Left-covariant Differential Calculi on $SL_q(N)$

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Abstract

We study $N^2 - 1$ dimensional left-covariant differential calculi on the quantum group $SL_q(N)$. In this way we obtain four classes of differential calculi which are algebraically much simpler as the bicovariant calculi. The algebra generated by the left-invariant vector fields has only quadratic-linear relations and posesses a Poincaré-Birkhoff-Witt basis.
We use the concept of universal (higher order) differential calculus associated with a given left-covariant first order differential calculus. It turns out that the space of left-invariant k-forms has the dimension $\binom{N^2-1}{k}$ as in the case of the corresponding classical Lie group $SL(N)$.

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1 Introduction

After the seminal work [6] of S. L. Woronowicz, bicovariant differential calculi on quantum groups (Hopf algebras) have been extensively studied in the literature. There is a well developed general theory of such calculi. Bicovariant differential calculi on the quantum group $SL_q(N), N \geq 3$, have been recently classified in [3]. All calculi occuring in this classification have dimension N^2, i. e. their dimension does not coincide with the dimension $N^2 - 1$ of the corresponding classical Lie group. On the other hand, the first example of a non-commutative differential calculus on a quantum group was Woronowicz' $3D$-calculus on $SU_q(2)$ [5]. This is a three dimensional left-covariant calculus which is not bicovariant. The $3D$-calculus is algebraically much simpler and in many respects nearer to the classical differential calculus on $SU_q(2)$ than the four dimensional bicovariant calculi on $SU_q(2)$. This motivates to look for N^2-1 dimensional left-covariant differential calculi on $SL_q(N)$. The main aim of our approach is to follow the classical situation as close as possible. Let us briefly explain the basic idea of the approach given in this paper. As in [3], we assume that the differentials $\mathrm{d}u^i_j$ of the matrix entries u^i_j generate the left module of 1-forms. Hence the differential d can be expressed as $\mathrm{d}a = \sum (X_{ij} * a)\omega(u^i_j)$ for $a \in SL_q(N)$, where $\omega(u^i_j) := \sum_n \kappa(u^i_n)\mathrm{d}u^n_j$ are the left-invariant Maurer-Cartan forms and X_{ij} are linear functionals on $SL_q(N)$ such that $X_{ij}(1) = 0$. In our approach the functionals X_{ij} will be chosen from the algebra of regular functionals $\mathcal{U}_R$ as defined in [2], 2. 2. We assume that the vector space of left-invariant 1-forms has dimension $N^2 - 1$ and that $X_{ij}(u^r_s) = \delta_{ir}\delta_{js}$ for $i \neq j$. The main property of these calculi is that there are only quadratic-linear relations between the left-invariant vector fields. The corresponding formulas show that these calculi are very close to the classical differential calculi on $SL(N)$ in many respects. Their enveloping algebras of the Lie algebra have Poincaré-Birkhoff-Witt property. The dimensions of the vector spaces of left-invariant k-forms $\Gamma^{\wedge k}_{\mathrm{inv}}$ is equal to the corresponding dimension $\binom{N^2-1}{k}$ in the classical case. Moreover, the associated higher order calculi (Γ, d) give the ordinary calculus on $SL(N)$ in the limit $q \to 1$.
This paper is organized as follows. Section 2 contains some general results about left-covariant differential calculi on quantum groups which will be needed later. In Section 3 we develop four classes of left-covariant first order differential calculi (Γ_r, d), $r = 1, \dots, 4$. We start with the choice of left-invariant functionals X_{i} and compute all relations between them. In Sections 4 we briefly describe the construction of the universal higher order differential calculus associated with a given left-covariant first order calculus on a quantum group. We apply this construction to the calculi (Γ_r, d), $r = 1, \dots, 4$.
Throughout this paper q is a nonzero complex number such that $q^2 \neq 1$ and we abbreviate $\lambda := q - q^{-1}$.

