Exercise Sheet 10

Discussion on 12.01.23

Exercise 1 (Globally continuous, piecewise differentiable functions)
Let $\Omega \subseteq \mathbb{R}^n$ be a Lipschitz domain and let $(\Omega_j)_{j=1,\dots,J}$ be Lipschitz domains that are disjoint subsets of Ω with $\overline{\Omega} = \overline{\Omega}_1 \cup \cdots \cup \overline{\Omega}_J$. Let $u : \Omega \to \mathbb{R}$ with $u|_{\Omega_j} \in \mathcal{C}^1(\overline{\Omega}_j)$ for all $j = 1, \dots, J$. Show that u is weakly differentiable if and only if $u \in \mathcal{C}(\overline{\Omega})$.

Exercise 2 (Solutions to PMP)
In the setting of the Poisson model problem, let $V = H^1_D(\Omega)$, $a : V \times V \to \mathbb{R}$ defined by $a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x$ and $F(v) = \int_\Omega fv \, \mathrm{d}x + \int_{\Gamma_N} gv \, \mathrm{d}s$. Furthermore assume that Γ_D has positive $(d-1)$ dimensional measure. Prove that there exists a unique solution $u \in V$ to

$$a(u, v) = F(v) \quad \text{for any } v \in V,$$

and that this solution satisfies

$$\|u\|_{H^1(\Omega)} \leq (1 + C_P^2) \max\{1, C_\gamma\} \big(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Gamma_N)}\big).$$

Exercise 3 ($H^1(\Omega)$ is not embedded in $\mathcal{C}^0(\overline{\Omega})$)
Let $\Omega = B_{1/2}(0) \subseteq \mathbb{R}^2$ and $u : \Omega \to \mathbb{R}$ with

$$u(x) := \ln\Big(\big|\ln|x|\big|\Big).$$

Prove that $u \in H^1(\Omega) \setminus \mathcal{C}(\Omega)$.
Hint: Calculate the (classical) partial derivatives on $B_{1/2}(0) \setminus \{0\}$ and show that they lie in $L^2(\Omega)$ by transformation to polar coordinates.

Exercise 4 (Euler formulas)
Let $\mathcal{T}$ denote a regular triangulation of the simply connected bounded domain Ω with nodes $\mathcal{N}$, edges $\mathcal{E}$ and interior edges $\mathcal{E}(\Omega)$. Prove that

$$|\mathcal{N}| + |\mathcal{T}| = 1 + |\mathcal{E}|, \qquad 2|\mathcal{T}| + 1 = |\mathcal{N}| + |\mathcal{E}(\Omega)|,$$

where $|\bullet|$ denotes the number of elements in a set. How can these formulas be generalized for multiply connected domains?