \in \{2, \infty\}$ hold?

- $n \geq 3$:

  Does for any metric $N > 2$ hold?

We introduce *equivariant cohomology* $H^*_{S^1}\left(\Lambda, \Lambda^0; \mathbb{Q}\right)$ with respect to the S^1-action on the free loop space ΛM :

Let $ES^1 \longrightarrow BS^1 = ES^1/S^1$ be an *universal S^1-bundle.* This is a principal S^1-bundle with a contractible total space ES^1. The base space BS^1 is called a *classifying space*.

Then the *homotopy quotient* is defined as

$$\Lambda_{S^1} = \Lambda \times_{S^1} ES^1$$

and the equivariant cohomology as:

$$H^*_{S^1}\left(\Lambda, \Lambda^0; \mathbb{Q}\right) := H^*\left(\Lambda_{S^1}; \mathbb{Q}\right)$$

Via a *classifying map* $f : \Lambda_{S^1} \longrightarrow BS^1$ of the S^1-bundle $\Lambda \times ES^1 \longrightarrow \Lambda_{S^1}$ the relative cohomology $H^*_{S^1}\left(\Lambda, \Lambda^0; \mathbb{Q}\right)$ can be seen as a $H^*\left(BS^1\right)$-module.

Let $\eta \in H$ be a generator,i.e. $H^*\left(BS^1\right) \cong \mathbb{Q}[\eta]$.
Using rational homotopy theory one can show that there is a cohomology class $z \in H^{n+1}\left(\Lambda S^n, \Lambda^0; \mathbb{Q}\right)$ which is not a torsion element, i.e.

$$\eta^k \cdot z \neq 0$$

for all $k \geq 1$.

Faddell-Rabinowitz index and Resonance, III

For $a > 0$ let $j_a : \left(\Lambda^a, \Lambda^0\right) \longrightarrow \left(\Lambda, \Lambda^0\right)$ be the inclusion. Then we define a function $d_z : \mathbb{R}^+ \longrightarrow \mathbb{N}_0$:

$$d_z(a) := \min \left\{ k \in \mathbb{N} ; \eta^k \cdot j_a^*(z) = 0 \right\}$$

Definition

We define the *global index interval* $[\underline{\sigma}_z, \overline{\sigma}_z]$ by:

$$\underline{\sigma}_z = \liminf_{a \to \infty} \frac{d_z(a)}{a} \, ; \, \overline{\sigma}_z = \limsup_{a \to \infty} \frac{d_z(a)}{a}$$

Fadell Rabinowitz index and resonance IV

Theorem (R. 1984)

Let F be any Finsler metric on S^n and let $[\underline{\sigma}_z, \overline{\sigma}_z]$ be the global index interval *of the class z.*

a) If $t \in [\underline{\sigma}_z, \overline{\sigma}_z]$ then there is a sequence c_i of prime closed geodesics with

$$2t = \lim_{i \to \infty} \overline{\alpha}_i$$

b) For every $\epsilon > 0$ we have the following estimate:

$$\sum_c \frac{1}{\alpha_c} \geq \frac{1}{2},$$

where we sum over all prime geometrically distinct closed geodesics c whose mean average index $\overline{\alpha}_c$ satisfies: $\overline{\alpha}_c \in (2\underline{\sigma}_z - \epsilon, 2\overline{\sigma}_z + \epsilon)$.

Faddell-Rabinowitz index and Resonance, V

Corollary

If there are only finitely many geometrically distinct closed geodesics for a metric on S^n*, then* $\underline{\sigma}_z = \overline{\sigma}_z = \sigma$ *and*

$$\sum_{c:\overline{\alpha}_c = 2\sigma} \frac{1}{\alpha_c} =\geq 1/2$$

where we sum over all prime geometrically distinct closed geodesics with $\overline{\alpha}_c = 2\sigma$*.*

String topology and Closed Geodesics

String Theory:	Particles are made of vibrating bits of strings (very tiny)
Configuration spaces of String theory:	Spaces of paths or loops
String Topology:	Algebraic and topological description of intersection theory on the free loop space (M.Chas and D.Sullivan 1999)

