C^*-Algebras - Selected Topics

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Basics on C^*-Algebras and Multiplier Algebras

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7. C^*-Algebras Generated by Unbounded Operators

Lecture 1:

A readable introduction into C^*-algebras is chapter 5 of

- R.V. Kadison, P.V. Ringrose, Fundamentals of the Theory of Operator Algebras I, Academic Press, 1983.

Many important examples are treated in

- K.R. Davidson, C^*-Algebras by Example, Fields Inst. Monographs, AMS, 1996.

A good general textbook in

- G. I. Murphy, C^*-Algebras and Operator Theory, Academic Press, London, 1990.

Multiplier theory and K-theory can be found in

- N.E. Wegge-Olsen, K-Theory and C^*-Algebras, Oxford Univ. Press, 1993.

A standard monograph on C^*-algebras in

- G.K. Pedersen, C^*-Algebras and Their Automorphism Groups, Academic Press, London 1979.

1. Basics on C^*-Algebras

1.1 Definition and Basic Examples

Definition 1: A **-algebra* A is a complex algebra with an *algebra involution* $a \to a^*$ of A (i.e. $(\lambda_1 a_1 + \lambda_2 a_2)^* = \overline{\lambda_1} a_1^* + \overline{\lambda_2} a_2^*, (a^*)^* = a, (a_1 a_2)^* = a_2^* a_1^*$).

Definition 2: A *Banach algebra* A is a complex algebra which is a Banach space with norm $\|\cdot\|$ such that

$$\|ab\| \leq \|a\|\|b\| \text{ for all } a, b, \in A.$$

Definition 3: A C^**-algebra* A is a Banach algebra which is a $*$-algebra and satisfies

$$\|a^* a\| = \|a\|^2 \text{ for all } a \in A. \tag{1}$$

Equation (1) is called C^*-condition.

Since $\|a\|^2 = \|a^* a\| \leq \|a^*\|\|a\|$, (1) implies that $\|a\| = \|a^*\|$.

Definition 4: Let A_1 and A_2 be C^*-algebras.
A **-homomorphism* $\phi : A_1 \rightarrow A_2$ is an algebra homomorphism such that $\phi(a^*) = \phi(a)^*, a \in A_1$. A **-isomorphism* of A_1 and A_2 is a bijective *-homomorphism.

A map $\phi : A_1 \rightarrow A_2$ is called *isometric* if $\|\phi(a)\|_2 = \|a\|_1$ for all $a \in A_1$.

Basic Example 1: $C_0(X)$

Let X be a locally compact Hausdorff space and $A = C_0(X)$ the set of continuous functions on X vanshing at infinity. $(A = C(X)$ if X is compact.)

A is a commutative C^*-algebra with involution
$f(x) \rightarrow \overline{f(x)}$ and norm $\|f\| = \sup\limits_{x \in X} |f(x)|$.

Basic Example 2: concrete C^*-algebra

Let $\mathbb{B}(\mathcal{H})$ be the algebra of bounded operators on a Hilbert space $\mathcal{H}$. It is a C^*-algebra with adjoint operation and operator norm.

Any normclosed *-subalgebra A of $\mathbb{B}(\mathcal{H})$ is a C^*-algebra. These C^*-algebras are called *concrete C^*-algebras*.

1.2 First Representation Theorem

Let A be a commutative C^*-algebra.
Let X_A be the set of all characters of A (i.e. non-zero homomorphism $\varphi{:}A{\to}\mathbb{C}$).
Any $\varphi{\in}X_A$ is continuous with norm 1.
We equipp X_A with the relative weak *-topology of A'. Neighbourhood basis of φ:

$$\mathcal{U}_{a_1,\ldots,a_r,\varepsilon}(\varphi){=}\{\psi{\in}X_A{:}|\varphi(a_j){-}\psi(a_j)|{<}\varepsilon\}$$

X_A is locally compact.

The map $A \ni a \to \hat{a} \in C_0(X_A)$ defined by

$$\hat{a}(\varphi) = \varphi(a), a \in A, \varphi \in X_A,.$$

is called the *Gelfand transform* of A.

Theorem 5: I.M. Gelfand (Mat. Sbornik **9** (1941)- this paper is the birthplace of Banach algebra theory)
If A is an abelian C^*-algebra, then the Gelfand transform is an isometric $*$-isomorphism of A onto $C_0(X_A)$.

1.3 Second Representation Theorem

Theorem 6: I.M. Gelfand, M. Naimark (Mat. Sbornik **12**(1943) - reprint in Cont. Math. **167** (1994), AMS)

Every (unital) C^*-algebras A is isometric $*$-isomorph to a concrete C^*-algebra (that is, to a norm closed $*$-subalgebra of some $\mathbb{B}(\mathcal{H})$).

Remarks:

1. Gelfand / Naimark proved Theorem 2 under the additional assumption that A is *symmetric.*
(This means that $(1 + a^*a)^{-1} \in A$ for all $a \in A$.)
It was removed by the works of M. Fukamiya (1952), J.L. Kelley/R-L. Vaught (1953) and I. Kaplansky (1952), see also Schatz' MR-review of Fukamiya's paper.

2. Moreover, they conjectured that (1) can be replaced by $\|a^*a\| = \|a^*\|\|a\|, a \in A$. This was proved by Ono (1958).

3. This paper was submitted at August 22, 1941. This day is the birthday of C^*-algebras!

1.4 Spectrum and Some Other Concepts

I. Let A be a unital Banach algebra.

Definition 7: The *spectrum* of $a \in A$ is the set
$\sigma_A(a) \equiv \sigma(a) = \{\lambda \in \mathbb{C} : \lambda \cdot 1 - a \text{ is not invertible in } A\}$

Basic properties:

1. $\sigma(a)$ is a nonempty compact set.
2. $\lambda \to (\lambda \cdot 1 - a)^{-1}$ is an A-valued analytic function on $\rho(a) := \mathbb{C} \backslash \sigma(A)$.
3. $\sigma(p(a)) = p(\sigma(a))$ for each polynomial p.
4. If A is commutative, then
   $\sigma(a) = \sigma(\hat{a}) \equiv \{\varphi(a); \varphi \in X_A\}$.
5. If B is a unital C^*-subalgebra of a (unital) C^*-algebra A, then $\sigma_B(b) = \sigma_A(b)$ for each $b \in B$.

II. *Adjoining a unit*

Let A be a non-unital C^*-algebra. On $\tilde{A}{:=}A{\oplus}\mathbb{C}$ define

$(a,\lambda)(b,\mu) := (ab + \lambda b + \mu a, \lambda\mu), (a,\lambda)^* = (a^*, \bar{\lambda})$, $\|(a,\lambda)\| = \sup_{\|b\|\leq 1} \|ab + \lambda b\|$.

Then $\tilde{A}$ is a unital C^*-algebra (with unit $(0,1)$) which contains A as a C^*-subalgebra $((a,0) \cong a)$.

Define the spectrum of an element $a{\in}A$ by $\sigma_A(a){:=}\sigma_{\tilde{A}}(a)$.

III. *Positive elements*

Definition 8: $A_+ := \{a \in A : a = a^* \text{ and } \sigma(a) \subseteq [0, +\infty)\}$ is the set of *positive elements* of A.

- $A_+ = \{a \in A : a = b^*b \text{ for some } b \in A\}$.
- (Existence and uniqueness of positive square root)

If $a \in A_+$, then there exists a unique $b \in A_+$ such that $a = b^2$. We denote b by $a^{1/2}$.

- If $a, b \in A_+$, then $a + b \in A_+$, so A_+ is a wedge.

IV. *Approximate units*

Definition 9: An *approximate unit* for a C^*-algebra A is a net $\{e_i; i \in I\}$ of elements $e_i \in A, e_i \in A_+$, $\|e_i\| \leq 1$ such that $e_i \leq e_j$ if $i \leq j$ and

$$\lim_i e_i a = \lim_i a e_i = a \text{ for all } a \in A.$$

- Each C^*-algebra A has an approximate unit.

(If A is separable, we can take a sequence.)

V. *Ideals and Quotients*

Ideal always means a two-sided closed ideal.

- If I is an ideal of a C^*-algebra A, then $I = I^*$ and the quotient algebra A/J is a C^*-algebra.

- Let $\phi : A_1 \to A_2$ be $*$-homomorphism of C^*-algebras.

Then:
$\|\phi(a)\|_2 \leq \|a\|_1$ for $a \in A_1$,
$\|\phi(a)\|_2 = \|a\|_1$ for $a \in A$ if ϕ is injective,
$\phi(A_1)$ is a C^*-algebra.

1.5 Positive Functionals and GNS-Construction

Definition 10: A *representation* of A on $\mathcal{H}$ is a $*$-homomorphism of A into $\mathbb{B}(\mathcal{H})$.
π is called *nondegenerate* if $\pi(A)\mathcal{H}$ is dense in $\mathcal{H}$.

Notation:
$\mathrm{Rep}(A, \mathcal{H})$ - set of nondegenerate representations of A on $\mathcal{H}$.