2 Left-covariant Differential Calculi on Quantum Groups

Our basic reference concerning differential calculi on quantum groups is [6]. Let $\mathcal{A}$ be a fixed Hopf algebra with comultiplication Δ, counit ε, antipode κ and unit element 1. Sometimes we use Sweedler's notation $\Delta^{(n)}(a) = a_{(1)} \otimes a_{(2)} \otimes \cdots \otimes a_{(n+1)}$. A *first order differential calculus* (briefly, a FODC) over $\mathcal{A}$ is a pair (Γ,d) of an $\mathcal{A}$-bimodule Γ and a linear mapping $\mathrm{d}\colon \mathcal{A} \to \Gamma$ such that $\mathrm{d}(ab) = \mathrm{d}a{\cdot}b + a{\cdot}\mathrm{d}b$ for $a, b \in \mathcal{A}$ and $\Gamma = \mathrm{lin}\{a\mathrm{d}b : a, b \in \mathcal{A}\}$. A FODC (Γ, d) is called *left covariant* if there is a linear mapping $\Delta_L\colon \Gamma \to \mathcal{A}\otimes\Gamma$ for which $\Delta_L(a\mathrm{d}b) = \Delta(a)(\mathrm{id}\otimes \mathrm{d})\Delta(b), a, b \in \mathcal{A}$.
Suppose that (Γ, d) is a left-covariant FODC over $\mathcal{A}$. Recall that the canonical projection of Γ into $\Gamma_{\mathrm{inv}} = \{\omega \in \Gamma : \Delta_L(\omega) = 1 \otimes \omega\}$ is defined by $P_{\mathrm{inv}}(\mathrm{d}a) = \kappa(a_{(1)})\mathrm{d}a_{(2)}$, cf. [5]. We abbreviate $\omega(a) = P_{\mathrm{inv}}(\mathrm{d}a)$. Then $\mathcal{R} := \{x \in \ker\varepsilon : \omega(x) = 0\}$ is the *right ideal* of $\ker\varepsilon$ *associated with* (Γ, d). Recall that $\mathcal{A}^\circ$ denotes the Hopf algebra of representative linear functionals on $\mathcal{A}$. If $\mathcal{A}$ is Hopf *-algebra then $\mathcal{A}^\circ$ becomes a Hopf *-algebra via $f^*(a) = \overline{f(\kappa(a)^*)}$, $f \in \mathcal{A}^\circ$, $a \in \mathcal{A}$. The vector space $\mathcal{X} := \{X \in \mathcal{A}^\circ : X(1) = 0$ and $X(a) = 0$ for $a \in \mathcal{R}\}$ is called the *quantum Lie algebra* of the FODC (Γ, d).

Lemma 1. *A vector space $\mathcal{X}$ of linear functionals on $\mathcal{A}$ is the quantum Lie algebra of a left-covariant FODC (Γ, d) if and only if $X(1) = 0$ and $\Delta X - 1\otimes X \in \mathcal{X}\otimes\mathcal{A}^\circ$ for all $X \in \mathcal{X}$.*

Proof. The necessity of the condition $\Delta X - 1\otimes X \in \mathcal{X}\otimes\mathcal{A}^\circ$ follows at once from formula (5.20) in [6]: $\Delta(X_i) = \sum_j X_j\otimes f^j_i + 1\otimes X_i$. To prove the sufficiency part, let us note that the above conditions imply that $\mathcal{R} := \{a \in \ker\varepsilon : X(a) = 0$ for $X \in \mathcal{X}\}$ is a right ideal of $\ker\varepsilon$. From the general theory (cf. Theorem 1.5 in [6]) we conclude easily that $\mathcal{R}$ is the right ideal associated with some left-covariant FODC (Γ, d) over $\mathcal{A}$. Since $\mathcal{X}$ is finite dimensional, $\mathcal{X}^{\perp\perp} = \mathcal{X}$. Hence, $\mathcal{X}$ is the quantum Lie algebra of (Γ, d). □
Obviously, if $\mathcal{X}$ is a quantum Lie algebra, so is $\mathcal{X}^*$.

