Exercise Sheet 12

Discussion on 26.01.24

Exercise 1 (P_2 is no C^1 element)
Consider a regular triangulation $\mathcal{T}$ and the P_2 finite element $(T, P_2(T), \mathcal{K}_T)$ for any $T \in \mathcal{T}$. Prove that this finite element is not a C^1 finite element in general.

Exercise 2 (Transformation of finite elements)
Let $(\widehat{T}, \widehat{\mathcal{P}}, \mathcal{K}_{\text{ref}})$ be a finite element and $\Phi_T : \widehat{T} \to T$ an affine diffeomorphism.

a) Show that $(T, \mathcal{P}, \mathcal{K})$ is a finite element, where

$$\begin{aligned} T &= \Phi_T(\widehat{T}), \\ \mathcal{P} &= \{\widehat{q} \circ \Phi_T^{-1} \,|\, \widehat{q} \in \widehat{\mathcal{P}}\}, \\ \mathcal{K} &= \{\chi | \chi(v) = \widehat{\chi}(v \circ \Phi_T) \text{ for all } v \in \mathcal{C}^\infty(T), \widehat{\chi} \in \widehat{\mathcal{K}}\}. \end{aligned}$$

b) Show that the corresponding interpolants I_T and $I_{\widehat{T}}$ and any $\widehat{v} \in W^{m,p}(\widehat{T})$ and $v := \widehat{v} \circ \Phi_T^{-1} \in W^{m,p}(T)$ satisfy

$$(I_T v) \circ \Phi_T = I_{\widehat{T}} \widehat{v}.$$

Exercise 3 (Barycentric coordinates)
Consider a triangle $T = \text{conv}\{P_1, P_2, P_3\}$ and the barycentric coordinates $\lambda_1, \lambda_2, \lambda_3 \in P_1(T)$ defined via $\lambda_j(P_k) = \delta_{jk}$ for $j, k = 1, 2, 3$.

a) Prove that any $\alpha, \beta, \gamma \in \mathbb{N}_0$ satisfy

$$\int_T \lambda_1^\alpha \lambda_2^\beta \lambda_3^\gamma \, \mathrm{d}x = 2|T| \frac{\alpha! \beta! \gamma!}{(2 + \alpha + \beta + \gamma)!}.$$

b) For the points $P_{j+3} := (P_j + P_{j+1})/2$, $j = 1, 2$ and $P_6 := (P_3 + P_1)/2$, find the nodal basis functions $\mu_j \in P_2(T)$ with $\mu_j(P_k) = \delta_{jk}$ for $j, k = 1, \ldots, 6$.

c) Compute the local mass matrices for $P_1(T)$ and $P_2(T)$, i.e.

$$M_1 = \left(\int_T \lambda_i \lambda_j \, \mathrm{d}x \right)_{i,j=1,\ldots,3} \in \mathbb{R}^{3\times 3} \qquad \text{and} \qquad M_2 = \left(\int_T \mu_i \mu_j \, \mathrm{d}x \right)_{i,j=1,\ldots,6} \in \mathbb{R}^{6\times 6}$$

Exercise 4 (Error and refinement)

a) Let $(\mathcal{T}_k)_{k\in\mathbb{N}}$ be a sequence of regular triangulations, where $\mathcal{T}_{k+1}$ is a refinement of $\mathcal{T}_k$ for any $k \in \mathbb{N}$. Furthermore, let $u \in H_0^1(\Omega)$ be the exact solution (Galerkinapproximation) and $u_k \in S_0^1(\mathcal{T}_k)$ the P_1 finite element solution to the Poisson model problem on each level $k \in \mathbb{N}$, where

$$S_0^1(\mathcal{T}_k) := \{v_h \in \mathcal{C}(\overline{\Omega}) \,|\, \ldots v_h|_T \in P_k(T) \text{ for all } T \in \mathcal{T}_k, v_h|_{\partial\Omega} = 0\}.$$

Prove that $\|\nabla(u - u_k)\|_{L^2(\Omega)}$ is a monotonically decreasing sequence.

b) Consider the criss-cross triangulation $\mathcal{T}_0$ and its refinement depicted in Figure 1. Prove that the P_1 finite element solutions to the Poisson model problem with $f \equiv 1$ and $u|_{\partial\Omega} \equiv 0$ on the triangulations coincide.

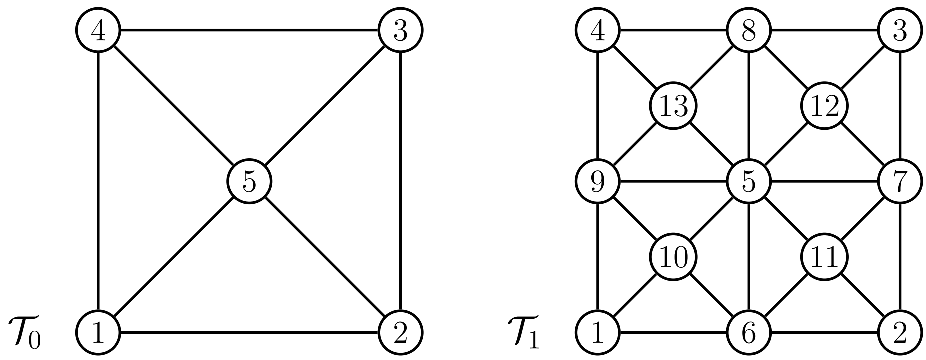

Figure 1: Criss-cross triangulation $\mathcal{T}_0$ (left) and its refinement $\mathcal{T}_1 = \texttt{bisec}(\texttt{bisec}(\mathcal{T}_0))$