A linear functional φ on A is called *positive* if $\varphi(a) \geq 0$ for all $a \in A_+$ (i.e. $\varphi(b^*b) \geq 0$ for $b \in A$).
A positive linear functional of norm one is a *state*.

If $\pi \in \mathrm{Rep}(A, \mathcal{H})$ and $\eta \in \mathcal{H}$, then $f_\varphi(a) = \langle \pi(a)\eta, \eta \rangle$ is a positive linear functional.

The *GNS-construction* shows that any positive linear functional f on A arises in this way. We sketch this construction. Assume for simplicity that A is unital.

Define $\langle a, b\rangle_0 := f(b^*a), a, b \in A$.
Since f is positive, $\langle \cdot, \cdot\rangle_0$ is a positive semidefinite sesquilinear form, so we have the Cauchy-Schwarz inequality

$$|\langle a, b\rangle_0|^2 \leq \langle a, a\rangle_0 \langle b, b\rangle_0,$$

that is, $|f(b^*a)| \leq f(a^*a)f(b^*b)$ for $a, b \in A$.
Hence $\mathcal{N}_f = \{a \in A : f(a^*a) = 0\}$ is a *left ideal*.

$\mathcal{D} = A/\mathcal{N}_f$ is a pre-Hilbert space with scalar product

$$\langle a + \mathcal{N}_f, b + \mathcal{N}_f\rangle := f(b^*a).$$

Set $\tilde{a} := a + \mathcal{N}_f$ and define $\pi(a)\tilde{b} := \widetilde{ab}$. Since $\mathcal{N}_f$ is a left ideal, $\pi(a)$ is well-defined. Moreover,
$\pi(a_1)\pi(a_2) = \pi(a_1a_2), \langle \pi(a)\tilde{b}, \tilde{c}\rangle = \langle \tilde{b}, \pi(a^*)\tilde{c}\rangle$,
$\pi(1) = I$ and $\pi(A)\tilde{1}$=$\mathcal{D}$.

If A is a C^*-algebra, f is continuous. From this it follows that all operator $\pi(a)$ are bounded, so π extends to a representation of A on the completion $\mathcal{H}$ of $\mathcal{D}$ and

$$\langle \pi(a)\tilde{1}, \tilde{1}\rangle = \langle \tilde{a}, \tilde{1}\rangle = f(1^*a) = f(a).$$

2. Irrational Rotation Algebras

2.1 Definition

Let $T \cong \{z(t) = e^{2\pi i t}; t \in \mathbb{R}\}$ be the 1-torus.
Let θ be a fixed irrational number. Put $q = e^{2\pi i\theta}$.

Define two unitaries U_0, V_0 on $L^2(T)$ by $(U_0 f)(t) = z(t)f(t)$ and $(V_0 f)(t) = f(t-\theta)$. Then

$$U_0 V_0 = q V_0 U_0. \tag{1}$$

Let $U = \oplus U_i$ and $V = \oplus V_i$ the direct sum of all equivalentce classes of irreducible pairs (U_i, V_i) of unitaries satisfying (1).

The C^*-algebra A_θ generated by U, V is called *irrational rotation algebra* or C^*-algebra of the *quantum torus*.

A_θ is the *universal C^*-algebra of the relation* (1,) i.e. if $\tilde{U}, \tilde{V}$ are unitaries satisfying (1) and $B = C^*(\tilde{U}, \tilde{V})$, then there is a $*$-homomorphism $\rho : A_\theta \to B$ such that $\rho(U) = \tilde{U}$ and $\rho(V) = \tilde{V}$.

2.2 Expectations and Trace on A_θ

For $(\lambda, \mu) \in T^2$ there is a $*$-automorphism $\rho_{\lambda,\mu}$ of A_θ such that

$$\rho_{\lambda,\mu}(U) = \lambda U, \rho_{\lambda,\mu}(V) = \mu V.$$

Define $\Phi_1(a) = \int\limits_0^1 \rho_{1,z(t)}(a)dt, a \in A$.

Proposition 11: *ϕ_1 is an expectation of A_θ onto $C^*(U)$, that is, Φ_1 is positive , $\|\Phi_1\| \leq 1$ and $\Phi_1^2 = \Phi_1$.*

$\Phi_1(U^k V^l) = \delta_{l0} U^k$
$\Phi_1(a) = \lim\limits_{n\to\infty} \frac{1}{2n+1} \sum\limits_{k=-n}^{n} U^k a U^{-k}$.

Similarly, $\Phi_2 : A_\theta \to C^*(V)$ is defined.

Corollary 12: *$\tau = \Phi_2\Phi_1 = \Phi_1\Phi_2$ is a faithful unital trace on A_θ and we have $\tau(U^k V^l) = \delta_{k0}\delta_{l0}$.*

Proposition 13: *τ is the unique trace on A_θ.*

Theorem 14: *The C^*-algebra A_θ is simple, i.e. A_θ has no nontrivial normclosed two-sided ideal.*

2.3 Projections in A_θ and Isomorphic Irrational Algebras

Projection: $p = p^*$ and $p^2 = p$
$C(T^2)$ has no projection $p \neq 0, 1$.

Theorem 15: M. Rieffel (Pacific J.M. **93**(1981))

For each $\alpha \in (\mathbb{Z} + \mathbb{Z}\theta) \cap [0,1]$ there a exists a projection $p_\alpha \in A_\theta$ such that $\tau(p_\alpha) = \alpha$.

Idea of proof: $p = M_g V + M_f + V^* M_g$,

Theorem 16: *Two irrational rotation algebras A_θ and A_θ are isomorphic as C^*-algebras if and only if $\theta \equiv \theta'$ mod $\mathbb{Z}$ or $\theta \equiv -\theta'$ mod $\mathbb{Z}$.*

Idea of proof: The K_0-group of A_θ is $\mathbb{Z} + \mathbb{Z}\theta$. Hence, if $A_\theta \cong A'_\theta$, then $\mathbb{Z} + \mathbb{Z}\theta = \mathbb{Z} + \mathbb{Z}\theta'$ which implies that $\theta - \theta' \in \mathbb{Z}$ or $\theta + \theta' \in \mathbb{Z}$.

3. Multiplier Algebras of C^*-Algebras

3.1 Three Definitions of $M(A)$

I. Let A be a complex algebra such that $Aa = 0$ implies $a = 0$. (Such algebras are called *semiprime*).
A *double centralizer* for A is a pair (L, R) of linear mappings of A into A such that for all $a, b \in A$,

$$L(ab) = L(a)b, R(ab) = aR(b), R(a)b = aL(b).$$

The set $DC(A)$ of double centralizers of A is an *algebra* with operations

$$\lambda(L, R)=(\lambda L, \lambda R), (L_1, R_1)+(L_2, R_2)=(L_1+L_2, R_1+R_2), (L_1, R_1)(L_2, R_2)=(L_1L_2, R_2R_1).$$

This algebra has a unit (Id, Id).

Example: For $c \in A$ set $L_c(a) = ca, R_c(a) = ac$.

Then $(L_c, R_c) \in DC(A)$.

The map $c \to (L_c, R_c)$ is an embedding of A into $DC(A)$. A is an *essential* ideal of $DC(A)$. (That is, A is an ideal and $xA{=}Ax{=}0$ for $x \in DC(A)$ implies $x{=}0$.)

II. Suppose A is a C^*-Algebra. Let $\mathcal{DC}(A)$ be the set of pairs (L, R) of mappings $L, R : A \to A$ such that

$$aL(b) = R(a)b \text{ for } a, b \in A.$$

Lemma 17: *Suppose* $(L, R) \in \mathcal{DC}(A)$. *Then:*

(i) L *and* R *are linear.*
(ii) $L(ab) = L(a)b$ *and* $R(ab) = aR(b)$ *for* $a, b \in A$,
(iii) L *and* R *are bounded and* $\|L\| = \|R\|$.

In particular, $DC(A) = \mathcal{DC}(A)$. With operations defined in I., $\mathcal{DC}(A)$ is a unital algebra. For $(R, L) \in \mathcal{DC}(A)$ define

$$(L, R)^* := (R^*, L^*), \text{ where } T^*(a) := T(a^*)^*, a \in A,$$

$$\|(L, R)\| := \|R\| = \|L\|.$$

Then $\mathcal{DC}(A)$ is a unital C^*-algebra and A is an essential ideal of $\mathcal{DC}(A)$.

Remark: This is the double centralizer algebra introduced by R. C. Busby (TAMS 132 (1968)).

Lemma 18: *Let B be a C^*-algebra which contains A as a $*$-ideal. Then there exists a unique $*$-homomorphisms $\pi : B \to \mathcal{DC}(A)$ such that $\pi(x) = x$ for $x \in A$. π is injective iff A is an essential ideal of B.*

This shows that $\mathcal{DC}(A)$ is the largest C^*-algebra which contains A as an essential ideal.

Lemma 19: *Let $\phi : A \to B$ be a surjective $*$-homomorphism of non-unital C^*-algebras A, B. Then ϕ has a unique extension $\tilde{\phi}$ to a $*$-homomorphism $\tilde{\phi} : M(A) \to M(B)$.*

In particular, representations of A extend uniquely to representations of $M(A)$.