3 Left-covariant first order Differential Calculi on $SL_q(N)$

Let $\ell^+ = (\ell^{+i}_{\ j})$ and $\ell^- = (\ell^{-i}_{\ j})$ be the $N\times N$ matrices of linear functionals $\ell^{+i}_{\ j}$ and $\ell^{-i}_{\ j}$ on $\mathcal{A} = SL_q(N)$ as defined in [2]. Recall that $\ell^\pm$ is uniquely determined by the properties that $\ell^\pm : \mathcal{A} \to M_N(\mathbf{C})$ is a unital algebra homomorphism and $\ell^{\pm i}_{\ j}(u^n_m) = p^{\mp 1}(\hat{R}^{\pm 1})^{in}_{mj}$ for $i, j, n, m{=}1, \ldots, N$, where p is an N-th root of q, $\hat{R}$ is the R-matrix of the quantum group $SL_q(N)$, and $u = (u^n_m)$ is the fundamental matrix corepresentation of $SL_q(N)$. In this section we define $N^2 - 1$ dimensional left-covariant FODC (Γ_r, d), $r = 1, \ldots, 4$, over $\mathcal{A} = SL_q(N)$. We set

$$X_{ij} \;=\; -\lambda^{-1}\ell^{+j}_{\ j}\ell^{-j}_{\ i}, \quad X_{ji} = q^{2\delta_{jN}}\lambda^{-1}g_i\ell^{-i}_{\ i}\ell^{+i}_{\ j},\; i < j,$$

$$
\begin{aligned}
X_i &= q^{-1}\lambda^{-1}(g_i-1), i=1,\dots,N-1,\ g_i=(\ell^{+i}_{\ i}\ell^{-N}_{\ N})^2 \quad \text{for } r=1,\\
X_{ij} &= -q^{-2\delta_{i1}}\lambda^{-1}h_j\ell^{+j}_{\ j}\ell^{-j}_{\ i},\quad X_{ji}=\lambda^{-1}\ell^{-i}_{\ i}\ell^{+i}_{\ j},\ i<j,\\
X_j &= q\lambda^{-1}(1-h_j), j=2,\dots,N,\ h_j=(\ell^{+1}_{\ 1}\ell^{-j}_{\ j})^2 \quad \text{for } r=2,\\
X_{ij} &= q^{2\delta_{jN}}\lambda^{-1}\kappa(\ell^{-j}_{\ i})\ell^{-i}_{\ i}g_i \quad X_{ji}=-\lambda^{-1}\kappa(\ell^{+i}_{\ j})\ell^{+j}_{\ j},\ i<j,\\
X_i &= q^{-1}\lambda^{-1}(g_i-1), i=1,\dots,N-1, \quad \text{for } r=3,\\
X_{ij} &= \lambda^{-1}\kappa(\ell^{-j}_{\ i})\ell^{-i}_{\ i},\quad X_{ji}=-q^{-2\delta_{i1}}\lambda^{-1}\kappa(\ell^{+i}_{\ j})\ell^{+j}_{\ j}h_j,\ i<j,\\
X_j &= q\lambda^{-1}(1-h_j), j=2,\dots,N, \quad \text{for } r=4.
\end{aligned}
$$