The Chas-Sullivan Loop Product, Part I

Let

$$\mathcal{F} = \{(\alpha, \beta) \in \Lambda \times \Lambda \, ; \, \alpha(0) = \beta(0)\}$$

be the *figure 8-space* of the compact manifold M. The evaluation map

$$\mathrm{ev} : \Lambda \longrightarrow M \, ; \, ev(c) = c(0)$$

is a locally trivial fiber bundle. The fiber is the *based loop space*

$$\Omega(M, p) := \{\alpha : ([0,1], \{0,1\}) \to (M, p)\}$$

The Chas-Sullivan product, Part II

The map

$$\mathrm{ev} : \mathcal{F} \longrightarrow M\,;\, (\alpha, \beta) \mapsto \alpha(0) = \beta(0)$$

is also a fiber bundle which is the pullback of the map

$$\mathrm{ev} \times \mathrm{ev} : \Lambda \times \Lambda \longrightarrow M \times M$$

via the *diagonal embedding*

$$\Delta : M \longrightarrow M \times M\,;\, x \mapsto (x, x)\,.$$

Hence we obtain the following *commutative diagram*

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{e} & \Lambda \times \Lambda \\ {\scriptstyle \mathrm{ev}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{ev}\times\mathrm{ev}} \\ M & \xrightarrow{\Delta} & M \times M \end{array}$$

The embedding

$$e : \mathcal{F} \longrightarrow \Lambda \times \Lambda\,;\,(\alpha,\beta) \mapsto (\alpha,\beta)$$

can be seen as an *embedding of codimension* n with *normal bundle* $(\mathrm{ev}^* \times \mathrm{ev}^*)\,(\nu_\Delta)$
There is a *tubular neighborhood* η_e of the embedding $e : \mathcal{F} \longrightarrow \Lambda \times \Lambda$ which is the inverse image of a tubular neighborhood η_Δ of the diagonal embedding $\Delta : M \longrightarrow M \times M$ hence

$$\eta_e = (\mathrm{ev} \times \mathrm{ev})^{-1}\,(\eta_\Delta)$$

Hence we can identify the *Thom space* $D(\eta_e)/S(\eta_e)$ of the tubular neighborhood η_e with the quotient space $(\Lambda \times \Lambda) \,/\, (\Lambda \times \Lambda - \eta_e)$ which defines a homomorphism

$$\begin{aligned} \tau_e \quad : \quad & H_k\left(\Lambda \times \Lambda\right) \longrightarrow \\ & H_k\left(\Lambda \times \Lambda, \Lambda \times \Lambda - \eta_e\right) \cong H_k\left(D(\eta_e), S(\eta_e)\right) \cong H_{k-n}\left(\mathcal{F}\right) \end{aligned}$$

The last isomorphism is the *Thom-isomorphism* of the vector bundle η_e, $S(\eta_e)$ resp. $D(\eta_e)$ is the *sphere* resp. *disc bundle* of the tubular neighborhood η_e.

The Chas-Sullivan product, Part V

For $\alpha, \beta \in \Lambda$ we denote by

$$\alpha \star \beta(t) = \begin{cases} \alpha(2t) & ; \quad 0 \le t \le 1/2 \\ \beta(2t+1) & ; \quad 1/2 \le t \le 1 \end{cases}$$

the *concatenation of loops.* This defines a mapping

$$\gamma : \mathcal{F} \longrightarrow \Lambda, \; ; \gamma\left((\alpha, \beta)\right) = \alpha \star \beta$$

Then we obtain the *Chas Sullivan product* as the following composition

$$H_k(\Lambda) \otimes H_l(\Lambda) \longrightarrow H_{k+l}(\Lambda \times \Lambda) \overset{\tau_e}{\longrightarrow} H_{k+l-n}(\mathcal{F}) \longrightarrow H_{k+l-n}(\Lambda)$$

(co)homology products in string topology

String-topology defines products in the (co)homology of the free loop space
(Chas-Sullivan 1999, Goresky-Hingston 2009):

$$\bullet \quad : H_j(\Lambda M) \otimes H_k(\Lambda M) \to H_{j+k-n}(\Lambda M)$$

$$\circledast \quad : H^j(\Lambda M, \Lambda^0 M) \otimes H^k(\Lambda M, \Lambda^0 M) \to H^{j+k+n-1}(\Lambda M, \Lambda^0 M)$$

These products generalize the *intersection product* in the homology of compact manifolds.