III. Let A be a non-degenerate C^*-subalgebra of $\mathbb{B}(\mathcal{H})$. (Non-degenerate means: $A\varphi{=}0$ implies $\varphi{=}0$.)
Define

$$M(A) = \{x \in \mathbb{B}(\mathcal{H}) : xA \subseteq A \text{ and } Ax \subseteq A\}.$$

Obviously, $M(A)$ is a unital C^*-algebra containing A.

Lemma 20: *The map $x{\to}(L_x, R_x)$ is an isometric *-isomorphism of $M(A)$ onto $\mathcal{DC}(A)$.*

A *unitization* of a C^*-algebra A is an embedding of A into a unital C^*-algebra B as an essential ideal.

Example: $A = C_0(X), X$ locally compact Hausdorff space.

Then unitizations of A are in one-to-one correspondence with compactifications of X. There are a smallest $(X{\cup}\{\infty\})$ and a largest $(\beta(X))$ compactification of X.

⇒ The multiplier algebra $M(A)$ is the largest unitization of A (by Lemma 18).

Definition 21: $C(A) := M(A)/A$ is called the *corona algebra* for A.

3.2 Examples

Example 1: $A = \mathcal{K}(\mathcal{H})$ *C^*-algebra of compact operators on $\mathcal{H}$.*

Then $M(A) = \mathbb{B}(\mathcal{H})$ and $M(A)/A = \mathbb{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is just the *Calkin algebra*.

Example 2: *Commutative C^*-algebras $C_0(X)$*

X is a locally compact Hausdorff space.

A C^*-algebra is called *σ-unital* if it has a countable approximate unit.

$C_0(X)$ is σ-unital if and only if X is σ-compact
(i.e. X is a countable union of compact subsets of X)

In what follows we assume that X is σ-compact.
(This is stronger than the separability of $C_0(X)$).

Lemma 22: *$M(C_0(X)) = C_b(X) = C(\beta(X))$, where $\beta(X)$ is the Stone-Czech compactification of X.*

$$C(C_0(X)) = C_b(X)/C_0(X) \cong C(\beta X \backslash X)$$

The compact space $\beta X \backslash X$ is called *corona space* of X.

3.3 Size of Multiplier Algebras

Let A be a C^*-algebra.

Lemma 23: *A is σ-unital iff A contains a strictly positive element h (that is, $\varphi(h) > 0$ for every state φ of A).*

(If $h \in A, \|h\| \leq 1$, is strictly positive, then

$$\{e_n := h^{1/n}; n \in \mathbb{N}\} \text{ and } \{e'_n = (h + 1/n)^{-1}h, n \in \mathbb{N}\}$$

are countable approximate units for A.)

Theorem 24: Akemann-Pedersen-Tomiyama (JFA **13** (1973))

Let A be a non-unital σ-unital C^*-algebra and let h be a strictly positive element of A. Let $B = C^*(h)$ and $X = \sigma(h)\backslash\{0\}$. Then:

(i) *$C_b(X)$ is non-separable.*

(ii) *$M(B)$ and $C(B) = M(B)/B$ are non-separable.*

(iii) *$M(A)$ is non-separable.*

3.4 Strict Topology

Let A be a C^*-Algebra. The *strict topology* on $M(A)$ is the locally convex topology β defined by the seminorms

$p_a(x) = \|xa\| + \|ax\|, x \in M(A), a \in A.$

That is, $x_j \to x$ in $M(A)[\beta]$ iff $x_j a \to xa$ and $ax_j \to ax$ in A for all $a \in A$.

Lemma 25: (i) $M(A)[\beta]$ *is complete.*
(ii) A *is dense in* $M(A)[\beta]$.

That is, $M(A)$ is the completion of A in the strict topology.

The following Stone-Weierstraß theorem (due to Woronowicz) is needed in Lecture 3.

Proposition 26: *Let A be C^*-algebra and Q is unital *-subalgebra of $M(A)$ which is closed in the strict topology. Suppose that Q seperates representations of A (i.e. if $\pi_1, \pi_2 \in \mathrm{Rep}(A, \mathcal{H}), \pi_1 \neq \pi_2$, then there exists $q \in Q$ such that $\pi_1(q) \neq \pi_2(q)$).*
Then $Q = M(A)$.

3.5 Some Results of Pedersen

For a subset B of $\mathbb{B}(\mathcal{H})$, let B^m (resp. B_m) denote the set of all $T = T^* \in \mathbb{B}(\mathcal{H})$ for which there is an increasing (resp. decreasing) net (T_i) from B such that
$T\varphi = \lim T_j\varphi, \quad \varphi \in \mathcal{H}$.
Let $\tilde{A} := A \oplus C \cdot 1$ be the *minimal unitization* of A.

Theorem 27: $M(A)_{sa} = (\tilde{A}_{sa})^m \cap (\tilde{A}_{sa})_m$.

We recall Tietze's extension theorem:

Let X be a normal topological Hausdorff space.
(*Normal* means that given disjoint closed subsets A_1 and A_2 of X there exist disjoint open subsets U_1 and U_2 of X such that $A_1 \subseteq U_1$ and $A_2 \subseteq U_2$. That is, X is a T_4-*space.)*

Then for each bounded continuous function f on a closed subset Y of X there exists a continuous function $\tilde{f}$ on X such that $f(y)=\tilde{f}(y)$ for $y\in Y$ and $\sup_{x\in X} |\tilde{f}(x)| = \sup_{y\in Y} |f(y)|$.

How to generalize Tietze's theorem to C^*-algebras?

$A = C_0(X)$. Then A σ-unital implies normality of X.

Closed subsets of $X \cong$ closed ideals of $C_0(X)$
$\cong$ surjective quotients of $C_0(X)$.

Theorem 28: *Let* $\rho{:}A{\to}B$ *be a* surjective **-homomorphism between* σ*-unital* C^**-algebras* A *and* B*. Then there exists a unique* surjective *extension* $\tilde{\rho}$ *to a *-homomorphism of* $M(A)$ *and* $M(B)$*.*

Technical fact used in the proof:

Lemma 29: *Suppose* $(e_n; n \in \mathbb{N})$ *is a monotone selfadjoint approximate unit for* A *and* $(T_n; n \in \mathbb{N})$ *is a bounded sequence in* $M(A)$. *Then the sum*

$$\sum_{n=0}^{\infty}(e_n - e_{n-1})^{1/2} T_n (e_n - e_{n-1})^{1/2}$$

of elements in $M(A)$ *converges in the strict topology to an element* $T \in M(A)$.

Idea of proof: Set $u_n = (e_n - e_{n-1})^{1/2}, u_0 = 0$.
Suppose $0 \leq T_n \leq I$. Each summand is positive and dominated by u_n^2 and $\sum u_n^2 = I$. Hence $T \in (A_+)^m$.

Since $I - T = \sum_{n=0}^{\infty} u_n (I - T_n) u_n$,

$T \in I - (A_+)^m = I + (-A_+)_m \subseteq (\tilde{A}_{sa})_m$.

Hence $T \in (A_+)^m \cap (\tilde{A}_{sa})_m \subseteq M(A)_{sa}$.

3.6 Corona Algebras

Let A be a σ-unital C^*-Algebra.

Theorem 30: *The corona algebra* $C(A)$ *has the countable Riesz separation property, that is, if* (a_n) *and* (b_n) *are increasing resp. decreasing sequences from* $C(A)_{sa}$ *such that* $a_n \leq a_{n+1} \leq b_{n+1} \leq b_n$ *for all* $n \in \mathbb{N}$*, then there exists* $c \in C(A)_{sa}$ *satisfying* $a_n \leq c \leq b_n$*, for all* n*.*

Theorem 31: *Each corona algebra is a* SAW^**-algebra, that is, given two element* $a, b \in C(A)_+$ *such that* $ab = 0$*, there exists* $c \in C(A)_+$ *such that* $ac = a$ *and* $cb = 0$*.*

Theorem 32: (S. Sakai)
If δ *is a derivation of a simple* C^**-algebra* A*, then exists* $x \in M(A)$ *such that* $\delta(a) = xa - ax$*,* $a \in A$*.*

Lecture 2:

C^*-**Algebras and Operators on Hilbert** C^*-**Modules**

Literatur:

1. E.C. Lance: Hilbert C^*-modules, A toolkit for operator algebraists, Cambridge UP, 1995.

2. V.M. Manuilov, E.V. Troitsky: Hilbert C^*-modules, AMS 2005.

Pioneering work by I. Kaplansky (1953; commutative case), W. Paschke (1973) and M. Rieffel (1974).