Let $\mathcal{X}_r$, $r=1,\dots,4$, denote the linear span of the functionals X_{ij}, $i\neq j$, $i,j=1,\dots,N$, and X_n, $n=1,\dots,N-1\,(2,\dots,N)$, for $r=1,3$ $(r=2,4)$.
Computing the coproducts of these functionals we obtain for $r=1$ and $i<j$:

$$
\begin{aligned}
\Delta X_{ij} &= 1\otimes X_{ij}+\sum_{m=i}^{j-1}X_{mj}\otimes\ell^{+j}_{\ j}\ell^{-m}_{\ i},\\
\Delta X_{ji} &= g_i\otimes X_{ji}+\sum_{m=i+1}^{j}X_{mi}\otimes g_i\ell^{-i}_{\ i}\ell^{+m}_{\ j},\\
\Delta X_i &= 1\otimes X_i+X_i\otimes g_i,\ i=1,\dots,N-1,
\end{aligned}
$$

for $r=2$ and $i<j$:

$$
\begin{aligned}
\Delta X_{ij} &= h_j\otimes X_{ij}+\sum_{m=i}^{j-1}X_{mj}\otimes h_j\ell^{+j}_{\ j}\ell^{-m}_{\ i},\\
\Delta X_{ji} &= 1\otimes X_{ji}+\sum_{m=i+1}^{j}X_{mi}\otimes\ell^{-i}_{\ i}\ell^{+m}_{\ j},\\
\Delta X_j &= 1\otimes X_j+X_j\otimes h_j,\ j=2,\dots,N.
\end{aligned}
$$

These formulas show that $\Delta X-1\otimes X\in\mathcal{X}_r\otimes\mathcal{A}^\circ$ for all $X\in\mathcal{X}_r$, $r=1,2$. The computations for $r=3,4$ are quite similar. Therefore, by Lemma 1 the vector space $\mathcal{X}_r$, $r=1,\dots,4$, defines a left-covariant FODC over $\mathcal{A}$.
Using the pairing between $SL_q(N)$ and $\mathcal{U}_R$ we verify that $X_{ij}(u^k_l)=\delta_{ik}\delta_{jl}$ for $i\neq j$ and $X_i(u^k_l)=\delta_{kl}(\delta_{ik}-q^{-2}\delta_{kN})$, $i=1,\dots,N-1$, $r=1,3$, and $X_j(u^k_l)=\delta_{kl}(\delta_{jk}-q^2\delta_{1k})$, $j=2,\dots,N$, $r=2,4$.

Now we will consider the $*$-structures. In case $SL_q(N,\mathbf{R})$, $|q|=1$, we get $\mathcal{X}_r^*=\mathcal{X}_r$, $r=1,\dots,4$. Hence, all four calculi are $*$-calculi. In case $SU_q(N)$, $q\in\mathbf{R}$, we get $\mathcal{X}_1^*=\mathcal{X}_3$ and $\mathcal{X}_2^*=\mathcal{X}_4$. Hence, none of them are $*$-calculi with respect to $SU_q(N)$.

We shall study the calculus $\mathcal{X}_3$ in more detail. Let $\{\omega_{ij},\omega_n : i\neq j,\ n=1,\dots,N-1\}$ (resp. $\{u^i_j, u^n_n-1 : i\neq j, n=1,\dots,N-1\}$) be the dual basis of $\Gamma_{\rm inv}$ (resp. of $\ker\varepsilon/\mathcal{R}$) to the basis $\{X_{ij},X_n\}$ of $\mathcal{X}$.
Let ϑ denote the Heaviside symbol, i.e. $\vartheta(m)=1$ if $m>0$ and $\vartheta(m)=0$ if $m\leq 0$.

The commutation rules between matrix entries and 1-forms are as follows:

$$
\begin{aligned}
\omega_{ij} u^l_k &= q^{\delta_{ik}+\delta_{jk}-2\delta_{Nk}} u^l_k \omega_{ij} + \lambda \vartheta(k-j) u^l_j \omega_{ik}, \\
\omega_{ji} u^l_k &= q^{-\delta_{ik}+\delta_{jk}} u^l_k \omega_{ji} - \lambda \vartheta(i-k) u^l_i \omega_{jk}, \\
\omega_i u^l_k &= q^{2\delta_{ik}-2\delta_{kN}} u^l_k \omega_i + q\lambda \vartheta(k-i) u^l_i \omega_{ik}.
\end{aligned}
$$