Let $\mathrm{cr}(X)$ be critical value of the homology class $X \in H_k(\Lambda M)$, i.e. the smallest number a, such that the homology class can be represented in the subset $\Lambda^{\leq a} = \{\gamma \in \Lambda M; E(\gamma) \leq 1/2a^2\}$.

Then the loop products $\bullet$ and $\circledast$ satisfy the following basic inequalities: (Goresky, Hingston 2009):

$$\mathrm{cr}(X \bullet Y) \leq \mathrm{cr}(X) + \mathrm{cr}(Y) \text{ for all } X, Y \in H_*(\Lambda) \tag{1}$$

$$\mathrm{cr}(x \circledast y) \geq \mathrm{cr}(x) + \mathrm{cr}(y) \text{ for all } x, y \in H^*(\Lambda, \Lambda^0). \tag{2}$$

There is a *non-nilpotent* element

$$\theta \in H_{3n-2}\left(\Lambda S^n; \mathbb{Z}\right)$$

and a non-nilpotent element

$$\omega \in H^{n-1}\left(\Lambda S^n, \Lambda^0 S^n\right).$$

(I.e. $\Theta^{\bullet m} \neq 0, \omega^{\circledast m} \neq 0$ for all $m \geq 1$.)

Using these non-nilpotent elements one can define the *global mean frequency* $\sigma = \sigma\left(M, g\right)$ of a Riemannian resp. Finsler metric on S^n :

$$\overline{\alpha}^{-1} := \frac{1}{2n-2} \lim_{k\to\infty} \frac{\mathrm{cr}(\Theta^{\bullet k})}{k} = \frac{1}{2n-2} \lim_{k\to\infty} \frac{\mathrm{cr}(\omega^{\circledast k})}{k}$$

Theorem (Resonance theorem, HINGSTON-R. 2013)

A Riemannian or Finsler metric on $S^n, n > 2$ determines a global mean frequency $\overline{\alpha} > 0$ with the property that

$$\deg(X) - \overline{\alpha}\,\mathrm{cr}(X)$$

is bounded as X ranges over all nontrivial homology or cohomology classes on Λ. Therefore the countably infinite set of points $(\mathrm{cr}(X), \deg(X))$ in the (l, d)-plane lies in bounded distance from the line $d = \overline{\alpha} l$.

Theorem (Density Theorem)

Let $\overline{\alpha} = \overline{\alpha}_g$ be the global mean frequency of a Riemannian or Finsler metric g on S^n, $n > 2$. For any $\varepsilon > 0$ we have the following estimate for the sum of inverted average indices α_c of geodesics on (S^n, g) :

$$\sum_c \frac{1}{\alpha_c} \geq \begin{cases} \frac{1}{n-1} & ; \quad n \text{ odd} \\ \frac{1}{2(n-1)} & ; \quad n \text{ even} \end{cases}$$

where we sum over a maximal set of prime, geometrically distinct closed geodesics γ whose mean frequency $\overline{\alpha}_c =: \alpha_c/\ell(c)$ satisfies: $\overline{\alpha}_c \in (\overline{\alpha} - \varepsilon, \overline{\alpha} + \varepsilon)$

String-topology and closed geodesics, Part V

As an application one obtains:

Theorem (Hingston-R. 2013:)

For a Riemannian (resp. Finsler) metric of positive sectional (resp. flag) curvature $1/4 < K \leq 1$ (resp. $\lambda^2/(1+\lambda^2) < K \leq 1$) on an odd-dimensional sphere S^n with global mean frequency $\overline{\alpha}$ we obtain:

- *If there are only finitely many closed geodesics then there are two resonant closed geodesics c_1, c_2 with eigenfrequency $\overline{\alpha}_1 = \overline{\alpha}_2 = \overline{\alpha}$.*
- *If there are no resonant closed geodesics then there is a sequence $c_1, c_2, \ldots$ of closed geodesics with*

$$\overline{\alpha} = \lim_{k\to\infty} \frac{\alpha_k}{L(c_k)}$$