Areas of applications of Hilbert C^*-modules

- *Noncommutative geometry*
  Hilbert C^*-modules can be considered as
  "noncommutative vector bundles"
  Index theory of elliptic operators over C^*-algebras
  (A.S. Mishchenko)

- *K- and KK-theory* (G.G. Kasparov)
  Unbounded Fredholm modules

- *Induced representations of C^*-algebras* (M. Rieffel)
  Morita equivalence of C^*-algebras (L.G. Brown, P. Green, M. Rieffel)

- *C^*-algebras of noncompact quantum groups*
  (S.L. Woronowicz)
  Unbounded operators affiliated with C^*-algebras
  (S. Baaj, P. Julg, S.L. Woronowicz)
  Unbounded regular operators on Hilbert C^*-modulars

4. Hilbert C^*-Modules and C^*-Algebras

4.1 Basic definitions

Definition 1: A *(right) pre-Hilbert A-module* is a vector space E which is a right A-module
(with $\lambda(xa) = (\lambda x)a = x(\lambda a)$ for $\lambda \in \mathbb{C}, x \in E, a \in A$)
equipped with an „A-valued inner product“ $\langle \cdot, \cdot \rangle : E \times E \to A$, that is, for $x, y_1, y_2, y \in E, \alpha_1, \alpha_2 \in \mathbb{C}, a \in A$:

(i) $\langle x, \alpha_1 y_1 + \alpha_2 y_2 \rangle = \alpha_1 \langle x, y_1 \rangle + \alpha_2 \langle x, y_2 \rangle$,

(ii) $\langle x, ya \rangle = \langle x, y \rangle a$,

(iii) $\langle x, y \rangle^* = \langle y, x \rangle$,

(iv) $\langle x, x \rangle \geq 0$. If $\langle x, x \rangle = 0$, then $x = 0$.

From (ii) and (iii) we get $\langle xa, y \rangle = a^* \langle x, y \rangle$.

Proposition 2: *Cauchy-Schwarz inequality*

$$\langle x, y\rangle^* \langle x, y\rangle \leq \|\langle x, x\rangle\|_A \langle y, y\rangle \text{ for } x, y \in E.$$

Set $\|x\|_E := \|\langle x, x\rangle\|_A^{1/2}$. From Proposition 2 if follows that

$$\|\langle x, y\rangle\|_A \leq \|x\|_E \|y\|_E.$$

Moreover, $\|xa\|_E \leq \|x\|_E \|a\|_A$ for $x \in E, a \in A$.

Clearly, $\|\cdot\|_E$ is a norm on E.

If $(E, \|\cdot\|_E)$ is complete, then E is called a (right) *Hilbert A-module* or a *Hilbert C^*-module over A*.

Let $\langle E, E\rangle = \mathrm{Lin}\{\langle x, y\rangle; x, y \in E\}$. The closure of $\langle E, E\rangle$ is a two-sided ideal of A. The *Hilbert A-module* E is called *full* if this ideal is all of A, i.e. $\langle E, E\rangle$ is dense in E.

The guiding principle of the theory is to think of Hilbert C^*-modules as generalizations of complex Hilbert spaces.

However, there are fundamental differences:

- There is no orthogonal decomposition $E_1^\perp \oplus E_1 = E$ for a closed Hilbert A-submodule E_1.
  $E_1^\perp$ may be $\{0\}$ even if $E_1 \neq E$ and E_1 closed.

- Bounded operators do not have adjoints in general.

4.2 Examples

Example 1: $\underline{E = A}$, $\langle x, y\rangle = x^* y$

Then $\|x\|_E = \|\langle x, x\rangle\|_A^{1/2} = \|x^* x\|_A^{1/2} = \|x\|_A$.

E is full (take an approximate unit).

Example 2: Finite sums of Hilbert A-modules

$$E = E_1 \oplus \cdots \oplus E_n, \quad \langle (x_i), (y_i)\rangle = \textstyle\sum_{i=1}^n \langle x_i, y_i\rangle$$

If $E_1 = \cdots = E_n = A$, then E is denoted by A^n.

$$\|(x_1, \ldots, x_n)\|_E^2 = \|\sum_{i=1}^n x_i^* x_i\|_A^2 \left(\neq \sum_{i=1}^n \|x_i\|_A^2\right)$$

Example 3: Standard Hilbert A-module $\mathcal{H}_A$

$\mathcal{H}_A = \{(x_n) : x_n \in A$ and $\sum\limits_{i=1}^{\infty} x_n^* x_n$ converges in $A\}$

(This is weaker than requiring $\sum_n \|x_n\|^2 < \infty$!)

$\mathcal{H}_A$ is a vector space and right A-module. Define

$$\langle (x_n), (y_n) \rangle := \sum_{n=1}^{\infty} x_n^* y_n \tag{1}$$

Then $\mathcal{H}_A$ *becomes a Hilbert A-module*.

Convergence of (1) in A follows from the Cauchy-Schwarz inequality applied to A^k:

$$\|\sum_{n=1}^{k} x_n^* y_n\|^2 \leq \|\sum_{n=1}^{k} x_n^* x_n\| \|\sum_{n=1}^{k} y_n^* y_n\|.$$

4.3 Adjointable Operators on Hilbert C^*-modules

In what follows E, F and G are (right) Hilbert A-modules.

I. $\mathcal{L}(E, F)$

Let $\mathcal{L}(E, F)$ be the set of all maps $t : E \to F$ for which there exists a map $t^* : F \to E$ such that

$$\langle tx, y\rangle = \langle x, t^*y\rangle \text{ for } x \in E, y \in F.$$

Such maps t are called *adjointable*.

Some basic properties: $t \in \mathcal{L}(E, F), s \in \mathcal{L}(F, G)$

1. *t is linear and A-linear*, i.e. $t(xa)=t(x)a$ for $x\in E, a\in A$.

2. *t is bounded.*

3. *t^* is uniquely determined by t and $(t^*)^* = t$.*

4. *$st \in \mathcal{L}(E, G)$ and $(st)^* = t^*s^*$.*

5. *$\mathcal{L}(E) := \mathcal{L}(E, E)$ is a C^*-algebra.*

6. *A closed submodule E_1 of Hilbert A-module E is orthocomplementable (i.e $E_1 \oplus E_1^{\perp} = E$) iff E_1 is the range of a bounded adjointable operator.*

Example: Let $F = A = C[0,1]$ and

Let $E = \{f \in F : f(0) = 0\}$. Define $t(f) = f$ for $f \in E$, that is, t is the inclusion map, so t is bounded.

Statement 1: t is not adjointable.

Statement 2: $E^{\perp} := \{g \in F : \langle f, g\rangle = 0 \text{ for } f \in E\} = \{0\}$

II. $\mathcal{K}(E,F)$

For $x \in E$ and $y \in F$ define a map $\Theta_{x,y} : F \to E$ by

$$\boldsymbol{\Theta}_{\mathbf{x},\mathbf{y}}(\mathbf{z}) = \mathbf{x}\langle \mathbf{y}, \mathbf{z}\rangle, \mathbf{z} \in \mathbf{F}.$$

Then $\Theta_{x,y} \in \mathcal{L}(E,F)$.
Let $\mathcal{K}(E,F)$ be the closed linear subspace of $\mathcal{L}(E,F)$ spanned by $\Theta_{x,y}$, where $x \in E, y \in F$.

$\mathcal{K}(E) := \mathcal{K}(E,E)$ is a closed two-sided ideal of $\mathcal{L}(E)$.

Remarks: 1. $\Theta_{x,y}^{*} = \Theta_{y,x}$ and $\Theta_{x,y}\Theta_{u,v} = \Theta_{x,v\langle u,y\rangle}$.
2. Let $E = F$. Then
$\langle \Theta_{x,x}(z), z\rangle = \langle x, \langle x,z\rangle, z\rangle = \langle x,z\rangle^{*}\langle x,z\rangle \geq 0$,
$\Theta_{x,x}\Theta_{x,x} = \Theta_{x,x\langle x,x\rangle}$, and $\Theta_{x,x}^{2} = \Theta_{x,x}$ if $\langle x,x\rangle = 1$.
3. $\Theta_{x,x}$ is not compact in general!

4.4 Examples

$A \cong B$ denotes isomorphic C^*-algebra A and B.

1. $\underline{\mathcal{K}(A) \cong A \text{ and } \mathcal{L}(A) \cong M(A).}$

2. $\underline{\text{If } A \text{ is unital, then } \mathcal{L}(A) = \mathcal{K}(A) = B(A).}$

3. $\underline{\mathcal{L}(A^n) \cong M_n(M(A)) \text{ and } \mathcal{K}(A^n) \cong M_n(A).}$

4. $\underline{\mathcal{L}(\mathcal{H}_A) \cong M(A \otimes \mathcal{K}) \text{ and } \mathcal{K}(\mathcal{H}_A) \cong A \otimes \mathcal{K}.}$

5. **Theorem 2:** Green 1978, Kasparov 1980
For every Hilbert A-module E, $\quad M(\mathcal{K}(E)) \cong \mathcal{L}(E)$.

4.5 Kasparov's Stabilisation Theorem

A Hilbert A-module E is called *countably generated* if there exist a countable subset M such that closed A-submodule generated by M is E.

Two Hilbert A-modules E and F are called *unitarily equivalent* (Symbol: $E \cong F$) if there exists a unitary operator $u \in \mathcal{L}(E, F)$, that is, $u^*u = I_E$ and $uu^* = I_F$.

Theorem 3: T.G. Kasparov (JOT 4 (1980))
If E is countably generated Hilbert A-module, then $E \oplus \mathcal{H}_A \cong \mathcal{H}_A$.