Note that the commutation rules are closed in the upper $(i<j)$ and lower $(i>j)$ triangle of left invariant 1-forms, respectively, while the commutation rules on the diagonal ω_i are *not* closed.
The associated right ideal $\mathcal{R}_3$ of $\ker \varepsilon$ is generated by the following set of quadratic-linear terms:

$$
\begin{aligned}
& u^i_j u^k_l \text{ for } i \neq j,\ i \neq l,\ k \neq j,\ k \neq l; \quad u^i_j u^j_i \text{ for } i \neq j; \\
& u^i_a u^a_j \text{ for } i > a,\ j > a \text{ and for } i < a,\ j < a; \\
& u^i_a u^a_j - \lambda u^i_j \text{ for } i < a < j \text{ and for } i > a > j; \\
& u^a_a u^i_j - q^{2\delta_{ia}-2\delta_{aN}} u^i_j \text{ for } i < j, \quad u^a_a u^i_j - u^i_j \text{ for } i > j; \\
& (u^i_i - q^{-2\delta_{iN}})(u^i_i - q^{2-2\delta_{iN}}),\ i = 1, \ldots, N; \\
& (u^i_i - 1)(u^j_j - q^{-2\delta_{jN}}) \text{ for } i < j,\ \mathcal{V} - \varepsilon(\mathcal{V}).
\end{aligned}
$$

Note that $\mathcal{V} := \sum_{i=1}^N q^{2\delta_{iN}} u^i_i$ is the only linear element of $\mathcal{A}$ that is annihilated by $\mathcal{X}_3$. We abbreviate $q_j := q^{2\delta_{jN}}$ and $q_{nij} := q^{2(\delta_{ni}-\delta_{nj}+\delta_{jN})}$. Throughout let $i<j<k<l$. Then the commutation rules between the $\{X_{\mathrm{i}}\}_{\mathrm{i}\in I}$, $I = \{(i,j),\ n\colon i \neq j,\ n = 1, \ldots, N-1\}$ are as follows:

$$
\begin{aligned}
& X_{ij} X_{il} - q q_l X_{il} X_{ij} = 0, \\
& X_{il} X_{jl} - q X_{jl} X_{il} = 0, \\
& X_{ij} X_{jl} - q q_l X_{jl} X_{ij} - q q_l \lambda X_{il} X_j = q_l X_{il}, \\
& X_{ij} X_{kl} - q_l X_{kl} X_{ij} = 0, \\
& X_{jk} X_{il} - q_l X_{il} X_{jk} = 0, \\
& X_{ik} X_{jl} - q_l X_{jl} X_{ik} - q_l \lambda X_{il} X_{jk} = 0,
\end{aligned}
$$

$$
\begin{aligned}
& X_{ji} X_{li} - q X_{li} X_{ji} = 0, \\
& X_{li} X_{lj} - q X_{lj} X_{li} = 0, \\
& X_{ji} X_{lj} - q^{-1} X_{lj} X_{ji} = -q^{-1} X_{li}, \\
& X_{ji} X_{lk} - X_{lk} X_{ji} = 0, \\
& X_{kj} X_{li} - X_{li} X_{kj} = 0, \\
& X_{ki} X_{lj} - X_{lj} X_{ki} - \lambda X_{li} X_{kj} = 0,
\end{aligned}
$$

$$
X_{ij} X_{li} - q^{-1} q_l^{-1} X_{li} X_{ij} + q_l^{-1} \lambda X_{lj} X_i = -q^{-1} q_l^{-1} X_{lj},
$$

$$
\begin{aligned}
&X_{il}X_{lj} - q^{-1}q_l^{-1}X_{lj}X_{il} = X_{ij},\\
&X_{ij}X_{lj} - q^{-1}q_l^{-1}X_{lj}X_{ij} = 0,\\
&X_{ij}X_{lk} - q_l^{-1}X_{lk}X_{ij} = 0,\\
&X_{il}X_{kj} - X_{kj}X_{il} = 0,\\
&X_{ik}X_{lj} - q_l^{-1}X_{lj}X_{ik} + q_l^{-1}\lambda X_{lk}X_{ij} = 0,
\end{aligned}
$$

