Exercise Sheet 11

Discussion on 19.01.23

Exercise 1 (Minimum angle condition)
For any triangle T and node $z \in \mathcal{N}(T)$, denote by $\sphericalangle(T, z)$ the interior angle of T at z. Prove that any family $(\mathcal{T}_k)_{k\in\mathbb{N}}$ of regular triangulations with

$$0 < \omega_0 \leq \inf_{k\in\mathbb{N}} \min_{T\in\mathcal{T}_k} \min_{z\in\mathcal{N}(T)} \sphericalangle(T, z) \tag{1}$$

is shape regular. Furthermore, find an example of a family of triangulations that does not satisfy (1) and is not shape regular.

Exercise 2 (Equivalences for shape regularity)
Consider a family of triangulations with the minimum angle condition (cf. Exercise 1) and let $\mathcal{T}$ be any triangulation of this family. Let $z \in \mathcal{N}$ and define $\mathcal{T}(z) := \{T \in \mathcal{T} \mid z \text{ is a node of } T\}$. Prove

$$\#\mathcal{T}(z) \lesssim 1.$$

Furthermore, for a triangle $T \in \mathcal{T}(z)$ with diameter h_T and an edge $E \in \mathcal{E}$ of T, prove

$$|E| \approx |T|^{1/2} \approx h_T,$$

where $|E|$ denotes the length of E and $|T|$ denotes the volume of T. Here, $A \lesssim B$ abbreviates, that there exists a constant $C < \infty$ that may depend on the minimum angle ω_0, but not on other properties of the family of triangulations. The formula $A \approx B$ abbreviates $A \lesssim B \lesssim A$.

Exercise 3 (Q_k-FEM)
Let $T = [0, 1]^2 \subseteq \mathbb{R}^2$, $k \in \mathbb{N}$, $0 = t_0 < t_1 < \cdots < t_k = 1$ and define $Q_{ij} := (t_i, t_j) \in T$. Furthermore, let $\mathcal{K} = \{\chi_{ij} \mid i, j = 0, \ldots, k\}$ with $\chi_{ij}(v) := v(Q_{ij})$ for $v \in C^\infty(T)$. Prove that $(T, Q_k(T), \mathcal{K})$ is a finite element (where $Q_k(T)$ is the space of polynomials of partial degree $\leq k$).

Exercise 4
Let $T \subseteq \mathbb{R}^2$ be a triangle and let $0 < t < 1/2$. Let $E = \operatorname{conv}\{a, b\} \subseteq T$ be an edge of T with midpoint $\operatorname{mid}(E) = (a + b)/2$ and define

$$p_E^\pm := \operatorname{mid}(E) \pm t(a - b) \in E,$$
$$\chi_{p_E^\pm}(v) := v(p_E^\pm) \qquad \text{for all } v \in \mathcal{C}^\infty(T).$$

Define $\mathcal{K} := \{\chi_{p_E^\pm} \mid E \text{ is an edge of } T\}$. Show that $(T, P_2(T), \mathcal{K})$ is *not* a finite element.