In the proof the following proposition is used.

Proposition 4: *If there exists $T \in \mathcal{L}(E, F)$ such that t and t^* have dense ranges, then $E \approx F$.*

4.6 Imprimitivity Bimodules and Morita Equivalence

Let A and B C^*-algebras.

Definition 5: An $A{-}B$-*imprimitivity bimodule* is a vector space E which is a full right Hilbert A-module and a full left Hilbert B-module such that

(i) $\langle bx, y\rangle_A = \langle x, b^* y\rangle_A$ and $\langle xa, y\rangle_B = \langle x, ya^*\rangle_B$
for $a \in A, b \in B$ and $x, y \in E$.

(ii) $x\langle y, z\rangle_A = \langle x, y\rangle_B z$ for $x, y, z \in E$.

Remarks:
(i) says that B and A act by adjointable operators on E_A resp. E_B. This implies $(bx)a{=}b(xa)$ and $(\lambda b)(xa){=}b(x(\lambda b))$ for $b \in B, a \in A, x \in E, \lambda \in \mathbb{C}$, that is, E is indeed an $A{-}B$-bimodule.
(ii) It follows then $\|\langle x, x\rangle_A\|^{1/2} = \|\langle x, x\rangle_B\|^{1/2}$ for $x \in E$.

Definition 6: Two C^*-algebras A and B are called (strongly) *Morita equivalent* (Symbol $A \sim_M B$) if there exists an $A{-}B$-imprimitivity module.

(One can also define: A and B are Morita equivalent if there is a full Hilbert A-module such that $B \cong \mathcal{K}_A(E)$.)

Morita equivalence is indeed an equivalence relation.

Example 1: $A \sim_M A \otimes \mathcal{K}$
We define a left Hilbert $A \otimes \mathcal{K}$-module structure on the standard right Hilbert A-module $\mathcal{H}_A$:

$$(a_{ij}) \cdot (x_j) := (\sum_j a_{ij} x_j)$$

$$\langle (x_i), (y_j) \rangle_{A \otimes \mathcal{K}} := (x_i^* y_j)$$

Then $\mathcal{H}_A$ becomes an A–$A \otimes \mathcal{K}$-imprimitivity module.

Example 2: Let $\phi : B \to A$ be an isomorphism.
Then A becomes an $A - B$-imprimitivity module by

$$xa = x \cdot a, bx = \phi(b)x, \langle x, y \rangle_A = x^* y,$$

$$\langle x, y \rangle_B = \phi^{-1}(xy^*) \text{ for } x, y, a \in A, b \in B.$$

Theorem 7: L.G. Brown/P. Green/M. Rieffel (Pacific J. M. **71**(1977))
Two σ-unital C^*-algebras A and B are Morita equivalent if and only if they are stable isomorphic, that is, $A \otimes \mathcal{K}$ and $B \otimes \mathcal{K}$ are isomorphic.

Let E be an imprimitivity $A - B$-bimodule.
Having a representation of A, one can associate a representation of B by *Rieffel's induction*.

This implies that Morita equivalent C^*-algebras have *isomorphic categories of representations* on Hilbert space. Also, they have *isomorphic ideal lattices.*

5. Unbounded Operators on Hilbert C^*-Modules

5.1 Basic Definitions and Examples

Let E and F be Hilbert A-modules.
By an *operator* $t{:}E{\to}F$ we mean an A-linear map of a dense right A-submodule $\mathcal{D}(t)$ into F. For such an operator define
$\mathcal{D}(t^*){=}\{y{\in}F$: there exist $z{\in}E$ such that $\langle tx, y\rangle{=}\langle x, z\rangle$ for
all $x \in \mathcal{D}(t)\}$.

z is then uniquely determined by x.
Setting $t^*y = z$, we obtain an A-linear map t^* such that $\langle tx, y\rangle = \langle x, t^*y\rangle$ for $x \in \mathcal{D}(t), y \in \mathcal{D}(t^*)$.
t is called *closed* if $\mathcal{G}(t) = \{(x, tx); x \in \mathcal{D}(t)\}$ is closed in $E \oplus F$.
t is called *selfadjoint* if $t = t^*$.

Suppose that t is densely defined and closed. Then:

1. *t^* is closed.*

2. *$I + t^*t$ is closed, $(I + t^*t)\mathcal{D}(t^*t)$ is closed in E,*
   $\|(I + t^*t)x\| \geq \|x\| + \|tx\|$ for $x \in \mathcal{D}(t^*t)$.

Proof. All fact follow at once from this inequality.
We prove this inequality:

$$\langle x, x\rangle \leq \langle x, x\rangle + \langle tx, tx\rangle = \langle (I + t^*t)x, x\rangle,$$

$$\|x\|_E^2 = \|\langle x, x\rangle\|_A \leq \|\langle (I+t^*t)x, x\rangle\|_A \leq \|(I+t^*t)x\|_E \|x\|_E$$

by the Cauchy-Schwarz inequality, so $\|x\|_E \leq \|(I+t^*t)x\|_E$.

$$\|tx\|_E^2 \leq \|\langle (I + t^*t)x, x\rangle\|_A \leq \|(I + t^*t)x\|_E \|x\|_E,$$

so $\|tx\| \leq \|(I + t^*t)x\|$. □

Definition 8: An operator t is called *regular* if t is closed, $\mathcal{D}(t^*)$ is dense in F and $\operatorname{ran}(I + t^*t)$ is dense in E.

In the paper by M. Hilsum, K-theory **3**(1989), 401-440, there is an example of a closed (even selfadjoint) operator which is not regular!

5.2 Regular Operators on Hilbert C^*-modules

Lemma 9: (Paschke) *If* $T \in \mathcal{L}(E)$ *satisfies* $\langle Tx, x\rangle \geq 0$ *for* $x \in E$*, then* $T \in \mathcal{L}(E)_+$.

Proof. The polarization identity, applied to the sesquilinear form $(x, y) \to \langle Tx, y\rangle$, implies that $T = T^*$.
In the C^*-algebra $\mathcal{L}(E)$ we write $T = R - S$ with $R, S \in \mathcal{L}(E)_+$ and $R \cdot S = 0$. Then $\langle Ry, y\rangle \geq \langle Sy, y\rangle$, so

$$0 \leq \langle S\sqrt{S}x, S\sqrt{S}x\rangle = \langle SSx, Sx\rangle \leq \langle RSx, Sx\rangle = 0,$$

hence $\langle S^3x, x\rangle = 0$. By polarization, $\langle S^3x, y\rangle = 0$ and so $S^3 = 0$. Since $S \geq 0, S = 0$. Thus $T = R$. □

Theorem 10: *Suppose that* $t : E \to F$ *is a regular operator. Then there exist operators* $z_t \in \mathcal{L}(E, F)$ *and* $q_t \in \mathcal{L}(E)_+$ *such that* $\mathrm{ran}\,(q_t) = \mathcal{D}(t), 0 \leq q_t \leq 1$,

$$q_t = (I + t^*t)^{1/2} = (I - z_t^* z_t)^{1/2},\ z_t = tq_t, t = z_t q_t^{-1}.$$

Proof. I. By property 2. above, $I + t^*t$ is a bijection of $\mathcal{D}(t^*t)$ onto E. Set $a = (I + t^*t)^{-1}$.

If $x = (I + t^*t)y$, then $\langle ax, x\rangle = \langle y, (I + t^*t)y\rangle \geq 0$.

In particular, $a = a^*$ (by polarization) and $a \in \mathcal{L}(E)_+$ by Lemma 9. Set $q_t = a^{1/2}$.

From $\|ax\| = \|y\| \leq \|(I + t^*t)y\| = \|x\|$ we get $\|a\| \leq 1$

and so $0 \leq q_t \leq 1$.

II. *There is a bounded* $z_t : E \to F$ *such that* $z_t = tq_t$.

Proof.

$\langle tq_t^2x, tq_t^2x\rangle = \langle t^*tax, x\rangle \le \langle (I+t^*t)ax, ax\rangle = \langle x, ax\rangle = \langle q_tx, q_tx\rangle$

and hence

$$\|tq_t(q_tx)\| \le \|q_tx\| \text{ for } x \in t.$$

Define

$$z_t(q_tx) = tq_t^2x,\ x \in E.$$

Then $\|zy\| \le \|y\|$ for $y \in qE$. Since

$$q_tE \subseteq q_t^2E = (I + t^*t)^{-1}E \subseteq \mathcal{D}(t^*t)$$

is dense, z_t extends by continuity to a bounded operator on E and $\|z_t\| \le 1$.

We show that $\mathrm{ran}(q_t) \subseteq \mathcal{D}(t)$. Then we have $z_t = tq_t$. For let $x \in E$. Since q_tE is dense in E, there is a sequence $\{y_n\}$ such that $q_ty_n \to x$. Then $q_t^2y_n \to q_tx$ and $z_t(q_ty_n) = t(q_t^2y_n) \to z_tx$. Since t is closed, $q_tx \in \mathcal{D}(t)$. $\square$

III. $z_t \in \mathcal{L}(E,F)$ *and* $z_t^* = \overline{q_t t^*}$.

Proof. $\|q_t t^* x\|^2 = \|\langle q_t t^* x, q_t t^* x\rangle\| = \|\langle x, t q_t q_t t^* x\rangle\| \leq \|x\| \|t q_t\| \|q_t t^* x\| \leq \|q_t t^* x\| \|x\|$,
so $\|q_t t^* x\| \leq \|x\|$ for $x \in \mathcal{D}(t^*)$.

