Effective resistance metric and random walks on networks

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Finite weighted graph

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\[ C(x, y) = 0 \iff \text{there is no 'edge' between } x \text{ and } y. \]  

We only consider simple graphs, i.e. \( C(x, x) = 0 \) for all \( x \in V \).
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Interpretations

1. Network of streets: $C(x, y) \equiv$ width of street between $x$ and $y$
2. Electrical network: $1/C(x, y) \equiv$ resistance between $x$ and $y$
Random walk

Let \( X = (X_k)_{k \in \mathbb{N}} \) be the Markov chain with state space \( V \) with Markov kernel

\[
P[X_{k+1} = y \mid X_k = x] = \frac{C(x, y)}{C_x} =: P(x, y)
\]

where \( C_x := \sum_v C(x, v) \).

We only consider connected graphs. Hence, \( X \) is irreducible.
Laplacian

\[ \Delta := P - I \]

is the Laplacian associated with \( P \).

\[(\Delta f)(x) = \sum_y \frac{C(x, y)}{C_x} f(y) - f(x)\]
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Energy form

The energy form of \((V, C)\) is

\[ \mathcal{E}(f) := \frac{1}{2} \sum_{x, y} C(x, y)(f(x) - f(y))^2 \]
Effective resistance

Let \( x, y \in V \) and \( \phi^{xy} : V \to \mathbb{R} \) be the unique solution of the Dirichlet problem

\[
-\Delta \phi^{xy} = \frac{1}{C_x} \mathbb{1}_x - \frac{1}{C_y} \mathbb{1}_y
\]

\( \phi^{xy}(y) = 0 \)

Then the *effective resistance* between \( x \) and \( y \) is defined to be

\[
R(x, y) := \phi^{xy}(x)
\]

\( \phi^{xy} \) is the electric potential which induces the flow of 1 ampere from \( x \) to \( y \).
Equivalent definitions

- probabilistic: Let \( \tau_y = \inf \{ k \geq 0 : X_k = y \} \). Then

\[
R(x, y) = \frac{1}{C_x} \mathbb{E}_x \left[ \sum_{k=0}^{\tau_y-1} 1_x(X_k) \right]
\]
Equivalent definitions

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- variational:

$$R(x, y) = \sup \left\{ \frac{|u(x) - u(y)|^2}{\mathcal{E}(u)} : u : V \to \mathbb{R}, \mathcal{E}(u) > 0 \right\}$$
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**Theorem**

The effective resistance $R : V \times V \to \mathbb{R}$ is a metric on $V$. 
Potential as metric data

\[ \phi^{xy}(z) = \frac{1}{2}(R(x, y) + R(y, z) - R(x, z)) \geq 0 \forall x, y, z \in V \]
The finite case

The infinite case

Future work

Definitions

Results

Potential as metric data

$$\phi^{xy}(z) = \frac{1}{2} (R(x, y) + R(y, z) - R(x, z)) \geq 0 \quad \forall \ x, y, z \in V$$

Reconstructing the graph from $R$

Let $y \in V$. Define $A_y \in \mathbb{R}^{V \times V}$ and $b_y \in \mathbb{R}^V$ by

$$A_y(x, z) := \begin{cases} 
\delta_{xy} \delta_{yz}, & x = y \text{ or } z = y \\
\phi^{xy}(z), & \text{otherwise}
\end{cases}$$

$$b_y(x) := 1 - \delta_{xy}. $$

Then the vector $C(\cdot, y) \in \mathbb{R}^V$ satisfies

$$A_y \cdot C(\cdot, y) = b_y.$$
Proposition

Let \((V, C_1), (V, C_2)\) be simple weighted graphs with associated effective resistances \(R_1\) and \(R_2\), resp. Then

\[ R_1 = R_2 \Rightarrow C_1 = C_2. \]
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**Theorem**

Let \((V, d)\) be a finite metric space. Then, \(d\) is the effective resistance of some finite weighted graph if and only if for all \(y \in V\)

1. \(\det A_y > 0\)
2. \(A_y \cdot \mathcal{X} = b_y\) admits a non-negative solution, i.e. \(\mathcal{X}(x) \geq 0 \ \forall \ x \in V\).
Weighted graph (locally finite)

$(V, C)$ as before, $V$ countable, $C$ satisfying

$$C_x = \sum_y C(x, y) < \infty$$

for all $x \in V$. 
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Random walk, Laplacian, Energy form

All as before, domain of $\Delta$ and $\mathcal{E}$

$$\mathcal{D}(V, C) := \{ u : V \to \mathbb{R} : \mathcal{E}(u) < \infty \}$$

space of functions of finite Dirichlet energy.
Problem with infinity

The Dirichlet problem

\[-\Delta u = \frac{1}{C_x} \mathbf{1}_x - \frac{1}{C_y} \mathbf{1}_y\]

\[u(y) = 0\]

has in general more than one solution \(\Leftrightarrow\) how to define \(R(x, y)\)?
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The Dirichlet problem

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has in general more than one solution $\rightsquigarrow$ how to define $R(x, y)$?

Def. of effective resistance

Let $x, y \in V$ and define

\[R(x, y) := \sup \left\{ \frac{|u(x) - u(y)|^2}{\mathcal{E}(u)} : u \in \mathcal{D}(V, C), \mathcal{E}(u) > 0 \right\} .\]
Consistency

Let $V_n \subseteq V_{n+1} \subset V$, $|V_n| < \infty$ be an exhaustion of $V$, i.e. $\bigcup V_n = V$, such that the induced subgraph $G_n = (V_n, C|_{V_n \times V_n})$ of $(V, C)$ is connected. Then the effective resistance $R_n$ of $G_n$ satisfies

$$R(x, y) = \lim_{n \to \infty} R_n(x, y).$$

In particular, $R$ is a metric on $V$. 
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Proposition

For any finite $W \subseteq V$, the restriction $R \upharpoonright W \times W$ is again an effective resistance, i.e. there exists a finite weighted graph on $W$ with effective resistance $R \upharpoonright W \times W$. 
**Proposition**

Let \((V, d)\) be an infinite metric space s.t. for every finite \(W \subset V\), \(d \upharpoonright W \times W\) is an effective resistance. Furthermore, let \((V_n)\) be an exhaustion of \(V\) and \(G_n = (V_n, C_n)\) be the weighted graph with effective resistance \(d \upharpoonright V_n \times V_n\). Then,

\[
C(x, y) := \lim_{n \to \infty} C_n(x, y)
\]

exists and is independent of the choice of \((V_n)\).
Proposition

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exhaustion of \(V\) and \(G_n = (V_n, C_n)\) be the weighted graph with effective resistance \(d \upharpoonright V_n \times V_n\). Then,

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exists and is independent of the choice of \((V_n)\).

Problem: In general, \((V, C)\) can be disconnected.
Future work

- Under which assumptions to \((V, d)\) is \((V, C)\) a well-defined graph with effective resistance \(d\)?
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- Is this graph unique?
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- Is this graph unique?
- Finite case: \(R_n \to R\) point-wise \(\iff\) \(E_n \to E\) in the \(\Gamma\)-sense. Possible extension to infinite case?
Thank you for your attention!