$$
\begin{aligned}
&X_{ji}X_{il} - qX_{il}X_{ji} = X_{jl},\\
&X_{li}X_{jl} - q^{-1}q_lX_{jl}X_{li} + q_l\lambda X_{ji}X_l = -q^{-1}q_lX_{ji},\\
&X_{ji}X_{jl} - q^{-1}X_{jl}X_{ji} = 0,\\
&X_{ji}X_{kl} - X_{kl}X_{ji} = 0,\\
&X_{li}X_{jk} - q_lX_{jk}X_{li} = 0,\\
&X_{ki}X_{jl} - X_{jl}X_{ki} + \lambda X_{kl}X_{ji} = 0,
\end{aligned}
$$

$$
X_{il}X_{li} - q_l^{-1}X_{li}X_{il} = X_i - X_l,
$$

$$
\begin{aligned}
&X_nX_m - X_mX_n = 0,\\
&X_nX_{ij} - q_{nij}X_{ij}X_n = q^{-1}\lambda^{-1}(q_{nij}-1)X_{ij},\\
&X_nX_{ji} - q_{nij}^{-1}X_{ji}X_n = q^{-1}\lambda^{-1}(q_{nij}^{-1}-1)X_{ji}.
\end{aligned}
$$

Summarizing the preceding we obtain

1. $\mathcal{X}_r$, $r = 1, \ldots, 4$, is an N^2-1 dimensional left-covariant FODC on $SL_q(N)$.
2. The relations of the algebra $\mathcal{U}_r$ generated by $\mathcal{X}_r$ are quadratic-linear. To be more precise, there are complex matrices $\sigma = (\sigma^{\mathfrak{ij}}_{\mathfrak{kl}})$ and $C = (C^{\mathfrak{k}}_{\mathfrak{ij}})$, $\mathfrak{i}, \mathfrak{j}, \mathfrak{k}, \mathfrak{l} \in I$, such that

   $$X_{\mathfrak{i}}X_{\mathfrak{j}} - \sum_{\mathfrak{k},\mathfrak{l}} \sigma^{\mathfrak{ij}}_{\mathfrak{kl}} X_{\mathfrak{k}}X_{\mathfrak{l}} = \sum_{\mathfrak{m}} C^{\mathfrak{m}}_{\mathfrak{ij}} X_{\mathfrak{m}},\ \ \mathfrak{i},\mathfrak{j} \in I.$$

   The matrix σ is diagonalizable and has eigenvalues -1 and 1 with multiplicities $\binom{N^2-1}{2}$ and $\binom{N^2}{2}$, respectively. The structure constants C are antisymmetric, i. e. $C\sigma^{\mathfrak{t}} = -C$.
3. The algebra $\mathcal{U}_r$ has a linear basis

   $$\{X_{\mathfrak{i}_1}^{m_1} \ldots X_{\mathfrak{i}_n}^{m_n} : \mathfrak{i}_1 < \mathfrak{i}_2 < \ldots < \mathfrak{i}_n,\, \mathfrak{i}_k \in I, m_k \in \mathbf{N}_0,\, k = 1, \ldots, n\}, \quad n = N^2 - 1,$$

   with respect to an arbitrary order of the index set I.