Hence $\overline{q_t t^*}$ is a bounded operator from F to E. Clearly, $z_t = t q_t$ has the adjoint $\overline{q_t t^*}$. □

IV. $\mathcal{D}(t) = \mathrm{ran}(q_t)$ *and* $t = z_t q_t^{-1}$.

Proof. Let $x \in \mathcal{D}(t)$ and $y \in E$. Then

$$\begin{aligned} &\langle y, (q_t z^* t + q_t^2) x\rangle = \langle z q_t y, t x\rangle + \langle q^2 y, x\rangle \\ &= \langle t^*(t q_t) q_t y, x\rangle + \langle q_t^2 y, x\rangle = \langle (I + t^* q_t^2 y, x\rangle = \langle y, x\rangle. \end{aligned}$$

Hence $x = q_t z^* t x + q_t^2 x \in \mathrm{ran}(q)$. Then converse inclusion was shown in II.

V. $q_t = (I - z_t^* z_t)^{1/2}$.

Proof. $z^* z q_t x \subseteq q_t t^* t q_t^2 x = q_t (I - q_t^2) x = (I - q_t^2) q_t x$. Since $q_t E$ is dense, $z_t^* z_t = 1 - q_t^2$. □

Corollary 11: *Let* $t : E \to F$ *be regular. Define a unitary* $v \in \mathcal{L}(E \oplus F, F \oplus E)$ *by* $v(x, y) = (y - x)$.
Then $\mathcal{G}(t) \oplus v\mathcal{G}(t^*) = E \oplus F$, *i.e. the graph of* t *is a orthocomplemented submodule.*

Proof. Define $p = \begin{pmatrix} q_t^2 & q_t z_t^* \\ z_t q_t & z_t z_t^* \end{pmatrix}$. Using the properties from Theorem 10 one shows that p is a projection such that $\operatorname{ran} p = \mathcal{G}(t)$ and $\operatorname{ran}(I - p) = v\mathcal{G}(t^*)$. □

Proposition 12: *If* $t : E \to F$ *is closed (and densely defined),* t^* *is densely defined and* $\mathcal{G}(t) \oplus \mathcal{G}(t)^\perp = E \oplus F$, *then* t *is regular.*

Proof. Let p be the projection of $E \oplus F$ onto $\mathcal{G}(t)$. Write

$$p = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}.$$

Then one verifies that $b = ta$ and $I - a = t^* b$. Hence $I - a = t^* t a$, so $(I + t^* t)a = I$ and $I + t^* t$ is surjective. □

Corollary 13: *Suppose that* $t : E \to F$ *is regular. Then* t^* *is regular,* $t^{**} = t$ *and* t^* *is self adjoint and regular.*

Proposition 14: *Let* $t : E \to F$*. Suppose that* $a \in \mathcal{L}(F)$ *and* $b \in \mathcal{L}(E)$ *are both invertible and* $c \in \mathcal{L}(E, F)$*. Then* t *is regular if and only if* $atb + c$ *is regular.*

We have shown the following result which is the fundamental theorem on regular operators.

Theorem 15: *Let* $\mathcal{R}(E, F)$ *denote the set of regular operators form* E *to* F *and let* $\mathcal{Z}(E, F)$ *the set of all* $z \in \mathcal{L}(E, F)$ *such that* $\|z\| \leq 1$ *and* $\mathrm{ran}(I - f^* f)$ *is dense in* E*.*

The map $t \to z_t$ *given by Theorem 10 is a bijection of* $\mathcal{R}(E, F)$ *onto* $\mathcal{Z}(E, F)$*. Moreover,* $(z_t)^* = z_{t^*}$*.*

5.3 Semiregular Operators

Let E and F be Hilbert C^*-modules for a C^*-algebra A.

Definition 16: (A. Pal 1999)

An operator $t : E \to F$ is called *semiregular* if t is closable and $\mathcal{D}(t^*)$ is dense in E.

Suppose t is a semiregular operator on the Hilbert C^*-module $E = F = A$. Let $\pi \in \mathrm{Rep}(A, \mathcal{H})$.

We define an operator on the C^*-algebra $\pi(A)$ by

$$\mathcal{D}(\pi(t)) = \pi(\mathcal{D}(t)) \text{ and } \pi(t)\pi(a) = \pi(ta), a \in \mathcal{D}(t).$$

Then $\pi(t)$ *is a well-defined semiregular operator on the Hilbert* C^**-module* $\pi(A)$*.*

Lecture 3:

Unbounded Operators and C^*-Algebras

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6. Unbounded Operators Affiliated With C^*-algebras

Motivation:

In noncommutative geometry a „noncommutative space“ are often described by a (noncommutative) C^*-algebra A.

Elements of A are interpreted as

„function on the noncommutative space vanishing at infinity“.

If the space is noncompact, then there are also

„unbounded functions on the noncommutative space“.

They are usually given by unbounded operators on some Hilbert space.

Problem: How to bring unbounded operators in relation to C^*-algebras?

Example 1: $A = C_0(\mathbb{R}^n)$

„A should be generated by the coordinate functions $x_k, k = 1, \ldots, n$“

Example 2: Quantum complex plane:
$z^* z = q^2 z z^*, 0 < q < 1$

Representations: $z e_n = q^n \lambda e_{n+1}, \lambda \in (q, 1]$, on $l^2(\mathbb{Z})$

Example 3: Quantum q-CCR algebra :
$z z^* - q z^* z = 1, 0 < q < 1$

Unbounded Representations:
$z e_n = \left(\frac{q^n \lambda + 1}{1-q}\right)^{1/2} e_{n-1}, \lambda \in (q, 1]$, on $l^2(\mathbb{Z})$

Bounded representations:
$z e_n = \left(\frac{q^n - 1}{q-1}\right)^{1/2} e_{n-1}$, on $l^2(\mathbb{N}_0)$,
$z = e^{i\varphi}(1-q)^{-1/2}, \varphi \in (0, 2\pi]$, on $\mathbb{C}$.

C^*-approach to noncompact quantum groups

6.1 Main Idea and Definitions

Let A be a non-degenerate C^*-algebra on $\mathcal{H}_0$ and let $T \in \mathcal{C}(\mathcal{H}_0)$.

We want to develop a concept „$T\eta A$“ such that the following holds:

For any $\pi \in \mathrm{Rep}(A, \mathcal{H})$ there should be an operator $\pi(T) \in \mathcal{C}(\mathcal{H})$ *canonically* associated with π.

I. *First assume that $\rho(T) \neq \phi$ and $\lambda \in \rho(T)$.*

Suppose for a moment that $(T - \lambda I)^{-1} \in A$.

Then it is natural to define

$\pi(T-\lambda I)\pi((T-\lambda I)^{-1}a)\varphi := \pi(a)\varphi, a \in A, \varphi \in \mathcal{H}$.

$\pi(T) := \pi(T-\lambda I) + \lambda I$

In order to have $\pi(T-\lambda I)$ densely defined, $\pi(T - \lambda I)^{-1}A)\mathcal{H}$ has to be dense in $\mathcal{H}$.

Example 1: Let $A = C_0(\mathbb{R})$ realized on $\mathcal{H}_0 = L^2(\mathbb{R})$ by multiplication operators. Let $f \in C(\mathbb{R})$ be real-valued.
Then $(f+i)^{-1} \notin C_0(\mathbb{R})$ in general,
but $(f+i)^{-1} \in C_b(\mathbb{R}) = M(C_0(\mathbb{R}))$. $\square$

Hence we should assume that $(T - \lambda I)^{-1} \in M(A)$.

Proposition 1: *For* $x \in M(A)$*, the following are equivalent:*

(i) xA *is dense in* A*.*
(ii) $\pi(x)\mathcal{H}$ *is dense in* $\mathcal{H}$ *for each* $\pi \in \mathrm{Rep}(A, \mathcal{H})$*.*
(iii) $\pi(x)\mathcal{H}$ *is dense in* $\mathcal{H}$ *for each irreducible* $\pi \in \mathrm{Rep}(A, \mathcal{H})$*.*

Since $\pi(x)^*\pi(x)\mathcal{H}$ is dense in $\mathcal{H}$ iff $\pi(x)\mathcal{H}$ is dense in $\mathcal{H}$, we obtain the following

Corollary 2: *For* $x \in M(A)$*,* xA *is dense in* A *if and only if* x^*xA *is dense in* $\mathcal{H}$*.*

Natural assumptions:
$(T - \lambda I)^{-1} \in M(A)$ and $(T - \lambda I)^{-1}A$ is dense in A.

Then $\pi(T)$ as defined above has a dense domain for any $\pi \in \mathrm{Rep}(A, \mathcal{H})$.

