Exercise Sheet 9

Discussion on 05.01.23

Exercise 1 (Finite-Differences for the Poisson problem (five-point stencil))
Let $\Omega = (0,1)^2$, $\Gamma_D = \partial\Omega$ and $u_D = 0$. Let u denote the solution to the Poisson problem

$$\begin{aligned} -\Delta u = f \qquad &\text{in } \Omega, \\ u|_{\Gamma_D} = 0. \end{aligned}$$

Let $x_{j,m} := (j\Delta x, m\Delta x)$ be as in exercise 2 and 3 from sheet 8. The finite difference approximation $(U_{j,m})_{0\leq j,m\leq J}$ is defined by

$$\begin{aligned} -\Delta_h U_{j,m} = f(x_{j,m}) \qquad &\text{für } 1 \leq j,m \leq J-1, \\ U_{0,m} = U_{J,m} = U_{j,0} = U_{j,J} = 0 \qquad &\text{for } 0 \leq j,m \leq J. \end{aligned}$$

Let $u \in \mathcal{C}^4(\overline{\Omega})$. Use exercise 3 from sheet 8, to prove the error estimation

$$\sup_{0\leq j,m\leq J} |u(x_{j,m}) - U_{j,m}| \leq \frac{\Delta x^2}{12} \|u\|_{\mathcal{C}^4([0,1]^2)}.$$

Exercise 2 (Weak derivatives are necessary)
Let $\Omega = (-1,1) \subseteq \mathbb{R}$ and $v_\varepsilon, \operatorname{sgn} : \overline{\Omega} \to \mathbb{R}$ with $v_\varepsilon(x) := \sqrt{|x|^2 + \varepsilon^2} - \varepsilon$ and

$$\operatorname{sgn}(x) := \begin{cases} \frac{x}{|x|} & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

Prove that v_ε converges to $v = |\bullet|$ and the gradient of v_ε converges to sgn in $L^2(\Omega)$ for $\varepsilon \searrow 0$. Use this to show that $C^1(\overline{\Omega})$, equipped with the scalar product $(\bullet, \bullet)_{L^2(\Omega)} + (\nabla \bullet, \nabla \bullet)_{L^2(\Omega)}$, is *not* a Hilbert space.

Exercise 3 (Energy minimization)
Let V be a Hilbert space with scalar product $(\bullet, \bullet)_V$, $a : V \times V \to \mathbb{R}$ a symmetric bilinear form with $0 \leq a(v,v)$ for any $v \in V$. Given $f \in V$, define $G(v) := \frac{1}{2}a(v,v) - (f,v)_V$ for any $v \in V$. Prove that the following two statements are equivalent for $u \in V$:

1. $G(u) = \min_{v\in V} G(v)$,
2. $a(u,v) = (f,v)_V$ for any $v \in V$.

Exercise 4 (Regularity of solutions)
Let $\gamma \in (\pi, 2\pi)$ and $\Omega := \{(r\cos\varphi, r\sin\varphi) \,|\, 0 < r < 1, 0 < \varphi < \gamma\}$ and $u(r,\varphi) = r^{\pi/\gamma}\sin(\varphi\pi/\gamma)$ in polar coordinates on Ω. Prove that $u : \Omega \to \mathbb{R}$ solves the Poisson problem $-\triangle u = 0$ on Ω with right-hand side $f \equiv 0$ and respective boundary data, but $u \notin C^1(\overline{\Omega})$ for $\gamma \in (\pi, 2\pi)$.
Hint: You may use the formula for the Laplacian in polar coordinates,

$$\Delta u = \partial^2 u/\partial r^2 + r^{-1}\partial u/\partial r + r^{-2}\partial^2 u/\partial\varphi^2.$$