4 Higher Order Calculi

Given a left-covariant FODC (Γ, d) on $\mathcal{A}$. Let $\mathcal{R}$ be the associated right ideal of (Γ, d) and $\mathcal{S}\colon \mathcal{R} \to \Gamma_{\mathrm{inv}}\otimes\Gamma_{\mathrm{inv}}$ the linear mapping defined by $\mathcal{S}(r) = \omega(r_{(1)}) \otimes \omega(r_{(2)})$. Let S_2

be the subbimodule of $\Gamma \otimes_{\mathcal{A}} \Gamma$ generated by $\mathcal{S}(\mathcal{R})$. Furthermore, let S be the ideal of the tensor algebra $\Gamma^{\otimes} = \sum_n \Gamma^{\otimes n}$ generated by S_2.
Then $\Gamma^{\wedge} = \Gamma^{\otimes}/S$ is called *universal exterior algebra associated with* (Γ, d). By construction, $\Gamma^{\wedge}$ is a left-covariant bimodule and a $\mathbf{Z}$-graded algebra.

Proposition 2. *Let* (Γ, d) *be a left-covariant FODC over* $\mathcal{A}$ *and let* $\Gamma^{\wedge}$ *be the universal exterior algebra built over* (Γ, d)*. Then there exists one and only one linear mapping* $\mathrm{d}\colon \Gamma^{\wedge} \to \Gamma^{\wedge}$ *such that*

1. d *increases the grade by one.*
2. d *extends* $\mathrm{d}\colon \mathcal{A} \to \Gamma$.
3. d *fulfills the graded Leibniz rule.*
4. $\mathrm{d}^2 = 0$.
5. d *is left-covariant, i. e.* $\Delta_L(\mathrm{d}\theta) = (\mathrm{id} \otimes \mathrm{d})\Delta_L\theta$, $\theta \in \Gamma^{\wedge}$.

$(\Gamma^{\wedge}, \mathrm{d})$ is called *universal differential calculus associated with the left-covariant FODC* (Γ, d). Woronowicz' construction of higher order calculi for *bicovariant* FODC can be obtained by taking a quotient of the universal differential calculus $(\Gamma^{\wedge}, \mathrm{d})$.
The mapping $\mathrm{d}|_{\Gamma_{\mathrm{inv}}}$ is given by $\mathrm{d}\omega(a) = -\omega(a_{(1)}) \wedge \omega(a_{(2)})$.
We apply the above construction to the left-covariant FODC $\mathcal{X}_r$, $r = 1, \ldots, 4$. Fix an arbitrary linear order of I.

Theorem 3. *Let* $(\Gamma^{\wedge}, \mathrm{d})$ *be the universal left-covariant differential calculus associated with one of the FODC* (Γ_r, d), $r = 1, \ldots, 4$*. Then*

$$\dim(\Gamma_{\mathrm{inv}}^{\wedge k}) = \binom{N^2 - 1}{k}, \quad k \geq 1.$$

More precisely, the set $\{\omega_{\mathfrak{i}_1} \wedge \omega_{\mathfrak{i}_2} \wedge \ldots \wedge \omega_{\mathfrak{i}_k} : \mathfrak{i}_1 < \ldots < \mathfrak{i}_k, \mathfrak{i}_j \in I, j = 1, \ldots, k\}$ *is a basis of the vector space* $\Gamma_{\mathrm{inv}}^{\wedge k}$.

In particular, for $k = 2$ Theorem 3 states that the space of left-invariant 2-forms is equal to $\ker(\sigma + \mathrm{id})$ (where σ is the linear mapping $\Gamma_{\mathrm{inv}} \otimes \Gamma_{\mathrm{inv}} \to \Gamma_{\mathrm{inv}} \otimes \Gamma_{\mathrm{inv}}$ defined by $\omega_{\mathfrak{i}} \otimes \omega_{\mathfrak{j}} \mapsto \sum_{\mathfrak{k},\mathfrak{l}} \sigma_{\mathfrak{i}\mathfrak{j}}^{\mathfrak{k}\mathfrak{l}} \omega_{\mathfrak{k}} \otimes \omega_{\mathfrak{l}}$.) In contrast, there are examples of three dimensional quadratically closed FODC on $SU_q(2)$ whose space of left-invariant 2-forms is only two dimensional.

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