II. *General Case: Approach by S. L. Woronowicz*

Let $\mathcal{Z}(\mathcal{H})$ denote the set if all $Z \in \mathbb{B}(\mathcal{H})$ such that $\|Z\| \leq 1$ and $\ker(I - Z^*Z) = \{0\}$.

For $T \in \mathcal{C}(\mathcal{H})$ and $Z \in \mathcal{Z}(\mathcal{H})$ define

$$Z_T = T(I + T^*T)^{-1/2}, \quad T_Z = Z(I - Z^*Z)^{-1/2}.$$

Motivation:
$t \to z_t := t(1 + \bar{t}t)^{-1/2}$ is a bijective mapping of $\mathbb{C}$ onto the open unit disc $\mathbb{D}$ with inverse given by $z \to z(1 - \bar{z}z)^{-1/2}$.

Proposition 3: *The transformation* $T \to Z_T$ *is a bijective mapping of* $\mathcal{C}(\mathcal{H})$ *onto* $\mathcal{Z}(\mathcal{H})$ *with inverse given by* $Z \to T_Z$. *Both mappings preserve adjoints, that is,* $(Z_T)^* = Z_{T^*}$ *and* $(T_Z)^* = T_{Z^*}$.
Moreover, $(I + T^*T)^{-1/2} = (I - Z_T^* Z_T)^{1/2}$.

By the above definitions,

$$\mathcal{D}(T) = (I - Z_T^* Z_T)^{1/2}\mathcal{H},$$

$$T(I - Z_T^* Z_T)^{1/2}\varphi = Z_T\varphi$$

Idea:

Assume that $Z_T \in M(A)$ and $(I - Z_T^* Z_T)^{1/2} A$ is dense in A.

Define $\pi(T)\pi((I - Z_T^* Z_T)^{1/2} a)\varphi = \pi(Z_T a)\varphi$.

Definition 4: An operator $T \in \mathcal{C}(\mathcal{H})$ is *affiliated with* A if $Z_T \in M(A)$ and $(I - Z_T^* Z_T)A$ is dense in A.

Symbols: $T \eta A, \quad A^\eta = \{T \in \mathcal{C}(\mathcal{H}) : T \eta A\}$

Note that by Corollary 3, $(I - Z_T^* Z)A$ is dense in A iff $(I - Z_T^* Z_T)^{1/2} A$ is.

Suppose that $T \eta A$. Let $\pi \in \mathrm{Rep}(A, \mathcal{H})$.
Then $\|\pi(Z_T)\| \leq \|Z_T\|$. Since $\pi(I - Z_T^* Z_T)\mathcal{H}$ is dense in $\mathcal{H}$ by Proposition 1,
$\ker \pi(I - Z_T^* Z_T) = \ker(I - \pi(Z_T)^* \pi(Z_T)) = \{0\}$.
Hence $\pi(Z_T) \in \mathcal{Z}(\mathcal{H})$. By Proposition 3, there exists an operator $\pi(T) \in \mathcal{C}(\mathcal{H})$ such that $Z_{\pi(T)} = \pi(Z_T)$. By Definition 4, we have

$$\begin{aligned}
&\pi(T)\pi((I - Z_T^* Z_T)^{1/2} a)\varphi \\
&= \pi(T)(I - \pi(Z_T)^* \pi(Z_T))^{1/2} \pi(a)\varphi \\
&= \pi(Z_T)\pi(a)\varphi = \pi(Z_T a)\varphi, a \in A, \varphi \in \mathcal{H}.
\end{aligned}$$

6.2 Two Examples

Example 2: $A = C_0(X), X$ locally compact Hausdorff space. Then $A^\eta = C(X)$.

Proof. Suppose that A acts faithfully on $\mathcal{H} = L^2(X, \mu)$ by multiplication operators.

Then $M(A) = C_b(X)$.

For $f \in C_b(X)$ we have

f $\in \mathcal{Z}(\mathcal{H})$ iff $|f(x)| < 1$ for all $x \in X$ iff $(1 - \bar{f}f(x))C_0(X)$ is dense in $C_0(X)$.

Recall $T\eta A$ iff $Z_T{\in}M(A)$ and $(I{-}Z_T^*Z_T)A$ is dense in $\mathcal{H}$. Hence $A^\eta = C(X)$. ○

Example 3: $A = \mathcal{K}(\mathcal{H})$ compact operators
Then $\mathcal{K}(\mathcal{H})^\eta = \mathcal{C}(\mathcal{H})$.

Proof. $M(A) = \mathbb{B}(\mathcal{H})$. If $z \in \mathcal{Z}(\mathcal{H})$, then $(I - Z^*Z)^{1/2}\mathcal{H}$ is dense in $\mathcal{H}$. This implies that $(I - Z^*Z)^{1/2}\mathcal{F}(\mathcal{H})$ is dense in $\mathcal{F}(\mathcal{H})$ and hence $(I - Z^*Z)^{1/2}\mathcal{K}(\mathcal{H})$ is dense in $\mathcal{K}(\mathcal{H})$.

6.3 Some General Results

Proposition 5:
(i) $M(A) = \{T\eta A : T \in \mathbb{B}(\mathcal{H})\}$
(ii) *If* A *is unital, then* $A^{\eta} = A$*.*
(iii) *If* $T\eta A$*, then* $T^*\eta A$*.*

Proof of (i): If $x \in M(A)$, then $z_x = x(I - x^*x)^{-1/2} \in M(A)$. Since x is bounded, $\|Z_x\| < 1$ and $(I - Z_x^* Z_x)^{-1} \in M(A)$, so $x\eta A$.
Conversely, if $T\eta A$ and $T \in \mathbb{B}(\mathcal{H})$, then $\|Z_T\| < 1$ and
$T = Z_T(I - Z_T^* Z_T)^{-1/2} \in M(A)$.

The next propositions are more involved.

Proposition 6: *Functional calculus of normal operators: Let* $T\eta A$ *be normal. Then there exists a unique homomorphism* $\phi_T : C(\sigma(T)) \to A$ *such that* $\phi_T(f) = f(T)$*.*

Proposition 7:
(i) *If* $T\eta A$*, then* $T^*\eta A$*.*
(ii) *If* $x \in M(A)$ *and* $T\eta A$*, then* $(T + x)\eta A$*.*
(iii) *If* $x \in M(A)$ *is invertible and* $T\eta A$*, then* $xT\eta A$*.*

Proposition 8: *Suppose that* $T \in \mathcal{C}(\mathcal{H})$ *and* $\rho(T) \neq \phi$. *Then* $T \eta A$ *if and only if* $(T-\lambda I)^{-1} \in M(A)$ *and* $(T-\lambda I)^{-1} A$ *and* $(T^* - \overline{\lambda} I)^{-1} A$ *are dense in* A *for one [resp. for all]* $\lambda \in \rho(T)$.

Proposition 9 and Corollary 10 of Lecture 3 require additional considerations. It is planned treat them in a forthcoming research paper.

6.4 What is the Relation between Regular Operators and Affiliated Operators?

I. Let A be a C^*-algebra on a Hilbert space $\mathcal{H}$ and let $T \in \mathcal{C}(\mathcal{H})$.

Suppose $T\eta A$. Then $\mathcal{D}(T) = (I - Z_T^* Z_T)^{1/2}\mathcal{H}$. We consider T as operator on A with dense domain $(I - Z_T^* Z_T)^{1/2}A$. Then $(I + T^*T)A = (I - Z_T^* Z_T)A$ is dense in A, so T is a regular operator on the Hilbert C^*-module $E = A$.

Conversely, if T is regular on $E = A$, then $T\eta A$.

II. Let t be a closed operator on a Hilbert C^*-module E. Set $A = \mathcal{K}(E)$. Then $\mathcal{L}(E) = M(A)$ be the Green-Kasparov theorem.

Suppose t is regular. Then $z_t \in \mathcal{L}(E) = M(A)$. Since t is regular, $(I + t^*t)E \equiv (I - z_t^* z_t)E$ is dense in E and hence $(I - z_t^* z_t)\mathcal{K}(E)$ is dense in $\mathcal{K}(E)$, so $t\eta\mathcal{K}(E)$.

Conversely, if $t\eta\mathcal{K}(E)$, then t is regular.

- *Regular operators and affiliated operators are equivalent notions.*

7. C^*-algebras Generated by Unbounded Operators

7.1 Definition and Basic Results

Let A be a C^*-algebra and let $T_1, \ldots, T_n \eta A$.

Definition 11: (S. L. Woronowicz 1995)
The C^*-algebra A is *generated by the set* $\{T_1, \ldots, T_n\}$ if the following is true:
If $\pi \in \mathrm{Rep}(A, \mathcal{H})$ and B is a C^*-algebra on $\mathcal{H}$ such that $\pi(T_1), \ldots, \pi(T_n)$ are affiliated with B, then $\pi \in \mathrm{Mor}(A, B)$.

The next proposition says that the C^*-algebra generated by a given set is unique if it exists.

Proposition 12: *Let A and B be C^*-algebras and let $j \in \mathrm{Mor}(B, A)$ injective. Suppose that $S_1, \ldots, S_n \eta B$ and A is generated by $T_1 := j(S_1), \ldots, T_n := j(S_n)$. Then j is an isomorphism, that is, $j(B) = A$.*

Proof. We realize A faithfully on a Hilbert space $\mathcal{H}$ and identity B with $j(B)$. Since $j \in \mathrm{Mor}(B, A), \overline{BA} = A$.

Applying the above condition to $\pi = \mathrm{id}_A \in \mathrm{Rep}(A, \mathcal{H})$, we obtain $\mathrm{id}_A \in \mathrm{Mor}(A, B)$, that is, $\overline{AB} = B$.
Therefore, $A = B$. $\square$

Example 3: (A. Hertsch, Diplomarbeit, Leipzig)
Suppose α is irrational. For $(\varphi_n) \in l_2(\mathbb{Z})$, let

$$u(\varphi_n) = (e^{2\pi i\alpha}\varphi_n),$$
$$V(\cdots, \varphi_n, \cdots) = (\cdots, \varphi_{n-1}, \cdots).$$

Then $UV = e^{2\pi i\alpha} VU$.

Since $\ker(I - U) = \ker(I - V) = \{0\}$,

$$A := i(I + U)(I - U)^{-1}, B := i(I + V)(I - V)^{-1}$$

are selfadjoint operators on $l_2(\mathbb{Z})$.

It can be shown that the set $\{A, B\}$ does not generate a C^*-algebra!

Theorem 13: S.L. Woronowicz (1995) *Let* A *be a* C^**-algebra and* $T_1, \ldots, T_n \eta A$.

Assumptions:

(i) $\{T_1, \ldots, T_n\}$ *separates the representations of* A*, that is, if* $\pi_1, \pi_2 \in \mathrm{Rep}(A, \mathcal{H})$ *and* $\pi_1(T_k) = \pi_2(T_k)$ *for* $k = 1, \ldots, n$*, then* $\pi_1 = \pi_2$.

(ii) *There exists an element* $r \in A$ *such that:*
If $\pi \in \mathrm{Rep}(A, \mathcal{H})$ *and* $\pi(T_1), \ldots, \pi(T_k) \eta B$ *for a* C^**-algebra* B *on* $\mathcal{H}$*, then* $\pi(r)B$ *is dense in* B*.*

Then A *is generates by the set* $\{T_1, \ldots, T_n\}$*.*

Remark: Let r be a finite product of elements
$(I + T_k^* T_k)^{-1}, (I + T_k T_k^*)^{-1}$.
Since $\pi(T_k)\eta B$ implies that $\pi((I+T_k^* T_k)^{-1})B$ and $\pi((I+T_k T_k^*)^{-1})B$ are dense on B, then $\pi(r)B$ *is dense in* B. So assumption (ii) is satisfied if there exists such an element r in A.

Proof of Theorem 13:

Let $\pi \in \mathrm{Rep}(A, \mathcal{H})$ and let B be a C^*-algebra on $\mathcal{H}$ such that $\pi(T_1)\eta B, \ldots, \pi(T_n)\eta B$. By assumption (ii), $\overline{\pi(r)B} \supseteq B$ and hence $\overline{\pi(A)B} \supseteq B$.

To prove the opposite inclusion, set
$Q := \{x \in M(A) : \pi(x) \in M(B)\}$.
Then Q is a unital C^*-subalgebra of $M(A)$.
We show that Q separates the representations of A.
For let $\pi_1, \pi_2 \in \mathrm{Rep}(A, H)$ such that $\pi_1(q) = \pi_2(q)$ for all $q \in Q$. Since $\pi(T_k)\eta B$ and $\pi(Z_{T_k}) = Z_{\pi(T_k)}, Z_{T_k} \in Q$. Therefore, $\pi_1(Z_{T_k}) = \pi_2(Z_{T_k})$ and hence $\pi_1(T_k) = \pi_2(T_k)$ for $k = 1, \ldots, n$. By assumption (i), $\pi_1 = \pi_2$.

Thus Q satisfies the assumptions of the Stone-Weierstraß theorem proved in Lecture 2. Therefore, $Q = M(A)$. Hence $\pi(A) \subseteq M(B)$ and $\pi(A)B \subseteq B$.

Thus, $\overline{\pi(A)B} = B$ and $\pi \in \mathrm{Mor}(A, B)$. $\square$

7.2 Examples

1.) Let $T_1, \ldots, T_n$ be elements of the unital C^*-algebra A. Then A is generated by $\{T_1, \ldots, T_n\}$ according to Definition 11 iff $A = C^*(I, T_1, \ldots, T_n)$.

2.) Let X be a locally compact Hausdorff space. Let $f_1, \ldots, f_n \in C(X)$. Suppose that $\{f_1, \ldots, f_n\}$ separates points of X and $\lim\limits_{|x| \to \infty} \sum_k |f_k(x)|^2 = +\infty$. Then assumption (i) and (ii) are fulfilled. (Set $r = \pi_{k=1}^{n}(1 + |f_k(x)|)^{-1}$). Hence $C(X)$ is generated by $\{f_1, \ldots, f_n\}$.

Special case: $X = \mathbb{R}^n$

$C_0(\mathbb{R}^n)$ *is generated by* $\{x_1, \ldots, x_n\}$.

(It can be shown that the above assumptions are also <u>necessary</u> for $C_0(X)$ being generated by $\{f_1, \ldots, f_n\}$.)

3.) Let $T_1 \equiv P = -i\frac{d}{dx}$ and $T_2 \equiv Q = x$ on $\mathcal{H} = L^2(\mathbb{R})$. Then $r := (1+T_2^2)^{-1}(1+T_1^2)^{-1}(1+T_2^2)^{-1}$ is an integral operator with kernel

$$K(x,y) = \tfrac{1}{2}e^{-|x-y|}(1+x^2)^{-1}(1+y^2)^{-1}.$$

Since $K \in L^2(\mathbb{R}^2)$, r is compact, so $r \in \mathcal{K}(\mathcal{H}) \equiv A$. From the irreducibility of the Schrödinger representation it follows easily that $\{T_1, T_2\}$ separates representations of $\mathcal{K}(\mathcal{H})$.

Hence $\mathcal{K}(\mathcal{H})$ *is generated by* $\{P, Q\}$.

4.) Let T be a closed densely defined operator on $\mathcal{H}$ with non-empty resolvent set $\rho(T)$. Fix $\lambda \in \rho(T)$ and set $A := C^*((T-\lambda I)^{-1})$.

Assume that $(T-\lambda I)^{-1}A$ *and* $(T^*-\overline{\lambda}I)^{-1}A$ *are dense in* A.

Then $T\eta A$ by Proposition 8 from Section 6.

We check directly by using Definition 11 that
$A = C^*((T-\lambda I)^{-1})$ *is generated by* T.

Proof. Let $\pi \in \mathrm{Rep}(A, \mathcal{H})$. Assume that $\pi(T) \eta B$.
Then $R := \pi((T - \lambda I)^{-1}) = (\pi(T) - \lambda I)^{-1} \in M(B)$ and RB is dense in B. Since $R \in A$, $\overline{\pi(A)B} \supseteq B$.
Since $\mathcal{R} = \pi((T - \lambda I)^{-1}) \in M(B), \pi(A) \subseteq M(B)$ and so $\overline{\pi(A)B} \subseteq B$. Thus $\pi \in \mathrm{Mor}(A, B)$. $\square$

5.) Woronowicz/Napiorkowski (Reports MP **31**(1992))
Let G be a connected Lie group and $A := C^*(G)$.
Let $\{T_1, \ldots, T_n\}$ be a basis of the Lie algebra of G.
We consider $T_1, \ldots, T_n$ as linear operators on A.
Then it can be shown that $T_1, \ldots, T_n \eta C^*(G)$.
Also, one can show $r := (I + T_1^* T_1)^{-1} \cdots (I + T_n^* T_n)^{-1}$ belongs to A.

By Theorem 13, $C^*(G)$ *is generated by* $\{T_1, \ldots, T_n\}$.

Notations:

Let A be a C^*-algebra and $\mathcal{H}$ a Hilbert space.

A_{sa} - set of hermitean elements $a = a^*$ of A

$M(A)$ - Multiplier algebra of A

$\mathrm{Rep}(A, \mathcal{H})$ - nondegenerate representations of A on $\mathcal{H}$

$\mathbb{B}(\mathcal{H})$ - bounded operators on $\mathcal{H}$

$\mathcal{K}(\mathcal{H})$ - compact operators on $\mathcal{H}$

$\mathcal{K}$ - compact operators on $l^2(\mathbb{N})$

$\mathcal{C}(\mathcal{H})$ - densely defined closed operators on $\mathcal{H}$

$\mathrm{Mor}(A, B)$ - $*$-homomorphisms ϕ of A into a C^*-algebra B such that $\phi(A)B$ is dense in B.

$C^*(T_1, \ldots, T_n)$ - C^*-algebra generated by $T_1, \ldots, T_n$

Let X be a locally compact Hausdorff space.

$C_0(X)$ - continuous functions on X vanshing at infinity

$C_b(X)$ - bounded continuous functions on X